The Incompatibility of Individualism and Ordinalism

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A formal proof is offered of the fact that Individualism and Ordinalism imply the controversial condition A3 used by Kemp and Ng (1976) to show the non-existence of an individualistic social welfare function based only on ordinal preferences but objected to by Samuelson (1977) as unreasonable. Mayston's argument against our definition of ordinality is refuted; his notion of 'true ordinality' involves elements of cardinality and/or goes against individualism.

Key words: Social welfare function; individualism; ordinalism.

In an earlier paper (Kemp and Ng, 1976) it was shown that social welfare functions satisfying specified conditions do not exist, even when the set of individual preferences is given. Among the four conditions specified, the third (A3) has attracted unfavourable attention. It has been suggested that (i) condition (A3) is unreasonable; and it has been argued that (ii) if (A3) is relaxed, then there do exist individualistic social welfare functions based on only the ordinal properties of individual preferences (Samuelson, 1977, 1981; Mayston 1980) (Note 1; see section Notes, at the end of this paper). We have never claimed that (A3) is reasonable; to the contrary. But we do continue to deny the possibility of constructing individualistic social welfare functions which rely on ordinal-preference information only (and which satisfy some other non-controversial conditions).

Now we have already stated, in our reply to Paul Samuelson (Kemp and Ng, 1977), that (A3) is implied by Individualism and Ordinalism, so that if (A3) is abandoned, then so must be Individualism and/or Ordinalism. However, a formal proof was lacking. In the present note we supply the missing proof. The proof rests on a particular definition of ordinality. David Mayston (1980, 1981) has argued that the definition is inappropriate. However, we shall show that Mayston's notion of 'true ordinality' involves elements of cardinality and/or goes against individualism.

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1. Proof of the proposition

Let $X$ be the set of social alternatives, $Y = \{1, \ldots, i, \ldots, m\}$ the set of individuals, and $U$ the set of all numerically bounded functions defined on $X \times Y$. Any set of individual preferences (cardinal or ordinal) is then represented (Note 2) by some $u \in U$ where $u(x,i)$ is the utility of individual $i$ in state $x$. The problem of social choice is to rank each pair of alternatives such that the overall ranking $R$ over $X$ is logically consistent and satisfies whatever requirements are deemed reasonable.

We impose the requirements of Individualism and Ordinalism. Individualism is in turn an amalgam of what we shall call Weak Individualism and Independence.

Weak Individualism requires social choice to be a function of and only of individual (cardinal or ordinal) preferences, with all non-preference variables ruled out of court. In particular, weak individualism rules out the possibility that the social ranking of alternative social states depends directly on the objective specification of those states; any dependence must be indirect, through individual preferences (Note 3). Writing $u(x) = \{u(x,1), \ldots, u(x,m)\}$ as the vector of individual utilities given by the relevant $u$, we have the following formal definition.

**Weak Individualism.** For any given $u(x)$ and $u(y)$ with $x, y \in X$, $R(x,y) = f\{u(x), u(y), u(x^1), \ldots, u(x^p)\}$ with the form of $f$ as a function of the $u$'s being independent of $x, y, x^1, \ldots, x^p$.

Weak Individualism does not exclude the possibility that the social ranking of $x$ and $y$ is a function of $u(x^1), \ldots, u(x^p)$. To exclude that possibility is the role of the condition of Independence.

**Independence.** For any $x, y \in X$, the social ranking $R(x,y)$ is a function $h\{x, y; u(x), u(y)\}$; that is, it depends only on $x, y$ and individual utility levels at $x$ and $y$.

Independence captures the irrelevance aspect of Arrow’s Independence of Irrelevant Alternatives but does not require its ordering aspect (Sen, 1970, pp. 89 ff.). It is therefore extremely reasonable (Plott, 1972). Since social alternatives are mutually exclusive and each alternative is described in all relevant detail, one does not want the choice between $x$ and $y$ to be affected by preferences with respect to $z$. Combining Weak Individualism and Independence we obtain

**Individualism.** For all $x, y$ in $X$, $R(x,y) = g(u(x), u(y))$ with the form of $g$ fixed.

**Ordinalism.** If $u^1$ and $u^2$ are in $U$ and such that $u^2(x,i) = \psi(u^1(x,i))$ for all $x$ and all $i$, where $\psi$, is a positive monotone transformation of each $i$, then $u^1$ and $u^2$ are equivalent pieces of information ($u^1 - u^2$) and are to be treated similarly.

As the final preliminary, we recall, from Kemp and Ng (1976), condition (A3).
Condition (A3). The social ordering of any two social alternatives depends only on the m individual orderings of the alternatives.

Formally, for any (not necessarily distinct) x, y, z, w, if \((xR_i,y \equiv zR_i,w)\) for all \(i\), then \((xR_y \equiv zR_w)\), where \(R_i\) stands for one (not necessarily the same for all \(i\)) of \(P_i, I\), and contra \(P_i\), and \(R\) stands for one of \(P, I\) and contra \(P\). (\(P_i\) and \(I\), indicate the preference and indifference of the \(i\)th individual, \(P\) and \(I\) indicate social preference and indifference.)

