and $\lceil (d+1)/3 \rceil$, respectively, where d is the maximum degree of the "undirected" graph.

2 MODEL

We consider a sealed-bid auction with a set of n bidders $\mathbb{N} = \{1, 2, \dots, n\}$ and m identical items. Each bidder $i \in \mathbb{N}$ is interested in a single copy of the item. Each bidder i has a set of competitors $S_i \subseteq \mathbb{N} \setminus \{i\}$, whose winnings may decrease bidder i's valuation on the item. Suppose the overall winner set is $\mathbb{W} \subseteq \mathbb{N}$, we define the valuation function of each bidder i as follows: if $i \in \mathbb{W} : |\mathbb{W} \cap S_i| = t$, $v_i(\mathbb{W}) = v_i^t$; otherwise, $v_i(\mathbb{W}) = 0$. We assume that the valuation vector is private information to each bidder, but the competition relations are public information.

After collecting the bids from the bidders, the auctioneer determines a set of winners $\mathbb{W} \subseteq \mathbb{N}$ with $|\mathbb{W}| \leq m$, to each of whom one item is allocated, as well as a payment p_i is charged. Then, the quasi-linear *utility* of each bidder i is defined as $u_i = v_i(\mathbb{W}) - p_i$ if i wins an item, and $u_i = 0$ otherwise. The auctioneer's objective is to maximize *social welfare*, which is the sum of the winners' valuations $SW = \sum_{i \in \mathbb{W}} v_i(\mathbb{W})$.

In the auction, each bidder i reports a bid $b_i = \{\hat{v}_i^0, \hat{v}_i^1, \dots, \hat{v}_i^{|S_i|}\}$. We say a mechanism is *truthful* (also known as *incentive compatible* or *strategy-proof*) if for every bidder i and fixed bids of other bidders, bidder i maximizes her utility by bidding the true valuation vector $v_i(\cdot)$. We also require the mechanism to guarantee *individual rationality*, which means that for every bidder, bidding truthfully never results in negative utility.

Given the competitor sets, we construct a directed *competition graph* G, in which each vertex represents a bidder, and each edge $(i, j),\ i, j \in \mathbb{N}$ indicates that bidder j is in bidder i's competitor set.

3 RESULTS

3.1 Single-Competitor Graphs

We consider the case when the competition graph has out-degree one for every vertex. That is, every bidder has exactly one competitor. This assumption is reasonable in many scenarios, in particular in duopoly markets where two firms have dominant control and compete with each other.

We present a polynomial time allocation algorithm for the welfare maximization problem for this case.

Given a single-competitor/friend graph $G(V, E)$, we process each connected component in G separately. We discuss all possible allocation outcomes for bidder k: (1) bidder k does not win; (2) bidder k wins and competitor c_k does not win; (3) both bidder k and c_k win. We consider these three cases separately. We use modified bid vector to guarantee the allocation optimal outcome for bidder k. This allows us to ignore the edge (k, c_k), after which the remaining graph becomes a tree rooted at vertex k. In such a tree, parents are always children's competitors/friends. We design a dynamic programming algorithm to find an optimal allocation on such a tree.

Since the allocation algorithm is optimal, it is well known that combining it with VCG payments [4, 8, 13] can yield a truthful mechanism with the same performance guarantee.

THEOREM 3.1. *There is a polynomial time, truthful, and social welfare maximizing mechanism when the underlying graph has out-degree one for every vertex.*

3.2 Planar Graphs

We consider the case when the relation graph is a planar graph. Even with planar graphs, solving the allocation problem optimally is still NP-hard. Consider the digital goods auction in which for every bidder i, the valuation for winning the item is 1 only if none of this bidder's competitors win the item, otherwise the valuation is 0. Thus, in this situation welfare maximization on planar competition graphs becomes the maximum independent set problem on planar graphs which is known to be NP-complete [7].

We present a strategy-proof and $(1 + \epsilon)$-approximation mechanism for planar graphs. First we introduce a PTAS allocation algorithm for planar graphs. The algorithm consists of two parts. First, we show an optimal allocation algorithm on graphs with bounded treewidth. Next, we employ Baker's graph decomposition scheme to derive a PTAS algorithm. Finally, we show that this algorithm falls into the domain of *maximal-in-range* allocation rules, thus it can induce a truthful mechanism via a VCG type of payments [12]. Using a technique from [1] and a result from [6], we can extend our result from planar graphs to a larger family of graphs that exclude any fixed minor.

THEOREM 3.2. *There is a truthful, polynomial time and* $1 + \epsilon$*-approximation mechanism for bidders with planar graph.*

3.3 General Graphs

We consider the general case where the bidders' relations do not have any restrictions. We present two truthful and computationally efficient mechanisms. Both mechanisms employ a simple partition scheme that is very similar to that in the planar graph case. We first partition the graph into several small solvable subgraphs, then find the optimal allocation for each subgraph, finally pick the best allocation among these as the final winner set. Finally, combining them with VCG-type payments gives us the truthful mechanisms.

3.3.1 Partition into Small-Size Subgraphs. For the allocation, we first partition the relation graph randomly into $n/\log n$ subgraphs, each of which has $\log n$ vertices. Finally, we find the subgraph achieving maximum social welfare among all subgraphs, and output the allocation of this subgraph as the final winning set.

THEOREM 3.3. *There is a polynomial time, truthful and* $(n/\log n)$*-approximation mechanism for social welfare maximization.*

3.3.2 Partition into Low-Degree Subgraphs. Given graph G, let $\overline{G}$ be its undirected version. Suppose $\overline{G}$ has maximum degree d, through a different partition method, we can get a truthful mechanism with a different approximation guarantee. This partition is due to [9] by a local search strategy. Then by simply extending the out-degree 1 algorithm from Section 3.1, we can again compute the optimal allocation in each subgraph, and pick the best one and apply the VCG payments. Using exactly the same arguments as before, we can show that this mechanism is truthful and achieves $\lceil (d + 1)/3 \rceil$ approximation.

THEOREM 3.4. *There is a polynomial time, truthful and* $\lceil (d+1)/3 \rceil$*-approximation mechanism for social welfare maximization.*

4 CONCLUSION

In this paper, we study truthful and computationally efficient mechanisms under different restrictions on the underlying competitor graph structure. Our results include (1) a truthful and social welfare optimal mechanism when each bidder has only one competitor; (2) a truthful and $(1 + \epsilon)$-approximation mechanism when the relation graph is planar; (3) two truthful mechanisms when bidders have arbitrary relations, with approximation ratio $(n/\log n)$ and $\lceil (d+1)/3 \rceil$, respectively. An interesting open questions is to generalize these results to multi-item combinatorial auctions.

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