Do individuals optimize in intertemporal consumption/savings decisions? A liberal method to encourage savings*

Yew-Kwang Ng

Monash University, Clayton 3168, Australia

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Rational consumption/saving choice requires rough knowledge regarding how much a dollar saved now could be compounded to become after various numbers of years. An indicative questionnaire reveals that most people grossly underestimate the significance of compound interest. When told of the right figures, they indicate their willingness to save much more. A simple model of intertemporal optimization under plausible parameters (4-5% real rate of return) prescribe many times higher consumption at older ages than at younger ages, implying very high rates of savings when young.

1. Introduction: A problem of current account deficits or over-consumption?

The choice between current consumption and future consumption (or current savings) is an important economic decision affecting a host of other things. For countries like the U.S. and Australia facing substantial current account deficits, an increase in savings is regarded as desirable. However, if foreigners are willing to invest here or lend money to us and local people are willing to borrow, we have capital inflows which imply current account deficits in a system of flexible exchange rates. It is then often said that, it is O.K. if the capital inflows allow increases in real domestic investment which will increase our productive capacities in the future. Nevertheless, why should we care whether the capital inflows increase investment or consumption if it is the result of the optimizing choice of rational economic actors unless there are some significant distortions that bias the result persistently. Then, ideally, the problem of trade deficits should be solved by eliminating such distortions. However, are individuals rational optimizers in intertemporal choice of consumption versus savings. If they are not, foreign capital inflows may finance current consumption to the detriment of these non-optimizing individuals. Problems of current account deficits are then seen as over-

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consumption. The encouragement of savings may then correctly be regarded as desirable.¹

Economic analysis (see Appendix A) indicates a likely positive relationship between the real rate of interest and savings. However, such a relationship does not seem to apply in the real world. As an explanation, it has been suggested that, if an individual has a targeted amount for consumption at old age, an increase in interest rate may actually lead to a decrease in savings. However, a fixed targeted amount irrespective of opportunity costs is clearly not consistent with rational maximization behaviour. Rather, I suspect that, due to ignorance and/or irrationality, many individuals do not maximize with respect to intertemporal decisions regarding consumption/savings. The next section outlines the basic requirements for rational decisions regarding intertemporal consumption/savings. Appendix A provides a more rigorous solution for a simplified model. Section 3 reports on the results of a small exploratory survey which indicates that individuals are not rational in the sense of being quite ignorant of information on the relevant opportunity costs (amount of future consumption forgone) of current consumption that could be easily obtained. Moreover, such information may significantly affect individual decisions, making people willing to save more. The concluding section suggests the widespread use of such a survey as an educational exercise with beneficial effects.

2. The rational approach to optimal saving

To determine the optimal amount of saving, an economist would naturally use an intertemporal maximization approach. The objective function is the utility (as a function of consumption, perhaps assumed time independent for simplicity) of the decision maker through time, possibly discounted. The constraint is that the consumption trajectory be feasible. Factors affecting this include the income profile and the lending and borrowing rates of interest (perhaps assumed identical for simplicity). However, it is unrealistic to assume that an average consumer knows how to apply the calculus of variation or the maximum principle. In fact, few economists actually use the intertemporal maximization approach described in their own saving decisions. Nevertheless, to determine the optimal amount of saving rationally, something like the following calculation is necessary.

First, one should have a rough idea of the marginal utility of current consumption which depends on (usually diminishes with) the level of current consumption. Next, one should have a rough idea of the probabilities that

¹Encouragement of savings may also be regarded as socially desirable on the ground that the society has a longer survival expectancy than individuals, and hence a lower rate of time preference. This is a separate justification which increases the importance of the issue discussed in this paper.
one will survive until the various years into the future. A typical curve depicting these probabilities for a person aged zero is roughly depicted in fig. 1. The relevance of survival probabilities is obvious. If one expects to die shortly, there is not much point to save unless one places relatively high values on leaving bequests.

Thirdly, one should have a rough idea as to how much a dollar saved now can be compounded to become in real terms after various numbers of years. For this, one should estimate the real rates of return over the relevant periods as well as being aware of the significance of compound interest. If negative savings are involved, one should be aware of the costs of borrowing instead.

