I. INTRODUCTION

Quantum cascade lasers (QCLs), since their invention in 1994, have become important light sources in the terahertz (THz) and mid-infrared (IR) spectral ranges. QCLs overthrow the key working principles of diode lasers in that they do not involve a transition of the electrons from the conduction band into the valence band but depend on intersub-band transitions between quantized energy levels within the conduction band. Therefore, their emission wavelength can be engineered across the mid-IR (3–24 μm) and THz (1.2–5 THz or 60–250 μm) regions for important applications including, but not limited to, trace-gas absorption spectroscopy, optical free-space data communication, remote sensing and imaging, and local oscillators for submillimeter-wave astronomy.

For these applications a narrow-linewidth, single-mode, coherent and compact source is highly desired. Like all other lasers, narrow-linewidth QCLs can easily suffer from various noises, which play an important role in their laser performance especially in their spectral linewidth. An accurate analysis of the linewidth caused by these noises is necessary in order to ensure that QCLs can meet certain design specifications. Several research groups have measured the linewidth of QCLs ranging from a few hundreds of megahertz down to a few KHz or less, depending on the measurement techniques and the stabilization processes used. External noise factors such as mechanical vibrations, external environmental temperature variations, and bias-current fluctuations can be potentially totally removed by various frequency-stabilization techniques, such as phase-locking techniques. However, the intrinsic (fundamental) noises caused by, e.g., spontaneous emission, carrier noise, blackbody radiation, and thermodynamical fluctuation of temperature cannot be overcome due to the fundamental quantum limitations. So far several research groups have reported the investigations on intrinsic noises such as the intensity noise, and frequency noise induced by spontaneous emission, electron density fluctuation, and blackbody radiation in QCLs. Bartalini experimentally reported the property of an ultra-narrow intrinsic linewidth beyond the Schawlow-Townes limit in distributed feedback mid-IR QCLs in a single-mode continuous-wave operation. An intrinsic linewidth of ~510 Hz at temperatures in the range between 81 and 92 K was observed for \( I/I_\text{th} = 1.54 \) (\( I \) is the operation current and \( I_\text{th} \) is the laser threshold current) using the side of a Doppler-broadened molecular transition of CO2. Yamanishi had theoretically clarified the hidden reason for the narrow intrinsic linewidth based on the classical rate equations by considering the spontaneous radiation and blackbody radiation effects. The experimental linewidth values measured by Bartalini are smaller than predictions. The possible reason is that the microscopic phenomena such as the dynamics of polarization, tunneling effect, and many body interactions are not considered in Yamanishi’s theoretical calculation of linewidth. Considering these factors may give a reduced value in the linewidth calculations. Jirauschek has also theoretically predicted an extremely narrow intrinsic linewidth of Hz level (3 Hz for the double-metal waveguide QCLs at the 4.5-THz emission) or even below caused by spontaneous emission, stimulated transitions, and blackbody radiation for high power THz QCLs. However, another important contribution to the intrinsic linewidth of QCLs, i.e., the fundamental thermal noise caused by thermodynamical fluctuation of temperature within the laser cavity, has not yet been investigated. Even for a
cavity in the perfect thermal equilibrium with its surroundings, this thermal noise floor always exists. From our analysis the noise floor caused by fundamental temperature fluctuations is around 1 Hz for THz QCLs and can reach tens of Hz or even hundreds of Hz for mid-IR QCLs. The linewidth broadening caused by fundamental thermal fluctuations for both THz and mid-IR QCLs are comparable to those caused by spontaneous emission, stimulated emission, and blackbody radiation. Therefore, it is important to investigate the effects of the fundamental thermal noise in both THz and mid-IR QCLs.

On the other hand for many applications it is the intrinsic frequency noise spectrum of QCLs that is of primary interest. The studies of the intrinsic frequency noise of QCLs have been focused on high frequency at 10 MHz and even up to GHz. For most spectroscopic applications the detected signal is focused on high frequency at 10 MHz and even up to GHz. The studies of the intrinsic frequency noise of QCLs have been focused on high frequency at 10 MHz and even up to GHz.9 For most spectroscopic applications the detected signal is focused on high frequency at 10 MHz and even up to GHz.9

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The intrinsic frequency noise caused by spontaneous emission, stimulated emission, and blackbody radiation, predicted by Jirauschek19 in the high power regime in the THz QCLs. In this paper we focus on double-metal waveguides due to their high temperature performance, which is one of the most important research subjects in the field. We will first investigate the frequency noise and linewidth broadening in the THz QCLs. The results for mid-IR QCLs are shown in the next section. As for this section the stochastic heat source is first evaluated, and then the thermal fluctuation spectrum and the resultant frequency noise and linewidth broadening are analytically derived.

