

All-optical imprinting of geometric phases onto matter waves

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Traditional optical phase imprinting of matter waves is of a dynamical nature. Here we show that both Abelian and non-Abelian geometric phases can be optically imprinted onto matter waves, yielding a number of interesting phenomena such as wave-packet redirecting and wave-packet splitting. In addition to their fundamental interest, our results open up opportunities for robust optical control of matter waves.

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When the Hamiltonian of a quantum system adiabatically changes along a path in its parameter space, a nondegenerate (degenerate) eigenstate will acquire, on top of an obvious dynamical phase, an Abelian (non-Abelian) geometric phase [1,2]. The discovery of geometric phases has motivated many proposals on built-in fault-tolerant quantum logical operations [3]. Geometric phases also provide an elegant framework to understand the coupling between translational motion and internal degrees of freedom. In particular, a slowly moving system subject to an external field naturally causes adiabatic changes in its internal Hamiltonian, giving rise to an Abelian or non-Abelian vector potential for the translational motion [4–6]. Interestingly, one can also engineer laser-matter interaction to synthesize effective vector potentials from geometric phases [7–10].

One implicit assumption in geometric phase-based quantum gate studies is that the qubit systems are frozen in space. By contrast, here we examine the geometric phases acquired by a system coherently delocalized in space. In particular, we propose to optically imprint geometric phases at each local space point onto delocalized matter waves of ultracold systems. We show that the imprinted geometric phases can alter the ensuing matter-wave propagation dramatically, thus establishing another interesting and potentially powerful means of optical control. Because here the geometric phases are not induced by the translational motion itself, but by active manipulation of laser-matter interaction, the imprinting of geometric phases onto matter waves results in a new type of coupling between internal and translational motions, which can be implemented experimentally.

In generating matter-wave vortices with time-dependent magnetic fields, some early work did take advantage of a spatially dependent Abelian Berry phase [11]. Our work however represents a major extension from the vortex context, by demonstrating the imprinting of both Abelian and non-Abelian geometric phases with *all-optical* methods. We shall show that imprinting geometric phases may redirect atom wave packets, split a wave packet, or create interesting matter-wave patterns. The all-optical geometric phase imprinting is insensitive to the dynamical details. Hence it is superior to conventional optical phase imprinting approach to matter-wave engineering [12,13], where the imprinted

phase is proportional to the duration of optical imprinting and to the magnitude of an optical potential.

Consider first a two-level atom in a plane-wave laser field. In the rotating wave approximation (RWA), the internal Hamiltonian is given by $H_{RWA,2} = (\Delta/2)(|1\rangle\langle 1| - |2\rangle\langle 2|) + (\Omega_0/2)(e^{-ik^l \cdot \mathbf{R}}|1\rangle\langle 2| + e^{ik^l \cdot \mathbf{R}}|2\rangle\langle 1|)$, where $|1\rangle$ and $|2\rangle$ are the two internal states of the atom, Ω_0 is assumed to be real, \mathbf{k}^l is the wave vector of the laser field, Δ is the detuning from one-photon resonance, and \mathbf{R} is the coordinate of the atom. In the $(|1\rangle, |2\rangle)$ representation $H_{RWA,2}$ has the eigenstate

$$|\phi^+\rangle = \frac{1}{\chi(\Delta)} \begin{pmatrix} \sqrt{\Omega_0^2 + \Delta^2 + \Delta} \\ \Omega_0 e^{ik^l \cdot \mathbf{R}} \end{pmatrix}, \quad (1)$$

and an analogous eigenstate $|\phi^-\rangle$ orthogonal to $|\phi^+\rangle$, where $\chi(\Delta) = \sqrt{\Omega_0^2 + (\sqrt{\Omega_0^2 + \Delta^2} + \Delta)^2}$ is the normalization factor. Note that these internal eigenstates vary with the coordinate \mathbf{R} . Within the adiabatic approximation and in the absence of a trapping potential, the overall eigenstate (internal plus translational) for the system, denoted $|\Psi^\pm\rangle$, is given by $|\Psi^\pm\rangle = |\phi^\pm\rangle e^{i\mathbf{k} \cdot \mathbf{R}}$, where \mathbf{k} is a wave vector associated with the translational motion of the atom.