Fourthly, one should estimate the marginal utilities of consumption (including bequests) in the future. These depend partly on the degree of diminishing marginal utility and the levels of consumption in the future (and possibly also on current consumption if there is a significant intertemporal independency) and partly on the capacity to enjoy consumption which may increase with learning, new consumption possibilities, etc. and decrease with failing health.

The above four factors considered together then determine whether one has saved too little or too much. For example, suppose one can earn a real rate of return (after tax where relevant) of 5% per annum every year. Then, an additional dollar saved this year becomes more than seven dollars in real terms after 40 years ($11.5 and $18.7 respectively after 50 and 60 years).
Suppose one expects to survive at least another 40 years with probability \( \alpha \) and expects to have marginal utility of consumption then equalling \( \beta \) times the marginal utility of current consumption. Then one can increase one's expected utility by saving a little more now and consuming more 40 years later if \( \alpha \beta > 1/7 \). Of course, comparisons with other years in the future also have to be made for overall optimality.

An issue arises as to whether future utilities should be discounted. If the uncertainty of one's continued survival has not been taken into account, then future utility should obviously be discounted to account for this uncertainty. However, since we have already taken into account the probability of survival \( \alpha \), it is not necessary to discount further. It is true that some economists [e.g. Shepard and Zeckhauser (1982)] use the market rate of interest to discount future utility even after discounting it with the probability of survival. However, while discounting future consumption at the rate of interest may be reasonable, discounting future utility at this rate is questionable. A rational individual maximizing expected utility would, apart from accounting for survival probabilities, adopt a trajectory of consumption with lower marginal utilities of consumption in future years to account for the non-unity and different marginal rates of transformation between consumption of different years due to the rate of interest. Then, consumption in the future would automatically be discounted when reckoning in terms of utility. To discount future utilities again at the market rate of interest involves double discounting.
Table 1

<table>
<thead>
<tr>
<th>Age t</th>
<th>c(t)</th>
<th>i=4%</th>
<th>i=5%</th>
<th>i=2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>647</td>
<td>391</td>
<td>2,146</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1,121</td>
<td>828</td>
<td>2,494</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1,942</td>
<td>1,754</td>
<td>2,898</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>3,366</td>
<td>3,712</td>
<td>3,366</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>5,835</td>
<td>7,859</td>
<td>3,911</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>10,113</td>
<td>16,638</td>
<td>4,544</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>17,529</td>
<td>35,223</td>
<td>5,279</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>30,382</td>
<td>74,567</td>
<td>6,134</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>52,660</td>
<td>157,859</td>
<td>7,126</td>
<td></td>
</tr>
</tbody>
</table>

*i = interest rate, r = rate of depreciation in the capacity to enjoy consumption, T = life expectancy, ε = elasticity of utility with respect to consumption, Y = lifetime income valued at birth.

If an individual is impatient, he may discount future utilities on top of the uncertainty discount. However, this impatience discount rate need not equal the market rate of interest. Conceivably, people who are over-patient may have negative impatience discount rate. In my view, both impatience and over-patience are forms of irrational behaviour. It is possible that, in the real world, there are more impatient people than over-patient people or even inconsistent people [see, e.g. Strotz (1955-1956), Thaler (1981) and Kahneman, Slovic and Tversky (1982)]. Then we may wish to use a positive rate of impatient discount for our descriptive and predictive theories. However, we may retain a zero rate of impatient discount for our normative theory.

A more elegant solution of optimal savings/consumption is the inter-temporal optimization approach [Yaari (1964, 1965)] mentioned at the beginning of this section. A simplified model of this is presented in Appendix A. Under some plausible set of parameters (4% real interest rate, moderately high degree of diminishing marginal utility of consumption, life-expectancy of 80 years), it is shown that the optimal trajectory of life-cycle consumption involves very low consumption when young and very high consumption when old, as shown in table 1. Figures are for dollars valued at prices at birth. Thus, the higher values for later years reflect higher real values, not

