Optimal Binary Periodic
Almost-Complementary Pairs

Avik Ranjan Adhikary, Zilong Liu, Yong Liang Guan, Sudhan Majhi, Srdjan Z. Budishin

Abstract

A pair of sequences is called a periodic complementary pair (PCP) if the periodic auto-correlations of the constituent sequences sum up to zero for all non-zero time-shifts. Owing to the scarcity of PCPs, we investigate optimal binary periodic almost-complementary pairs (BP-ACPs), each displaying correlation property closest to that of PCP. We show that an optimal BP-ACP of even-length $N$ has zero out-of-phase periodic auto-correlation sums (PACSs) except at the time-shift of $N/2$ where the corresponding PACS has minimum magnitude of 4. We also show that for any arbitrary odd $N$, all the out-of-phase PACSs of an optimal BP-ACP should have identical magnitude of 2. A number of optimal BP-ACPs from analytical constructions as well as computer search are presented. In addition, our proposed optimal BP-ACPs for the even-length case lead to two new families of base-two almost difference families.

Index Terms

Golay Complementary Pair (GCP), Periodic Complementary Pair (PCP), Z-Complementary Pair (ZCP), Almost Complementary Pair (ACP), Almost Difference Family (ADF).
I. INTRODUCTION

Periodic complementary pairs (PCPs) were introduced by Bömer and Antweiler [1] in 1990 as a periodic analog of aperiodic Golay complementary pairs (GCPs) [2], [3]. By definition, a PCP is a pair of sequences whose periodic auto-correlations sums (PACSs) are zero for all non-zero time-shifts [1]. It is noted that binary GCPs are also PCPs but the converse may not be true [4], [5]. Both GCPs and PCPs suffer from the scarcity problem, i.e., all the admissible lengths up to 100 for binary GCPs [6] are: 2, 4, 8, 10, 16, 20, 26, 32, 40, 52, 64, 80, whereas for binary PCPs, the admissible lengths [4], [7] are: 2, 4, 8, 10, 16, 20, 26, 32, 34, 40, 50, 52, 58, 64, 68, 72, 74, 80, 82. Up to now, the smallest length with unknown existence of binary PCP is 90 [4].

The concept of aperiodic Z-complementary pair (ZCP) was proposed by Fan et. al. [8] in 2007 to obtain pairs as an alternative to GCPs in scenarios where the required sequence lengths are not in the form of $2^{\alpha}10^\beta26^\gamma$. An aperiodic ZCP has zero aperiodic auto-correlation sums (AACSs) for certain out-of-phase time-shifts around the in-phase position. In the literature, such a region is widely called zero correlation zone (ZCZ) [8], [9]. Subsequently, binary periodic ZCP (BP-ZCP) was introduced by Yuan et. al. to design optimal training sequences for channel sounding in a single-carrier space-time block coded system with two transmit antennas [10]. Recently, Ke and Zhou designed periodic Z-periodic complementary sequence sets, each consisting of two or more constituent sequences, for potential applications in multi-carrier code-division multiple-access (MC-CDMA) communications and multiple-input and multiple-output (MIMO) channel estimation [11]. Codebooks from (almost) complementary pairs over QAM constellations have been proposed in [12], [13], [14] to support high-rate orthogonal frequency division multiplexing (OFDM) communications with low peak-to-average power ratios.

Although BP-ZCPs have been proposed, their optimal pairs remain unknown. In 2014, Liu et. al. studied the best possible aperiodic odd-length binary ZCP (OB-ZCP) in [7]. Inspired by that work, we study optimal binary periodic almost-complementary pairs (BP-ACPs) each displaying correlation property closest to that of binary PCPs. Optimal BP-ACPs may be regarded as the best possible periodic counterpart of OB-ZCPs. We adopt an optimality criteria similar to that of optimal (aperiodic) OB-ZCP each having maximum possible ZCZ width and minimum possible out-of-zone AACSs. We will show that an optimal BP-ACP of even length can only have PACS of 4 or $-4$ at the time-shift of $N/2$ and zero at all other non-zero time-shifts, where
$N$ denotes the sequence length. These optimal BP-ACPs lead to two new families of base-two almost difference families (ADF) which have wide applications in partially balanced incomplete block design (BIBD) [15]. We will also prove that all the optimal BP-ACPs of odd lengths have out-of-phase PACSs of 2 or $-2$.