We begin with a test case that is well-known in laser chirping control [14] (which realizes a Landau-Zener process). Here the detuning parameter Δ is adiabatically varied from $\Delta_1 > 0$ to $\Delta_2 < 0$, with $|\Delta_1/\Omega_0| \gg 1$ and $|\Delta_2/\Omega_0| \gg 1$. Accordingly \mathbf{k}^l changes from \mathbf{k}_1^l to \mathbf{k}_2^l but with a fixed direction. The initial state is assumed to be $|\Psi^+(\Delta = \Delta_1)\rangle \approx (1, 0)^T e^{i\mathbf{k} \cdot \mathbf{R}}$. Assuming that the translational motion during the chirping process is negligible, which can be a good approximation for ultracold atoms, one finds the following geometric phase for an arbitrary \mathbf{R} :

$$\begin{aligned} \beta(\mathbf{R}) &= i \int_{\Delta_1}^{\Delta_2} \langle \phi^+ | \frac{\partial}{\partial \Delta} | \phi^+ \rangle d\Delta \\ &= - \frac{\mathbf{R} \cdot \hat{\mathbf{k}}^l}{\hbar c} \int_{\Delta_1}^{\Delta_2} \frac{\Omega_0^2}{\chi^2(\Delta)} d\Delta \approx - \mathbf{R} \cdot [\mathbf{k}_2^l - \mathbf{k}_1^l], \end{aligned} \quad (2)$$

where \mathbf{k}_r^l denotes the laser wave vector when $\Delta = 0$ [15], and throughout $\hat{\mathbf{k}}$ represents a unit vector along the vector \mathbf{k} . Given that the evolving state remains in the system eigen-

state, the dynamical phase is evidently \mathbf{R} independent. Imprinting the geometric phase $\beta(\mathbf{R})$ onto the adiabatic state $|\phi^+\rangle$ in Eq. (1) and neglecting the overall dynamical phase, one obtains the final state

$$|\Psi_f^+(\Delta_2)\rangle = \frac{e^{i[\beta(\mathbf{R})+\mathbf{k}\cdot\mathbf{R}]}}{\chi(\Delta_2)} \begin{pmatrix} \sqrt{\Omega_0^2 + \Delta_2^2} + \Delta_2 \\ \Omega_0 e^{ik_2^l \cdot \mathbf{R}} \end{pmatrix} \\ \approx e^{i[\beta(\mathbf{R})+\mathbf{k}\cdot\mathbf{R}]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(\mathbf{k}_r^l + \mathbf{k})\cdot\mathbf{R}}. \quad (3)$$

Equation (3) shows that the population between the two levels is adiabatically inverted and the atom momentum increases by $\hbar\mathbf{k}_r^l$. This result is expected because (i) the chirping scenario is long known to induce the absorption of one on-resonance photon and (ii) momentum conservation requires the atom momentum to increase by $\hbar\mathbf{k}_r^l$. But quite noteworthy, here the obvious consequence of momentum conservation is manifested as an effect of the spatially-dependent geometric phase $\beta(\mathbf{R})$. Note also that if we further change Δ along a reversed path until it reaches its initial value, then Eq. (2) gives a zero Berry phase and hence zero momentum change. This is again correct because the atom releases a photon during the reversed process. One can also carry out similar analysis if the chirped laser field possesses a nonzero orbital angular momentum. That is, the transfer of orbital angular momentum from photon to the matter wave (hence vortex generation) can also be interpreted as geometric phase imprinting.

As a second Abelian example, we examine the system of $H_{\text{RWA},2}$ for $\Delta=0$ and with $\hat{\mathbf{k}}_r^l$ being rotated adiabatically. We assume below both $\hat{\mathbf{k}}_r^l$ and \mathbf{k} are in the x - z plane. Let γ be the angle by which $\hat{\mathbf{k}}_r^l$ is rotated clockwise with respect to the x -axis. Then $\mathbf{k}_r^l(\gamma) = k_r^l [\cos(\gamma)\hat{e}_x - \sin(\gamma)\hat{e}_z]$, where $k_r^l \equiv |\mathbf{k}_r^l|$, \hat{e}_x and \hat{e}_z are the unit vectors along the x and z axes. For convenience we also define the polar coordinates (R, θ) for \mathbf{R} confined on the x - z plane, i.e., $x=R \cos(\theta)$ and $z=R \sin(\theta)$. Then, the eigenstate in Eq. (1) for $\Delta=0$ can be rewritten as

$$|\phi^+(\gamma)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{ik_r^l R \cos(\theta+\gamma)} \end{pmatrix}. \quad (4)$$