2These results as well as some others (e.g. Camerer 1987 on biases in probability judgments) lend support to our general point that many individuals may be far from rational.
higher prices. A change in $Y$ (life-time income valued at birth) only changes consumption levels at all ages proportionately. The intuitive explanation of the low-high consumption trajectory is that, with a real interest-rate ($i$) in excess of the rate ($r$) of depreciation of the capacity to enjoy income (plus time discount where relevant, but see discussion above), one can increase total utility by consuming less when young and more (with interests) when old until just offset by the diminishing marginal utility of consumption. A dollar compounded at 4% per annum becomes $15.6 after 70 years. This may be worthwhile despite the uncertainty of survival, and the lower marginal utility of consumption in the future (due to higher consumption and lower capacity). An increase (decrease) in the interest rate from 4% to 5% (2%), ceteris paribus, dramatically shift consumption in favour of older (younger) ages, as shown in the last two columns of table 1. This is not surprising when one realizes that a dollar becomes $23 after 80 years at 4% but becomes $49.6 at 5% and only $4.87 at 2%. It might be queried: How could consumption in a single year exceed the life-time income of $100,000. The answer is that this latter figure is valued at birth ($t=0$), which, if uncon-sumed, becomes $4,956,144 after 80 years at 5%.

Our numerical example assumes a positive rate of depreciation in the capacity to enjoy consumption. In fact, this capacity is likely to increase with learning until middle age before declining with failing health. Changes to account for this and other complications (e.g. utility of wealth ownership) often strengthen rather than weaken our case. Our assumption of the possibility to borrow and lend at the same market rate of interest makes the profile (besides the absolute amount) of life-cycle earnings irrelevant in affecting optimal life-cycle consumption. However, given the latter, the former determines the optimal profile of savings/dissavings.

So much for the theory; do individuals in real economies behave as described by our theory?

3. Are individual consumers/savers rational? An indicative survey

Partly to discover whether individuals are rational in their consumption/saving decisions, and partly as an educational exercise, I undertook a small survey with students in my first-year economics class. The sample is thus not representative of the population. However, one should expect that university students, especially those doing economics, should be able to estimate the correct answers to Question 2 in Questionaire A better than a random sample from the population (at least that from the same age group). About 2/5 of the 82 respondents are female and about 3/4 are aged 21 or under. There are no significant differences in the answers by gender. All respondents are handed Questionaire B after they have completed Questionaire A. I consciously refrained from exerting influences on the answers by the respon-
dents. However, I did have to explain some of the questions. In particular, Question 1 of Questionaire A had to be explained twice.

Reporting the average responses to Questions 1 and 2 of Questionaire A would be misleading as the averages are too much influenced by extreme answers. For example, some gave infinity as answers to some questions. Excluding answers of infinity would not quite solve the problem. Some gave $10^{12}$ for a question (last square of table 2 in Question 1) where most answered around 8–200. One answerd 10,000 for one (first square of Question 1) that most gave answers around 1.15 to 5. One gave $20^{10}$ for a question (last square of Question 2) that most answered around 6–100. Instead, table 2 reports the answers to Questions 1 and 2 of Questionaire A by one whose answers are fairly representative.

As can be seen from this rather representative answer, the numbers of dollars after 40–60 years that are required to induce the respondents to save substantially more than what they now saving are considerably lower than what could actually be reached at a rate of return of 5–10%. But they do not know that this is the case. Thus, this representative respondent estimated that a dollar compounded at 10% would grow to $4.2 after 60 years, but it actually would grow to $304.5. One cannot help wondering that a reason why people have not saved more when the real rate of interest is fairly high (5–10%) is that they really do not comprehend the significance of compound interest at moderately high rates after many years.

In Question 1 of Questionaire B, the correct answers to Question 2 of Questionaire A are given. The respondents are then asked whether they have under or over-estimated the significance of compound interests after many years. As summarized in table 3, 55 out of a total of 76 answers to this question (some apparently refused to admit gross underestimation explicitly) admitted gross underestimation of the significance of compound interest. Some answer 'approximately right' but in fact gave answers like 4.1 (instead of the correct answer 1,469.8), 4.7 (instead of 304.5), 4.9 (instead of 56,347.5). Except a very few who really grossly overestimated, virtually all grossly underestimated the significance of compound interest at moderate rates (5% and above) after many years (20–60 years). Many ticked 'somewhat underestimated' because they grossly underestimated the significance of compound interests after many years at 10–20% but overestimated at 2%.