II. NOTATIONS AND DEFINITIONS

Throughout this paper, a binary sequence is a vector over $\mathbb{Z}_2 = \{0, 1\}$. $\otimes$ denotes the Kronecker product. Let $c_L$ be a length-$L$ vector with identical entries of $c$. For example, $0_L$ denotes a vector of $L$ zeros. For a length-$N$ binary sequence $a = [a_0, a_1, \cdots, a_{N-1}]$, and $a$ denotes the reverse of $a$. The aperiodic auto-correlation function (AACF) of $a$ is defined as

$$\rho_a(\tau) := \begin{cases} \sum_{k=0}^{N-1-\tau} (-1)^{a_k+a_{k+\tau}}, & 0 \leq \tau \leq N-1; \\ 0, & \tau \geq N. \end{cases}$$

and the periodic auto-correlation function (PACF) is defined as

$$\theta_a(\tau) := \sum_{k=0}^{N-1} (-1)^{a_k+a_{(k+\tau) \mod N}}, \quad 0 \leq \tau \leq N-1.$$  \hspace{1cm} (2)

For ease of presentation, from now on, we shall drop the subscript “mod N” whenever a PACF is mentioned. For $a, b \in \mathbb{Z}_2$, $(-1)^{a+b} = 1 - 2(a \oplus b)$, where $\oplus$ denotes modulo 2 addition. Therefore, $\theta_a(\tau)$ can be rewritten as

$$\theta_a(\tau) = N - 2 \cdot \left[\sum_{i=0}^{N-1} a_i \oplus a_{i+\tau}\right].$$

**Definition 1 (BP-ZCP [10]):** A pair of binary sequences $a$ and $b$ of length $N$ is said to be a periodic ZCP of ZCZ width $Z$ if and only if

$$\theta_a(\tau) + \theta_b(\tau) = 0, \quad \text{for all } 1 \leq \tau \leq Z - 1.$$  \hspace{1cm} (4)

Clearly, we require $Z \geq 2$ for non-trivial ZCZ. If $Z = N$, $(a, b)$ reduces to a PCP [1]. Throughout this paper, $\theta_a(\tau) + \theta_b(\tau)$ is called the PACS at time-shift $\tau$.

**Definition 2 (Support and Characteristic sequence [7]):** Given a binary sequence $a$ of length $N$, we define the support of $a$ as the set $D = \{0 \leq i \leq N-1 : a_i = 1\}$. Conversely, given a support, a binary sequence can be obtained. Such a sequence is also called the characteristic sequence of the support.
Lemma 1 ([7]): The difference function of $D \subseteq \mathbb{Z}_N := \{0, 1, \ldots, N - 1\}$ is defined as $d_D(\tau) := |(\tau + D) \cap D|$, where $\tau \in \{0, 1, \ldots, N - 1\}$. Let $D$ be the support of a binary sequence $a$. Then, the PACF of $a$ can be written as

$$\theta_a(\tau) = N - 4(k - d_D(\tau)), \text{ where } k = |D|. \quad (5)$$

Definition 3 (Almost Difference Family (ADF) [7]): Let $D = \{D^0, D^1\}$ be a family of two subsets of $\mathbb{Z}_N$, where $D^0 \subseteq \mathbb{Z}_N$ and $D^1 \subseteq \mathbb{Z}_N$. Let $K = \{k_0, k_1\}$, where $k_0 = |D^0|$ and $k_1 = |D^1|$. $D$ is called an $(N, K, \lambda, t)$ ADF if and only if $d_D(\tau) = d_{D^0}(\tau) + d_{D^1}(\tau)$ ($\tau \in \mathbb{Z}_N \setminus \{0\}$) takes the value of $\lambda$ for $t$ times, and $\lambda + 1$ for $N - 1 - t$ times.