Consider an initial state $|\Psi^+\rangle = |\phi^+(\gamma=0)\rangle e^{i\mathbf{k}\cdot\mathbf{R}}$. As the adiabatic parameter γ varies from 0 to α , the acquired geometric phase at an arbitrary local point (R, θ) is found to be

$$\beta(R, \theta) = i \int_0^\alpha \langle \phi^+ | \frac{\partial}{\partial \gamma} | \phi^+ \rangle d\gamma = \frac{k_r^l}{2} [R \cos(\theta) - R \cos(\theta + \alpha)]. \quad (5)$$

Neglecting the translational motion during the adiabatic process, the final state $|\Psi_f^+(\alpha)\rangle$ can be obtained by imprinting $\beta(R, \theta)$ onto the adiabatic state $|\phi^+(\gamma=\alpha)\rangle e^{i\mathbf{k}\cdot\mathbf{R}}$, yielding

$$|\Psi_f^+(\alpha)\rangle = e^{i[\beta(R, \theta) + \mathbf{k}\cdot\mathbf{R}]} |\phi^+(\gamma=\alpha)\rangle \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ik_r^l R \sin(\theta+\alpha/2)\sin(\alpha/2)} \\ e^{ik_r^l R \cos(\theta+\alpha/2)\cos(\alpha/2)} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{R}}. \quad (6)$$

Unlike in the above test case that apparently involves the absorption of one on-resonance photon, here neither the population on states $|1\rangle$ and $|2\rangle$ nor the atom mechanical momentum changes. However, the geometric phases $\beta(R, \theta)$ imprinted onto the matter wave will induce differences in time evolution if we suddenly switch off the laser field, after which the matter waves associated with each of the two components of $|\Psi_f^+(\alpha)\rangle$ will evolve independently and freely. The momentum in free space for each component of $|\Psi_f^+(\alpha)\rangle$ can be found by taking the spatial derivative of that component. Without loss of generality we set $\mathbf{k}=0$. One then finds that upon a sudden laser field switch off, the state $|\Psi_f^+(\alpha)\rangle$ will split into two components: atoms with the internal state $|2\rangle$ will move in the direction of $\cos(\alpha/2)\hat{e}_x - \sin(\alpha/2)\hat{e}_z$ with a momentum $\hbar k_r^l \cos(\alpha/2)$, but the atoms on the state $|1\rangle$ will move in the direction of $\sin(\alpha/2)\hat{e}_x + \cos(\alpha/2)\hat{e}_z$ with a momentum $\hbar k_r^l \sin(\alpha/2)$. These motion characteristics are clearly α dependent and intriguing. In particular, if there is no geometric phase imprinting, $\alpha=0$, the $|1\rangle$ component is static and the $|2\rangle$ component moves along \hat{e}_x ; but if $\alpha=\pi$, then the $|2\rangle$ component is static and the $|1\rangle$ component moves along \hat{e}_x . This could be useful for filtering the internal states.

We have also carried out numerical wave-packet simulations based on the Hamiltonian $-\frac{\hbar^2 \nabla^2}{2m} + H_{\text{RWA},2}$ (m being the mass of the atom) that undergoes slow changes in its parameter γ . In our simulations we replace the plane-wave factor $\exp(i\mathbf{k}\cdot\mathbf{R})$ of $|\Psi^+\rangle$ by a Gaussian wave packet. Numerical results do confirm our predictions.