After providing some relevant information, Question 2 asks the respondents whether they are prepared to save more or less than their present

3For your information, the current rates of interests are: ten-year government bonds, 13–14%; fixed deposits of one to five years with banks, 15–17%. The current rate of inflation: 7.7%. Thus, if you don't have to pay taxes, you can get a virtually risk-free real return of about 8–9% (per annum; compoundable) for 1–5 years and about 6%, for 10 years. If you have to pay taxes, you may consider investing in low-tax superannuation, insurance bonds, etc. Some of these are paying fixed after-tax returns of about 14–15% per annum, fixed for four years (giving a real, after-tax return of about 7%). Those of shorter terms and variable rates pay even more.
As can be seen in the second half of table 3, most agreed to save substantially more, and 38.75% agreed to save twice as much or higher. Only one out of 80 ticked ‘saving less’. A low rate of saving could partly be due to individual ignorance (about the significance of compound interests) which could be relatively easily removed by exercises similar to the present survey, perhaps in high schools.

In the questionnaire, it is implicitly assumed that everyone undertakes some savings. However, one student declared that he did not save. I told him to answer the question by changing ‘saving 20% more’ into saving ‘$20 per month’ and so on. He still said that he could not be induced to save. I then said, ‘What if a dollar saved now could become a million dollars after five years?’ He then admitted that he would then be tempted to save. However, I emphasize that the figures asked for are not those that will be more than sufficient to induce the nominated increase in saving but the lowest figures that are just sufficient. This student handed in figures requiring 1,000/10,000 fold increase after five years to induce him to save $20/$100 per month respectively. I do not think that the willingness to forgo $900,000 after five years in order to avoid having to save $100 now as rational (in the sense of expected utility maximization) unless the individual concerned faces well over 90% probability of death within a few years or is already consuming at a starving level now. While the whole survey is anonymous, I can identify this student only because of his unusual response and his indication on his questionnaire that he did not save. Subsequently, I asked him whether he expected to die within a few years. He said that he expected to live until about seventy years old. Neither did he expect a dramatic collapse in his capacity to enjoy consumption. He just wanted to leave his future to take care of itself.

A pure theorist may say that the purported behaviour may be consistent with expected utility maximization if the (time-independent) utility function is such that marginal utility decreases sharply around current income. However, this is inconsistent with changes in current income to different levels and the observation that virtually all individuals have fairly positive marginal utilities at wide ranges of income levels. Alternatively, a pure theorist may hypothesize that non-savers may expect fastly depreciating capacity to enjoy consumption. This hypothesis can be easily rejected by talking to the non-savers concerned.

It may also be hypothesized that some individuals find it very utility-

Alternatively, you could invest your savings in real estates, including your own house. I estimated that the money I used to buy my house nearly 15 years ago (after house prices had peaked) has yielded a tax-free return of 18% per annum. In other words, every single dollar has now become 12 dollars (or 3.2 dollars in prices 15 years ago). Even investments in rental properties, after the capital gain tax and the current high interest rates, may be expected to yield an after-tax, long-run return of about 10% or more, thanks to negative-gearing and tax-deductability.
Table 2
A representative answer

Questionnaire A

1. To induce you to save 20% (also 50% and 100% respectively) more than what you are now saving, the real rate of return on your savings has to increase to a level such that a dollar saved now can be compounded to become at least

| No of dollars (at this year prices) | 1.4 | 2.0 | 3.2 | 4.2 | 6.0 |
| save 20% more After | 5 yrs | 10 yrs | 20 yrs | 40 yrs | 60 yrs |

| No. of dollars (at this year prices) | 3.0 | 5.0 | 7.5 | 9.0 | 12.0 |
| save 50% more After | 5 yrs | 10 yrs | 20 yrs | 40 yrs | 60 yrs |

| No. of dollars (at this year prices) | 4.0 | 9.0 | 15.0 | 20.0 | 25.0 |
| save 100% more (i.e. save twice the amount now saving) After | 5 yrs | 10 yrs | 20 yrs | 40 yrs | 60 yrs |

2. Without using any detailed calculation (either using calculator, written or mental calculation), complete the following table to indicate how much you think a dollar now would grow (i.e. would be compounded) to, after the respective number of years indicated, and at the respective annul (real) rate of interest indicated. (Figures in brackets are the correct answers.)