Lemma 2: Let $D = \{D^0, D^1\}$ be an ADF with parameters $(N, \{k_0, k_1\}, \lambda, t)$. Also, let $a$ and $b$ be the characteristic sequences of $D^0$ and $D^1$, respectively. Then,

$$\sum_{i=0}^{1} [k_i(k_i - 1)] = \lambda t + (\lambda + 1)(N - 1 - t), \quad (6)$$

and the PACS of sequences $a$ and $b$ is given as follows.

$$\theta_a(\tau) + \theta_b(\tau) = \begin{cases} 
2N, & \text{for } \tau = 0; \\
2N - 4 \left( \sum_{i=0}^{1} k_i - \lambda \right), & \text{for } t (\neq 0) \text{ residues mod } N; \\
2N - 4 \left( \sum_{i=0}^{1} k_i - \lambda - 1 \right), & \text{for remaining } N - 1 - t \nonumber \\
\quad \text{non-zero residues mod } N.
\end{cases}$$

III. OPTIMAL BINARY PERIODIC ALMOST-COMPLEMENTARY PAIRS

In this section, we characterize and construct optimal and near-optimal BP-ACPs.

Theorem 1: Any non-trivial BP-ZCP $(a, b)$ with $Z \geq 2$ should have an even length.

Proof: Consider $(a, b)$ with length $N$. Since $Z \geq 2$, we have

$$\theta_a(\tau) + \theta_b(\tau) = 0, \quad 1 \leq \tau \leq Z - 1. \quad (7)$$

Based on (3), we obtain

$$2N - 2 \sum_{i=0}^{N-1} \left[ (a_i \oplus a_{i+r}) + (b_i \oplus b_{i+r}) \right] = 0. \quad (8)$$
Then,

\[
N = \sum_{i=0}^{N-\tau-1} \left[ (a_i + a_{i+\tau}) + (b_i + b_{i+\tau}) \right] + \\
\sum_{i=N-\tau}^{N-1} \left[ (a_i + a_{i+\tau}) + (b_i + b_{i+\tau}) \right] \pmod{2}
\]

\[
\equiv \sum_{i=0}^{N-\tau-1} (a_i + b_i) + \sum_{i=N-\tau}^{N-1} (a_i + b_i) + \\
\sum_{i=N-\tau}^{N-1} (a_i + b_i) + \sum_{i=0}^{\tau-1} (a_i + b_i) \pmod{2}
\]

\[
\equiv 2 \cdot \sum_{i=0}^{N-1} (a_i + b_i) \pmod{2} \equiv 0 \pmod{2}.
\]

Thus, \( N \) should be even. \( \blacksquare \)

**Lemma 3:** For a BP-ZCP \((a, b)\) of even length \( N \), if it is not a PCP, then such a sequence pair has maximum ZCZ width of \( N/2 \).

**Proof:** The proof of this lemma can be obtained easily by noting that the value of the PACS of the sequence pair is symmetric about the in-phase position, i.e.,

\[
\theta_a(\tau) + \theta_b(\tau) = \theta_a(N - \tau) + \theta_b(N - \tau). \tag{10}
\]

**Theorem 2:** For a BP-ZCP of even length \( N \), when the maximum ZCZ is achieved, the minimum magnitude of PACS at \( \tau = N/2 \) should be 4.

