Predicates as processes: Linear implication is a branching-time causality-preserving precongruence

Ross Horne and Alwen Tiu

School of Computer Science and Engineering, Nanyang Technological University, Singapore

Abstract

We summarise recent advances in the predicates-as-processes paradigm. In particular, we note the non-commutative logic MAV1 featuring choice and a novel dual pair of nominal quantifiers. For fragments of MAV1, we have shown the soundness of implication with respect to both weak simulation and pomset trace inclusion; hence implication is a branching-time causality-preserving precongruence. The system is sufficiently expressive to soundly embed finite fragments of process models not limited to the π-calculus, session type systems and workflow modelling languages.

1 A roadmap for the predicates-as-processes paradigm

There are striking formal connections between linear logic and interaction and concurrency in process calculi. Attempts to formalise such connections could be classified along two major thrusts: the proofs-as-process [1, 4] and predicates-as-processes paradigms. We follow the predicates-as-processes paradigm where processes are embedded directly as predicates in a logical system. Early work [12], considered linear logic as the target system. However, such embeddings tended to encoded the semantics of sequentiality as a theory rather than a logical primitive; hence implication does not directly give rise to a preorder over processes.

The calculus of structures [8], a generalisation of the sequent calculus, is sufficiently expressive to extend linear logic with a non-commutative operator suitable for modelling sequentiality as a logical primitive. Preliminary work [3], established that linear implication in proof system BV is sound (but not complete) with respect to completed trace inclusion for a fragment of CCS with only prefix and parallel composition. This preliminary investigation can be tightened according to the following agenda:

- Where is linear implication precisely situated in the spectrum of process preorders?
- Can expressive process calculi be soundly embedded in extensions of BV?
- Given that there is a strong objective justification for this process preorder (cut elimination); are there also compelling applications for this process preorder?

Figure 1 elaborates on the first question above. In Fig. 1, process preorders are divided along two axis: the linear-time/branching-time axis, and the interleaving/causality-preserving axis [13]. At the top of Fig. 1 is trace inclusion, defined by subset inclusion over the set of all traces of a process. Trace inclusion is widely considered to be the coarsest preorder over processes; hence preorders should be sound with respect to trace inclusion, as indicated by the arrows in the figure. Along the linear-time/branching-time axis, trace inclusion can be refined by various weak simulations which have finer properties regarding the distributivity of non-deterministic choice, indicated by ⊕ in this work. In the other direction, along the interleaving/causality-preserving axis, models such as pomset traces [7] preserve causal relationships between events; ensuring, for example, that, unlike trace inclusion and weak simulation, we have action refinement and no autoconcurrency [2].

In recent work, we observe that linear implication is both branching-time and causality-preserving; hence is situated at the bottom of Fig. 1.
2 A first-order non-commutative logic with nominals

We recall a first-order extension of BV, called MAV1 [11], featuring a novel de Morgan dual pair of nominal quantifiers “new” and “wen” introduced to model private names as featured in the π-calculus. The syntax is presented in Fig. 2 and the proof system is defined by a term-rewriting system modulo an equational theory, where rules can be applied deep within any context. The semantics ensures that seq is non-commutative, whereas other operators are commutative.

Linear negation is defined by a function over predicates such that the first-order quantifiers ∃ and ∀ are de Morgan dual, as are the nominal quantifiers И and Э, the multiplicatives ⊗ and ||, the additives & and ⊕, and atoms α and α. The non-commutative operator seq and unit are self-dual in the sense that they are their own de Morgan duals. Linear implication P ⊸ Q is defined in terms of linear negation and par as P || Q.

A derivation is a sequence of zero or more rewrites and a proof of P, is a derivation rewriting P to the unit. When such a derivation exists, we say that P is provable, and write ⊢ P. A generalisation of cut elimination holds for MAV1 [9, 11] as follows.

**Theorem 2.1** (Horne et al. (2016)). If ⊢ C{ P ⊗ P } then ⊢ C{ 1 }.

Cut elimination is the corner stone of a proof calculus. A corollary of significance here is that linear implication defines a precongruence.

For the sub-calculus, MAV, without quantifiers, the following soundness result has been established [10].

**Theorem 2.2** (Horne, Mauw and Tiu (2017)). For series-parallel processes with choice, if ⊢ P ⊸ Q then the pomset traces of P are included in the pomset traces of Q.

In the above, pomset traces are defined in terms of ideals given by certain graph homomorphisms [7]. The result was stated for a workflow model called an attack tree, but covers related process models [5] and should extend easily to pomset trace semantics for CCS and the π-calculus.
Embeddings of the $\pi$-calculus require existential quantifiers for input binders and the nominal quantifier $\Pi$ for private name binders. The $\pi I$-calculus can be treated by instead using dual nominal $\mathcal{E}$ for input. Such embeddings we have shown to be sound with respect to weak simulation. We tighten this result further by considering complete progressing open simulation, which is a termination-sensitive weak simulation precongruence. The proof requires fundamental proof theoretic advances.

**Theorem 2.3** (Horne and Tiu (submitted)). For an embedding of the $\pi$-calculus and $\pi I$-calculus as predicates, if $\vdash P \to Q$, there is a complete progressing open simulation, $\preceq$, such that $P \preceq Q$.

A further consequence of cut elimination for sub-system MAV is that coherent protocols are multi-party compatible for a finite session type system [6]. For session types, the branching-time causality-preserving termination-sensitive precongruence given by linear implication defines a subtype system.

**References**