<table>
<thead>
<tr>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>At</td>
</tr>
<tr>
<td>2%</td>
</tr>
<tr>
<td>5%</td>
</tr>
<tr>
<td>10%</td>
</tr>
<tr>
<td>20%</td>
</tr>
</tbody>
</table>

reducing having to save anything. They rather have the carefree life of living from hand to mouth. However, I do not believe that the disutility of saving as such and the trouble of managing some assets (which could just be a saving account) are so high as to outweigh the benefits of some real returns,
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Table 3
Answers to Questionaire B.

<table>
<thead>
<tr>
<th>Question 1</th>
<th>Number</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) grossly underestimated</td>
<td>55</td>
<td>72.37</td>
</tr>
<tr>
<td>(ii) somewhat underestimated</td>
<td>15</td>
<td>19.74</td>
</tr>
<tr>
<td>(iii) approximately right</td>
<td>2</td>
<td>2.63</td>
</tr>
<tr>
<td>(iv) somewhat overestimated</td>
<td>1</td>
<td>1.32</td>
</tr>
<tr>
<td>(v) grossly overestimated</td>
<td>3</td>
<td>3.95</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>76</strong></td>
<td><strong>100.00</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) saving at least 3 times as much</td>
<td>15</td>
<td>18.75</td>
</tr>
<tr>
<td>(ii) saving about twice as much</td>
<td>16</td>
<td>20.00</td>
</tr>
<tr>
<td>(iii) around 50% more</td>
<td>16</td>
<td>20.00</td>
</tr>
<tr>
<td>(iv) around 20% more</td>
<td>19</td>
<td>23.75</td>
</tr>
<tr>
<td>(v) no change</td>
<td>13</td>
<td>16.25</td>
</tr>
<tr>
<td>(vi) saving less</td>
<td>1</td>
<td>1.25</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>80</strong></td>
<td><strong>100.00</strong></td>
</tr>
</tbody>
</table>

the convenience of a bank balance, and the insurance against rainy days. Rather, it is more convincing that such individuals do not save because they do not maximize their expected utility.

It may be objected that, whatever an individual prefers, it must increase his utility since utility is just a numerical indicator of preference. This is not so since there are more than one period involved. There is a utility function for each period. It is true that we need an intertemporal comparison of utility (such as the capacity to enjoy consumption used in Appendix A) but this can be supplied by the individual concerned (e.g. by his admission that his preference in the future is not going to be less intense as now). The choice in the current period may then be consistent or inconsistent with the maximization of expected utility intertemporally. In fact, even for a single period, we may still have non-utility-maximizing preferences if ‘utility’ is not used as an indicator of preference, but as a measure of the subjective well-being or happiness of an individual. (See Ng (1979/1983, section 1.4) on these two contrasting usages of ‘utility’ and the confusion caused on the issue of cardinal versus ordinal measurability of utility.)

This divergence from intertemporal expected utility maximization is quite different from other divergences from expected utility maximization such as the Allais’ paradoxes and Tversky’s intransitivities [Allais and Hagen (1979) and Tversky (1969)]. These latter divergences may largely be explained by the utilities/disutilities of excitement, regret, etc. and mistakes that a rational individual would be prepared to admit and correct, once discovered. The remaining divergences, if any, may however be akin to the intertemporal divergence based on irrationality focussed on here. [For an analysis of
irrationality, see Ng (1989)]. Rationality, like other desirable traits such as intelligence, is endowed in different individuals in different degrees. It simply is empirically false that all or even most individuals are perfectly rational. It seems more likely that people are more-or-less continuously distributed along a line representing different degrees of rationality, with perfectly maximizing individuals on one extreme and quite irrational individuals (including the insane) on the other.