We now discuss the imprinting of non-Abelian geometric phases. As in Refs. [10,16,17], we adopt the so-called ‘‘tripod scheme’’ of four-level atoms interacting with three on-resonance laser fields. The internal state is denoted as $|n\rangle$, $n=0-3$. Each of the three transitions $|0\rangle \leftrightarrow |1\rangle$, $|0\rangle \leftrightarrow |2\rangle$, and $|0\rangle \leftrightarrow |3\rangle$ is coupled by one laser field. This coupling scheme can be realized if states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are degenerate magnetic sublevels and the three coupling fields have different polarizations. Alternatively, states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are nondegenerate and the three coupling fields then carry different frequencies (same polarization is then allowed). For convenience we adopt the same configuration as in Ref. [17], where two laser beams are counterpropagating along the x axis and the third laser beam is along the z axis. The associated internal Hamiltonian under RWA is given by $H_{\text{RWA},4} = \sum_{n=1}^3 \Omega_n |0\rangle \langle n| + \text{H.c.}$, with $\Omega_1 = \Omega_0 \sin(\xi) / \sqrt{2} e^{-ik_r^l x}$, $\Omega_2 = \Omega_0 \sin(\xi) / \sqrt{2} e^{ik_r^l x}$, and $\Omega_3 = \Omega_0 \cos(\xi) e^{ik_r^l z}$, where the parameter ξ is set to satisfy $\cos(\xi) = \sqrt{2} - 1$. Note that even if the three lasers carry different frequencies, their effective k_r^l in the x - z plane can still be the same by tilting the laser beams out of the x - z plane.

The Hamiltonian $H_{\text{RWA},4}$ has two degenerate and spatially dependent dark (null-eigenvalue) states $|D_{1(2)}\rangle$, which are given by [17],

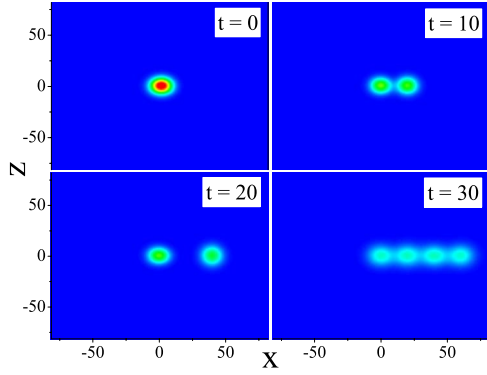


FIG. 1. (Color online) Numerical simulation of wave-packet splitting induced by imprinting non-Abelian geometric phases. Here t is in units of $m/(\hbar\kappa^2)$; x and z are in units of $1/\kappa$. See the text for details.

$$|D_1\rangle = (|\tilde{1}\rangle - |\tilde{2}\rangle)e^{-i\kappa'z/\sqrt{2}},$$

$$|D_2\rangle = [\cos(\xi)(|\tilde{1}\rangle + |\tilde{2}\rangle)/\sqrt{2} - \sin(\xi)|3\rangle]e^{-i\kappa'z}, \quad (7)$$

where $\kappa' \equiv k_r[1 - \cos(\xi)]$, $|\tilde{1}\rangle \equiv |1\rangle e^{ik_r(x+z)}$, and $|\tilde{2}\rangle \equiv |2\rangle e^{-ik_r(x-z)}$. Clearly, within the dark-state subspace the dynamical phase is always zero. So we focus on the time evolution in this dark subspace. Any state therein can be expanded as $c_1|D_1\rangle + c_2|D_2\rangle$. This expansion hence defines a dark state representation. In this representation, the mechanical momentum operator becomes [17,18]

$$\mathbf{P}^D = -i\hbar\tilde{\nabla} + \hbar\kappa(\sigma_x\hat{e}_x + \sigma_z\hat{e}_z), \quad (8)$$

where $\sigma_{x,z}$ are the Pauli matrices, $\kappa = \cos(\xi)k_r$, and $\tilde{\nabla}$ represents the gradient in the dark-state representation. As in Ref. [17], we also introduce an additional constant shift to state $|3\rangle$, denoted by a matrix operator V_s , such that the total effective Hamiltonian becomes $H_D^{\text{eff}} = \frac{(\mathbf{P}^D)^2}{2m}$. The eigenstates of H_D^{eff} are

$$|\Psi^{D,\pm}\rangle = \frac{1}{2} \begin{pmatrix} 1 \mp i e^{i\varphi_k} \\ -i \pm e^{i\varphi_k} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{R}}, \quad (9)$$

where φ_k is the angle between the x axis and the atom wave vector \mathbf{k} in Eq. (9). According to Eq. (8), it is straightforward to show that the eigenstates $|\Psi^{D,\pm}\rangle$ have the momentum $(\mathbf{k} \pm \kappa\hat{\mathbf{k}})\hbar$.