**Proof:** Based on (3), for \( \tau = N/2 \), we can write

\[
\theta_a(\tau) = N - 2 \cdot \sum_{i=0}^{N-1} a_i \oplus a_{i+N/2}. \tag{11}
\]

Thus,

\[
\theta_a(N/2) + \theta_b(N/2)
\]

\[
= 2N - 2 \cdot \sum_{i=0}^{N-1} \left[ a_i \oplus a_{i+N/2} + b_i \oplus b_{i+N/2} \right]. \tag{12}
\]

Conducting a derivation similar to that of Theorem 1, we have

\[
\frac{\theta_a(N/2) + \theta_b(N/2)}{2} = N - 2 \cdot \sum_{i=0}^{N-1} (a_i + b_i) \pmod{2}. \tag{13}
\]
Note that \( N \) is even. It then follows that

\[
\frac{\theta_a(N/2) + \theta_b(N/2)}{2} \equiv 0 \pmod{2}.
\]  

(14)

Since \( \theta_a(N/2) + \theta_b(N/2) \neq 0 \), the PACS should take values of the form \( \pm 4, \pm 8, \pm 12, \ldots \).

Thus, \( \theta_a(N/2) + \theta_b(N/2) \geq 4 \).

Corollary 1: For a binary sequence pair of odd length \( N \), the minimum magnitude of PACS at each non-zero time-shift should be 2.

Proof: The proof can be easily obtained using Theorem 1 of [7], since

\[
\theta_a(\tau) = \rho_a(\tau) + \rho_a(N - \tau), \quad 0 < \tau < N
\]  

(15)

where \( a \) is a sequence of length \( N \).

Now, we are ready to give the definition of optimal BP-ACPs for even- and odd- lengths.

Definition 4 (Optimal BP-ACP): A BP-ACP \((a, b)\) of length \( N \) is said to be optimal if

1) For even length: it has a ZCZ of width \( N/2 \) and the PACS at time-shift \( \tau = N/2 \) takes on the magnitude value of 4.

2) For odd length: all of its out-of-phase PACSs have identical magnitude of 2.

We also give the definition of near-optimal BP-ACPs of even-lengths below because they are relatively easier to be constructed.

Definition 5 (Near-optimal BP-ACPs): A BP-ZCP of even length \( N \) is said to be a near-optimal BP-ACP if it achieves the maximum ZCZ length of \( N/2 \) and the PACS at time-shift \( \tau = N/2 \) is larger than 4.

Next, we shall present two examples of optimal BP-ACPs.

Example 1: Let

\[
a = (+ + - - - - + - + - - - - -),
\]

\[
b = (- + - - - + + + + + - - - - - -),
\]

(16)

where + and − denote for 1 and −1, respectively. \((a, b)\) is a length-18 optimal BP-ACP with a ZCZ length of 9 because

\[
\left( \theta_a(\tau) + \theta_b(\tau) \right)_{\tau=0}^{17} = (36, 0_8, 4, 0_8).
\]  

(17)
Example 2: Let
\[ a = (+ + - + + - - - + + - + + + + + +), \]
\[ b = (+ - + - - - + + - + + - + + + + + +). \]
\((a, b)\) is a length-17 optimal BP-ACP because
\[ \left( \theta_a(\tau) + \theta_b(\tau) \right)_{\tau=0}^{16} = (34, 2, 2, 2, -2, -2, -2, -2, -2, 2, 2, 2, 2). \]

A. Necessary condition of optimal BP-ACPs

In this subsection, we give a necessary condition of optimal BP-ACPs which is useful for the computer search of such sequence pairs. For this, we associate the Z-transform of the sequence \( a \) of length \( N \) as
\[ a(z) = \sum_{i=0}^{N-1} (-1)^{a_i} z^i. \]

**Necessary condition:** For an optimal BP-ACP \((a, b)\) of even-length, denote the number of ones in \( a \) and \( b \) by \( g_0 \) and \( g_1 \), respectively, then
\[ N + \theta_a(N/2) + \theta_b(N/2) = 2 \left( (g_0 - g_1)^2 + (N - g_0 - g_1)^2 \right). \]