4. Concluding remarks

In the present age of electronic calculators, to know the amount a dollar will compound to after x years at y% involves just pressing the calculator a few times. However, few if any of the 82 respondents in the survey reported had bothered to find out. Yet such information is essential for making a rational decision regarding consumption/savings. It is very doubtful that people behave rationally in making intertemporal choices. It is not without reason that Pigou (1929, p. 25) complained about the 'faulty telescopic faculty', that Harrod (1948, p. 40) lamented on the 'conquest of reason by passion', and that Ramsey (1928, p. 543) deplored the 'weakness of imagination', all referring to the tendency of people not having enough concern for the future.

The fact that people do not behave in accordance with their own best interest does not imply that they have to be forced to do so. The undesirable effects of forcing people to do things against their will generally outweigh the benefits. The liberal tradition is to provide the relevant information so that people can make more enlightened choices. Though the costs of information provision also have to be taken into account, these are unlikely to be prohibitive if it is done through the normal educational system. The use of a survey like the one used here or some improved version of it may prove to be a very cost-effective way of encouraging people to save more for their own benefits.

Appendix A: A simple model of intertemporal optimal consumption/saving

Since our objective is to be indicative rather than realistic and definitive, we lose little by adopting a very simplified model, thus gaining illustrative clarity. Ignoring uncertainty and the associated pure time discount, an individual expects to live with certainty until age T and to drop dead then. She also knows her (exogenous) life-time income with certainty and can lend and borrow at the same market (real) rate of interest i. Thus, she maximizes at any age a, with respect to the control variable c(t).
where \( u \) = utility, \( c \) = consumption, \( t \) = time, and \( r \) is the rate of depreciation (with age) in the capacity to derive utility, not the rate of discount as commonly used.\(^4\) As usual, \( u' > 0 \), and \( u'' < 0 \).

Given non-satiation and no motive for leaving bequests, we write the budget constraint at age \( a \) as

$$
\int_a^T c(t) e^{-i(t-a)} dt = ye^{-ia} - \int_0^a c(t) e^{-i(t-a)} dt = Y(a),
$$

where \( i = \) market rate of interest, \( Y \) is the life-time earnings (plus gifts and inheritance received, if any) of the individual valued at age zero, and \( Y(a) \) is the life-time income left (i.e. unconsumed) at age \( a \) valued also at age \( a \), and a bar over \( c \) indicates its historical value.

The maximization of (1) subject to (2) gives the following first-order condition\(^5\)

$$
u'\{c^*(t)\} = e^{(r-i)(t-a)}u'\{c^*(a)\},
$$

which states that, from any age \( a \), the individual's optimal choice involves arranging his future consumption path such that the marginal utility of consumption (after accounting for the depreciating ability to enjoy consumption) decreases at the rate of interest \( i \). Given diminishing marginal utility \( (u'' < 0) \), consumption increases/decreases with time if \( i \) is larger/smaller than \( r \). Intuitively, if the rate of interest is larger than the rate of depreciation in the ability to enjoy consumption, (from a position of equal consumption through time) it is better to save now and consume more in the future until this advantage is offset by the diminishing marginal utility of consumption.

\(^4\)Shepard and Zeckhauser (1982) uses a rate of discount equal to the rate of interest and hence obtaining the simple solution that the optimal trajectory of consumption is constant through time. While discounting future consumption at the market rate of interest may be reasonable, discounting future utility at this rate is questionable, especially since the probability of survival has already been taken into account. Assuming expected utility maximization, we adopt no further discount on future utility. If desired, a discount rate can be easily added with only notational and computational complications. In fact, our \( r \) may be taken as the rate of depreciation (taken as zero by other analysts) plus the pure rate of time discount. Alternatively, a higher rate of time discount can be approximated by a smaller \( T \) in our model.