We can now state and prove our proposition.

Proposition. Individualism and Ordinalism imply (A3).

Proof. Let the precondition of (A3), \(xR_i,y \equiv zR_i,w\) for all \(i\), be satisfied; and let the social ranking of \(x\) and \(y\) be \(xR_y\). We must show that \(zR_w\). Let \(u^1 \in U\) represent the given set of individual preferences. Then, from Individualism, \(g(u^1(x), u^1(y))\) implies \(xR_y\). Since \(xR_i,y \equiv zR_i,w\) for all \(i\), we may choose \(u^2 \in U\) such that \(u^2(z) = u^1(x), u^2(w) = u^1(y)\). From Ordinalism, \(u^1\) and \(u^2\) are to be treated in the same way. Then, from Individualism, \(g(u^2(z), u^2(w))\) implies \(zR_w\).

2. Rebuttal of Mayston’s counter argument

We turn to Mayston’s (1980) counter argument. Consider Fig. 1(a), which depicts a simple, continuous, one-dimensional set of social alternatives and in which the arrows indicate the directions of preference for two individuals, \(A\) and \(B\). Suppose that the community is indifferent between \(x\) and \(y\), that is, that \(xI_y\). Then, if (A3) is accepted, \(zI_w\) and, by an earlier argument (Kemp and Ng, 1976), one deduces non-transitivity or violation of the Pareto principle. Mayston seeks to escape this conclusion by rejecting (A3). He writes, instead of \(zI_w\), \(zP_w\) because, for \(A\), the difference between \(z\) and \(w\) is less than the difference between \(x\) and \(y\) whereas, for \(B\), the opposite is true (‘secondary Pareto-dominance’). However, Mayston fails to notice that without making use of information concerning the intensity of individual preference (violating Ordinalism) and/or of non-preference information (violating Individualism) it is impossible to declare \(x\) and \(y\) socially equivalent in the first place without at the same time declaring \(z\) and \(w\) socially equivalent. And, as Mayston himself has shown, \(xI_y\) and \(zI_w\) are incompatible with secondary Pareto-dominance.

To further elucidate our argument, consider Fig. 1(b), which is obtained from Fig. 1(a) simply by compressing \(B\)’s utility scale. Evidently the two parts of Fig. 1 display the same ordinal information about preferences. In particular, the standing of \(x\) in relation to \(y\) is exactly the same as the standing of \(z\) against \(w\). Hence if \(x\) is judged to be socially equivalent to \(y\), then so must \(z\) be judged in relation to \(w\).
We conclude that Mayston's 'true ordinality' is infected with cardinality and/or anti-individualism.

Mayston's (1981) paper is vulnerable to much the same comment as his (1980) paper. However, it does enable us to further clarify the nature of our differences. Consider again the simple example of Fig. 1. Mayston believes that, by reference to more than immediate directions of individual preferences, but still within the framework of individualism and ordinalism, one can declare $x \succ y$ without at the same time declaring $z \succ w$. We hold that, with no more than ordinalism assumed, the standing of $x$ in relation to $y$ is exactly the same as the standing of $z$ in relation to $w$; hence $x \succ y$ implies $z \succ w$. Mayston replies that society can choose $x \succ y$ but deny $z \succ w$ by relying on an objective characteristic of the alternatives (the amount of chocolate consumed, for example). We regard such reliance as violating individualism. Mayston retorts that, if so, even the majority rule, indeed any conceivable rule, violates individualism. "It is difficult to imagine any rule (including majority rule) satisfying the condition that the social ranking shall depend upon individual preferences irrespective of the local weights placed upon divergent individual preferences" (Mayston, 1981). Of course, to obtain a specific SWF it is necessary to introduce some sort of interpersonal weighting. But, in our view, to be consistent with individualism, weights must be attached to individual preferences, ordinal or cardinal, not to objective characteristics of social states.

Notes

Note 1. However, note that in his new paper, Samuelson (1981, p. 235) states that his social welfare function "is not ... a function whose input variables are (just) individual rankings" (italics by Samuelson). In other words, something more than just
individual rankings are required. This is what we have been arguing: either cardinalism (going beyond just rankings) and/or non-individualism (going beyond individual preferences, ordinal or cardinal) has to be involved.

Note 2. We ignore the non-representability of lexicographic preferences. This difficulty has been rather exaggerated. With \( X \) uncountable, lexicographic preferences are extremely unlikely. With \( X \) countable, representation is ensured whether or not preferences are lexicographic (see Ng, 1979, p. 28).

Note 3. Samuelson (1977, 1981) makes direct use of objective amounts of consumption, arguing that (50, 50) is preferable to, say (90, 10). This may not seem unreasonable as it appeals to the egalitarian ethics. But suppose that, because of utility interdependence, different needs, etc., both individuals prefer (90, 10) to (50, 50). Obviously, direct dependence on objective specification of social states violates the Pareto principle in general. Even if we are confined to self-interested individuals, such a violation will be involved in the presence of factors other than individual consumption (e.g. public goods, regulations, non-economic factors); see Ng (forthcoming).

References


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