**Proof:** For \( z \neq 0 \), we have
\[ a(z)a(z^{-1}) + b(z)b(z^{-1}) = \sum_{\tau=0}^{N-1} (\theta_a(\tau) + \theta_b(\tau)) \cdot z^\tau \]
\[ = 2N + [\theta_a(N/2) + \theta_b(N/2)] z^{N/2}. \]

Now, setting \( z = 1 \) and recalling the definition of \( a(z) \), we get
\[ |a(1)|^2 + |b(1)|^2 = (N - 2g_0)^2 + (N - 2g_1)^2 \]
\[ = 2 \left[ (g_0 - g_1)^2 + (N - g_0 - g_1)^2 \right]. \]

Similarly, setting \( z = 1 \) into (21) and relating the result with that in (22), we have
\[ N + \frac{\theta_a(N/2) + \theta_b(N/2)}{2} = \left[ (g_0 - g_1)^2 + (N - g_0 - g_1)^2 \right]. \]

Recalling the definition of optimal BP-ACP of even length completes the proof. \( \blacksquare \)

**Remark 1:** We have done an exhaustive computer search for lengths up to 26. Although number 14 and 22 can be written in the form stated in the aforementioned necessary condition,
our exhaustive computer search shows that no optimal BP-ACPs of lengths 14 and 22 exist. A best possible near-optimal sequence pair of length-14 is given below.

\[
\begin{align*}
a &= (+ + + + + - + + + - + + -), \\
b &= (- - - - - + + + + + + - - -).
\end{align*}
\]

(a, b) has the maximum ZCZ length of 7 but the minimum non-zero PACS of 8 at time-shift of \( \tau = 7 \).

\[
(\theta_a(\tau) + \theta_b(\tau))_{\tau=0}^{13} = (28, 0, 8, 0_6).
\]

Remark 2: Based on Theorem 1 and the aforementioned necessary condition, we assert that no optimal BP-ACP exists for the following lengths (up to 100): 44, 46, 58, 68, 86, 90, 94.

B. Relationship with ADFs

Definition 6 (Type-I and Type-II optimal BP-ACPs): An optimal BP-ACP of even length \( N \) whose PACS is \( +4 \) (or \( -4 \)) at \( \tau = N/2 \) is said to be Type-I (or Type-II) optimal BP-ACP.

Theorem 3: Suppose that \( D = \{ D^0, D^1 \} \) consists of the supports of the Type-I optimal BP-ACP \((a, b)\) of even length \( N \), where \( g_0 = |D^0| \) and \( g_1 = |D^1| \). Then, \( D \) should be a \( \{ N, (D^0, D^1), \lambda = g_0 + g_1 - N/2, t = N - 2 \} \) ADF.

Proof: Applying the PACS formula in Lemma 2, we have \( 2N - 4(g_0 + g_1 - \lambda) = 0 \), meaning that \( \lambda = g_0 + g_1 - N/2 \). Substituting \( \lambda \) into (6), we obtain

\[
t = -(g_0^2 + g_1^2) - N^2/2 + N(g_0 + g_1) + N + N/2 - 1.
\]

Recalling (20) and relating it to (26), we arrive at \( t = N - 2 \).

Similarly, we have the following theorem.

Theorem 4: Suppose that \( D = \{ D^0, D^1 \} \) consists of the supports of the Type-II optimal BP-ACP \((a, b)\) of even length \( N \), where \( g_0 = |D^0| \) and \( g_1 = |D^1| \). Then, \( D \) should be a \( \{ N, (D^0, D^1), \lambda = g_0 + g_1 - N/2 - 2, t = 1 \} \) ADF.

IV. CONSTRUCTIONS OF OPTIMAL BP-ACPs

In this section, we will present some construction methods of optimal and near-optimal BP-ACPs.