\(^5\)We may view this as a calculus of variation problem or an optimal control problem in which case we may use

$$
\dot{Y}(t) - iY(t) - c(t)
$$

which may be obtained by differentiating \( Y(a) \) in (2) and replacing \( a \) by \( t \); we also have the terminal condition \( Y(T) \geq 0 \). In either case, the same first-order condition (3) can be obtained.
Eqs. 2 and 3 determine the optimal consumption path through to age $T$ from any age $a$ (including $a=0$), given $Y$, $i$, $r$, and $T$. To obtain explicit solutions, we need to have a specific utility function. Take the much used constant-elasticity form

$$u\{c(t)\} = \alpha \{c(t)\}^\varepsilon,$$  \hspace{0.5cm} (4)

where $\alpha$ and $\varepsilon$ are positive constants. With (4), we may solve, from (2) and (3), at any age $a$, for the optimal consumption path from age $a$ to $T$, for $\varepsilon i - r \neq 0$,

$$c(t) = (\varepsilon i - r) \cdot Y(a) \cdot \exp \left( \frac{(i-r)(t-a)}{1-\varepsilon} \right) \cdot (1-\varepsilon)^{-1} \cdot \left\{ \exp \left( \frac{(\varepsilon i - r)(T-a)}{1-\varepsilon} \right) - 1 \right\}^{-1}. \hspace{0.5cm} (5)$$

Putting $a=0$ in (5) to calculate the values of $c(t)$ for $t$ from 0 to $a$ and substituting the resulting solution into the second equation in (2), we have, after integration and simplification,

$$Y(u) = \frac{Y(\exp((\varepsilon i - r)(T-u)/(1-\varepsilon))-1)}{(1-\varepsilon)^{-1}} \cdot (1-\varepsilon) \cdot \left\{ \exp \left( \frac{(\varepsilon i - r)T}{1-\varepsilon} \right) - 1 \right\}. \hspace{0.5cm} (6)$$

Obviously, $Y(0) = Y$, and $Y(T) = 0$ as required. Substitute $Y(a)$ from (6) into (5) to give the final solutions for $c(t)$ (for $t$ from $a$ to $T$).

$$c(t) = \left[ Y(\varepsilon i - r) \exp \left( \frac{(i-r)(t-a)}{1-\varepsilon} \right) \left\{ \exp \left( \frac{(\varepsilon i - r)T}{1-\varepsilon} + ia \right) - \exp \left( \frac{(i-r)a}{1-\varepsilon} \right) \right\} \right] / \left[ (1-\varepsilon) \left\{ \exp \left( \frac{(\varepsilon i - r)(T-a)}{1-\varepsilon} \right) - 1 \right\} \left\{ \exp \left( \frac{(\varepsilon i - r)T}{1-\varepsilon} \right) - 1 \right\} \right]. \hspace{0.5cm} (7)$$

If precise values of $i$, $r$, $T$, and $Y$ are known, exact numerical values for

For the case $\varepsilon i - r = 0$, we have instead of (5),

$$c(t) = (T-a)^{-1} \cdot Y(a) \cdot \exp((i-r)(t-a)/(1-\varepsilon)). \hspace{0.5cm} (5a)$$

Eq. (6) should also be changed correspondingly.
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c(t) can be calculated for all ages from a (time of optimization) to T. In particular, from a = 0 (i.e. assuming optimization right from the beginning; not really unreasonable since parents may be viewed as optimizing on the behalf of the individual concerned), we have from (7),

\[ c(t) = \frac{Y(\bar{\epsilon} i - r) \exp((i - r)t/(1 - \bar{\epsilon}))}{(1 - \bar{\epsilon})\exp((\bar{\epsilon} i - r)T/(1 - \bar{\epsilon})) - 1}. \]  

The exact solution for c(t) for selected ages is reported in table 1 in the text for a plausible set of parametric values partly based on observation (e.g. \( \bar{i} \)) and commonsense (e.g. \( \bar{\epsilon} \) which must be between zero and one. Arthur uses values for \( \bar{\epsilon} \) from 0.4 to one; a higher \( \bar{\epsilon} \) is favourable to our case for higher savings when young.)

At the time of writing, the nominal interest rates for Australia/U.S. are about 16%/10% and inflation rates about 7%/4%. I use a lower real rate closer to the historical average [on which see Ibbotson and Sinquelield (1979)]. It is true that the after-tax return is considerably lower. However, there are many forms of accumulation (such as in low-tax superannuation and properties) that yield much higher returns.

References

Allais, Maurice and Ole Hagen, eds., 1979, Expected utility hypothesis and the Allais paradox: Contemporary discussions of decision under uncertainty with Allais' rejoinder (Reidel, Dordrecht).