Consider then two simple scenarios of laser field manipulation. In the first scenario, we adiabatically move all the laser beams along the negative z axis. The non-Abelian geometric phases thus induced are determined by

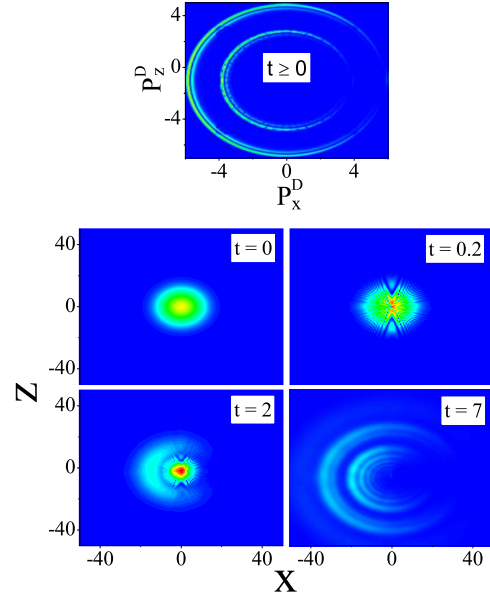


FIG. 2. (Color online) Simulation of the wave-packet dynamics after imprinting non-Abelian geometric phases via adiabatic rotation of the three laser beams in a tripod scheme. The wave packet plotted in momentum space (top panel, in units of $\hbar\kappa$) will remain the same for any $t \geq 0$. Length units (for the bottom four panels in x - z space) are the same as in Fig. 1.

$$\begin{aligned} i\frac{d}{dz} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= - \begin{pmatrix} i\langle D_1 | \frac{\partial}{\partial z} | D_1 \rangle & i\langle D_1 | \frac{\partial}{\partial z} | D_2 \rangle \\ i\langle D_2 | \frac{\partial}{\partial z} | D_1 \rangle & i\langle D_2 | \frac{\partial}{\partial z} | D_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned} \quad (10)$$

Because the matrix on the right side of Eq. (10) turns out to be independent of \mathbf{R} , one may naively conclude that the non-Abelian geometric phases induced here will not affect matter-wave propagation. But this is incorrect. To see this let us assume the initial state to be the $|\Psi^{D,-}\rangle$ state in Eq. (9) with $\varphi_k=0$. This initial state has a mechanical momentum $\hbar(k-\kappa)$ along the x axis, where $k \equiv |\mathbf{k}|$. Let the final state, a function of the displacement d_z along $-z$, be $|\Psi_f^D(d_z)\rangle$. It is enlightening to consider a specific case, e.g., $d_z = \frac{\pi}{4\kappa}$. Neglecting matter-wave propagation during the period of geometric phase imprinting, Eq. (10) then gives

$$|\Psi_f^D(\pi/4\kappa)\rangle = \left[\frac{1+i}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1-i}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] e^{ikx}. \quad (11)$$

The two components on the right side of Eq. (11) represent a superposition of the two eigenstates $|\Psi^{D,+}\rangle$ and $|\Psi^{D,-}\rangle$ in Eq. (9). Because these two eigenstates possess different mechanical momenta $(k+\kappa)\hbar$ and $(k-\kappa)\hbar$, we thus have the result that the non-Abelian geometric phases generated here can split matter waves.