Construction 1: Consider a pair of binary Barker sequences [5], [16] \((a, b)\) of lengths: a) 3 and 5; b) 5 and 7; c) 11 and 13. Then, the sequence pair \((c, d) = ([a, b], [a, -b])\) (where [a,
b] denotes the concatenation of a and b) are optimal BP-ACP of lengths a) $N=8$, b) $N=12$ and c) $N=24$, respectively. Each Barker sequence can be reverted or the two sequences can be exchanged to obtain more BP-ACPs.

**Construction 2:** Near-optimal BP-ACP $(c^{k+1}, d^{k+1})$ of length $N2^{k+1}$ can be generated from $(c^k, d^k)$ of length $N2^k$ by

$$(c^{k+1}, d^{k+1}) = \left([c^k, d^k], [c^k, -d^k]\right),$$

where $(c^0, d^0)$ is set to be a length-$N$ optimal BP-ACP obtained in **Construction 1**.

**Construction 3:** A near-optimal BP-ACP can be generated from Turyn’s construction [17] using an optimal (or near-optimal) BP-ACP and a binary GCP, both of which are over $\{1, -1\}$. Specifically, let $(a, b)$ be an optimal (or near-optimal) BP-ACP of length $N$, and $(c, d)$ be a length-$M$ binary GCP. Then $(e, f)$

$$e = a \otimes (c + d) / 2 + b \otimes (c - d) / 2,$$

$$f = b \otimes (c + d) / 2 - a \otimes (c - d) / 2,$$

will be a near-optimal BP-ACP of length-$MN$. In particular, $|\theta_e(N/2) + \theta_f(N/2)| = M |\theta_a(N/2) + \theta_b(N/2)| \geq 4M$.

**Construction 4:** Lüke reported a class of almost-perfect quadriphase sequences of length $N = p^J + 1 \equiv 2 \pmod{4}$ ($p$ odd prime, $J = 1, 2, \cdots$) which have zero out-of-phase periodic auto-correlations except at time-shift $\tau = N/2$ [18]. A near-optimal BP-ACP is obtained by applying the Gray mapping $\phi : Z_4 \rightarrow Z_2 \times Z_2$ to a Lüke’s almost-perfect quadriphase sequence, where $\phi(0) = (0, 0), \phi(1) = (0, 1), \phi(2) = (1, 1), \phi(3) = (1, 0)$.

**Example 3:** Consider Lüke’s almost-perfect quadriphase sequence of length-18 below.

$$s = (2, 0, 1, 0, 0, 0, 1, 0, 2, 2, 0, 1, 0, 3, 0, 1, 0, 2).$$

By applying the Gray mapping to $s$, we obtain

$$a = (-+++---+++---+++---),$$

$$b = (-+++---+++---+++---),$$

which have $(\theta_a(\tau) + \theta_b(\tau))_{\tau=0}^{17} = (36, 0_8, 32, 0_8)$.

**Remark 3:** Optimal BP-ACPs of odd-lengths widely exist: (1) A simple way to generate optimal BP-ACPs of odd-lengths is to pair two equi-length binary sequences (such as m-sequences, Hall sequences, generalized GMW sequences, etc) both having out-of-phase periodic
auto-correlations of $-1$; (2) All optimal OB-ZCPs reported in [7] are also optimal BP-ACPs but the converse may not be true; (3) All the base-two difference families $\{D^0, D^1\}$ (derived from the tool of cyclotomy) in [19] lead to optimal BP-ACPs by associating $D^0$ and $D^1$ with their characteristic sequences; (4) The generalized GMW sequence pair, the twin-prime sequence pair and the Legendre sequence pair reported in [20] are optimal BP-ACPs.

A list of best possible binary pairs of lengths up to 26 is given in Table I.

V. CONCLUSION

In this paper, we have studied and constructed optimal and near-optimal BP-ACPs displaying PACS property closest to PCPs. Our main contributions are: (1) We have shown that a binary sequence pair with non-zero ZCZ width must have even length; (2) We have proved that a length-$N$ optimal BP-ACP ($N$ even) has zero out-of-phase PACSs except at the time-shift of $\tau = N/2$, at which the PACS takes value of 4 or $-4$. We have shown that these new optimal BP-ACPs of even lengths lead to new base-two ADFs with parameters given in Theorems 3 and 4; (3) We have proved that any optimal BP-ACP of odd-length has out-of-phase PACSs of 2 or $-2$.