This result is also verified by our numerical simulations based on the full Hamiltonian $-\frac{\hbar^2\nabla^2}{2m} + H_{\text{RWA},4} + V_s$. In particular, in our simulations the initial state is chosen as a Gaussian

wave packet instead of a plane wave considered in Eq. (9). As an example we choose $\Omega_0/\hbar = 1200\hbar\kappa^2/m$. The duration of the imprinting process is chosen to be $0.063m/(\hbar\kappa^2)$. (In real units, for $m = 10^{-25}$ Kg, $\kappa \sim 10^6$ m $^{-1}$, the duration is ~ 60 μ S and $\Omega_0/\hbar \sim 10^6$ Hz. The cold-atom version of the Zitterbewegung oscillation [19] is negligible for these parameters.) Figure 1 depicts the simulation results for $\varphi_k = 0$ and $k = \kappa$. At $t = 0$ we have initiated the first non-Abelian geometric phase imprinting for $d_z = \pi/(4\kappa)$. As seen in Figs. 1(b) and 1(c), an initial wave packet of zero group velocity indeed splits into two parts: one part holds a zero group velocity and the other part moves along the x axis at an average mechanical momentum of $2\hbar\kappa$. Then, at $t = 20$, we move all the lasers up by $d_z = \pi/(4\kappa)$ to induce a second imprinting process. The right bottom panel of Fig. 1 shows that this further splits the wave packet into four copies of equal weights, a result also derivable from Eq. (10). Choosing a different d_z at $t = 20$ for the second imprinting process, we may even exchange the momenta between the two separated sub-wave-packets. Extending the strategy here, in principle one can split a wave packet into 2^n copies with an arbitrary integer n . This should be useful for atom optics applications such as atom interferometry. Other simulation results (not shown) also indicate that so long as Ω_0 is sufficiently large, then on one hand the matter-wave propagation during the adiabatic processes can be made negligible and on the other hand the adiabaticity in the internal motion is well maintained.

In the second scenario, we slowly rotate all the three laser beams clockwise in the x - z plane. The non-Abelian geometric phase at the local point (R, θ) is determined by

$$i \frac{d}{d\gamma} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = - \begin{pmatrix} i \langle D_1 | \frac{\partial}{\partial \gamma} | D_1 \rangle & i \langle D_1 | \frac{\partial}{\partial \gamma} | D_2 \rangle \\ i \langle D_2 | \frac{\partial}{\partial \gamma} | D_1 \rangle & i \langle D_2 | \frac{\partial}{\partial \gamma} | D_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ = \kappa R \begin{pmatrix} \cos(\theta + \gamma) & -\sin(\theta + \gamma) \\ -\sin(\theta + \gamma) & -\cos(\theta + \gamma) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad (12)$$

where $0 \leq \gamma \leq \alpha$ is the angle of rotation that serves as the

adiabatic parameter. An analytical solution to Eq. (12) is not found and hence we solve it computationally. After obtaining the numerical non-Abelian geometric phases for each point (R, θ) from Eq. (12), we imprint the spatially dependent geometric phases onto the system at $t = 0$ and then let it evolve under $H_D^{\text{eff}} = \frac{P_D^2}{2m}$ (also with the rotated laser fields). For the example shown in Fig. 2, the initial state is the state $|\Psi^{D,-}\rangle$ in Eq. (9) multiplied by a Gaussian profile, with $k = \kappa$ and $\varphi_k = -\pi/2$. This initial state hence has a zero group velocity according to Eqs. (8) and (9). As shown in Fig. 2 for $\alpha = 3\pi/2$, the phase imprinting substantially impacts the ensuing evolution. The evolving wave packet first displays interesting interference patterns, followed by a certain degree of self-focusing, and later it starts to display some moonlike patterns. As expected, the patterns in real space at later times begin to resemble the $t = 0$ wave-packet pattern plotted as a function of P_x^D and P_z^D , namely, the mechanical momenta along the x and z axes. Remarkably, even if we let $\alpha = 2\pi$, i.e., with all laser fields returning to their original configuration, similar results can still be obtained. We have checked that essentially the same results can be obtained using the full Hamiltonian $-\frac{\hbar^2 \nabla^2}{2m} + H_{\text{RWA},4} + V_s$. However, we find that in order to fully neglect matter-wave propagation during the imprinting, the duration of the adiabatic process here should be about 2 orders of magnitude shorter than in the first non-Abelian example. Hence Ω_0 must be about 2 orders of larger.

In summary, we have demonstrated the concept of all-optical imprinting of both Abelian and non-Abelian geometric phases onto matter waves. The adiabatic paths considered here are among the simplest. Exploring more complicated adiabatic paths might offer extensive laser control of matter-wave propagation. Extending this work to the imprinting of nonadiabatic geometric phases is also of considerable interest.

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