It is noted that the proposed optimal and near-optimal BP-ACPs of even lengths can be applied for optimal channel estimations and interference-free asynchronous CDMA communications. However, we have only found systematic constructions for near-optimal BP-ACPs of even lengths, as shown in Constructions 2-4. An interesting future work is to search systematic constructions of optimal BP-ACPs of even lengths.

REFERENCES

### TABLE I: Best possible binary pairs of lengths up to 26

| N  | $Z_{\text{max}}$ | Type    | $(\begin{smallmatrix}(-1)^a \\ (-1)^b\end{smallmatrix})$ | $|\theta_a(\tau) + \theta_b(\tau)|_{\tau=0}^{N-1}$ |
|----|------------------|---------|----------------------------------------------------------|-----------------------------------------------------|
| 2  | 2                | PCP     | $(++)$                                                   | $(4, 0_1)$                                           |
| 3  | 0                | OB-ZCP  | $(++)$                                                   | $(6, 2_2)$                                           |
| 4  | 4                | PCP     | $(++++)$                                                | $(8, 0_4)$                                           |
| 5  | 0                | OB-ZCP  | $(++++)$                                                | $(10, 2_4)$                                          |
| 6  | 3                | BP-ACP  | $(---++)$                                               | $(12, 0_2, 4, 0_2)$                                   |
| 7  | 0                | OB-ZCP  | $(++++)$                                                | $(14, 2_b)$                                          |
| 8  | 8                | PCP     | $(++++)$                                                | $(16, 0_7)$                                          |
| 9  | 0                | OB-ZCP  | $(---+++)$                                              | $(18, 2_8)$                                          |
| 10 | 10               | PCP     | $(-----+++)$                                            | $(20, 0_9)$                                          |
| 11 | 0                | OB-ZCP  | $(-----+++)$                                            | $(22, 2_10)$                                         |
| 12 | 6                | BP-ACP  | $(-----+++)$                                            | $(24, 0_5, 4, 0_5)$                                   |
| 13 | 0                | OB-ZCP  | $(-----+++)$                                            | $(26, 2_12)$                                         |
| 14 | 7                | BP-ACP  | $(-----+++)$                                            | $(28, 0_6, 8, 0_6)$                                   |
| 15 | 0                | OB-ZCP  | $(-----+++)$                                            | $(30, 2_{14})$                                       |
| 16 | 16               | PCP     | $(-----+++)$                                            | $(32, 0_{15})$                                       |
| 17 | 0                | OB-ZCP  | $(-----+++)$                                            | $(34, 0_{16})$                                       |
| 18 | 9                | BP-ACP  | $(-----+++)$                                            | $(36, 0_8, 4, 0_8)$                                   |
| 19 | 0                | OB-ZCP  | $(-----+++)$                                            | $(38, 2_{18})$                                       |
| 20 | 20               | PCP     | $(-----+++)$                                            | $(40, 0_{19})$                                       |
| 21 | 0                | OB-ZCP  | $(-----+++)$                                            | $(42, 2_{20})$                                       |
| 22 | 11               | BP-ACP  | $(-----+++)$                                            | $(44, 0_{10}, 8, 0_{10})$                             |
| 23 | 0                | OB-ZCP  | $(-----+++)$                                            | $(46, 2_{22})$                                       |
| 24 | 12               | BP-ACP  | $(-----+++)$                                            | $(48, 0_{11}, 4, 0_{11})$                             |
| 25 | 0                | OB-ZCP  | $(-----+++)$                                            | $(50, 2_{24})$                                       |
| 26 | 26               | PCP     | $(-----+++)$                                            | $(52, 0_{25})$                                       |