A. Stochastic heat source

The statistical property of the random internal heat source driving the temperature fluctuation can be evaluated by fluctuations-dissipation theorem32 from a Langevin diffusion equation, which has been used to investigate the thermal noise in solid-state devices.33-35 This stochastic equation is written as

\[ \frac{\partial \Delta T}{\partial t} - \frac{\eta}{\rho c} \nabla^2 \Delta T = \vec{V} \cdot F(\vec{r},t), \]

where \( \Delta T(\vec{r},t) = T(\vec{r},t) - \langle T(\vec{r},t) \rangle \), \( \langle \cdots \rangle \) means time or ensemble averaging, \( F(\vec{r},t) \) is the stochastic thermal-flux density of the internal heat sources of temperature fluctuation, \( \eta \) is thermal conductivity (W cm\(^{-1}\) K\(^{-1}\)), \( \rho \) is the density (g cm\(^{-3}\)), and \( c \) is the specific heat capacity (J g\(^{-1}\) K\(^{-1}\)).

According to the theory of thermodynamics,36,37 the total variance of temperature fluctuation is described by the following well-known formula

\[ \langle \Delta T^2 \rangle = \frac{k_B T_0^2}{C}, \]

where \( k_B \) is Boltzmann constant, \( C = \rho c V \) is heat capacity (J K\(^{-1}\)), \( V = Al \) is the volume, and \( T_0 \) is the equilibrium temperature.

II. ANALYSIS OF THZ QCLS

The QCLs can be designed for light emissions covering the mid-IR and THz regimes. The waveguide structures of mid-IR and THz QCLs are different because they are operated in different wavelength regimes. Mid-IR QCLs are typically based on a dielectric-waveguide structure in which the active region is embedded between semiconductor cladding layers with a smaller refractive index.7 For THz QCLs there are two types of waveguides used, namely, the surface-plasmon waveguide26,27 and the double-metal waveguide.3 The surface-plasmon waveguide sandwiches the active region with metal and a thin heavily doped semiconductor layer followed by a semi-insulating GaAs substrate. It has the advantages of high output powers and good beam patterns in terms of the beam divergence. The double-metal waveguide uses metal layers, which are placed just above and below the epilayer active region. It has a perfect optical confinement with a confinement factor close to 1 and an efficient heat removal,28-31 hence demonstrating the best temperature performance of THz QCLs. In this paper we focus on double-metal waveguides due to their high temperature performance, which is one of the most important research subjects in the field. We will first investigate the frequency noise and linewidth broadening in the THz QCLs. The results for mid-IR QCLs are shown in the next section. As for this section the stochastic heat source is first evaluated, and then the thermal fluctuation spectrum and the resultant frequency noise and linewidth broadening are analytically derived.
The stochastic heat source function $F(\vec{r},t)$ can be characterized by the following correlation function\(^{31}\)
\[
\langle F(\vec{r},t)F(\vec{r}',t') \rangle = F_0^2 \delta^3(\vec{r} - \vec{r}') \delta(t - t'),
\]
where $F_0$ is the constant and should be normalized in the following way to satisfy Eq. (2):
\[
\langle \Delta T^2 \rangle = \frac{1}{V^2} \left( \int_V \Delta T(\vec{r},t) d\vec{r} \right).
\]

$\Delta T(\vec{r},t)$ can be obtained by taking the Fourier transformation of Eq. (1)
\[
\Delta T(\vec{r},t) = \int \int \int \int \int \frac{d^2k}{(2\pi)^2} \sum_{\omega} \{ F(\vec{k},\omega)F^*(\vec{k}',\omega') \} e^{i\vec{k} \cdot \vec{r}-i\omega t}.
\]

Comparing Eq. (2) with Eq. (6), we have
\[
F_0^2 = 2DV k_b T_0^2 / c.
\]

**B. Equilibrium thermal fluctuation spectrum in THz QCLs**

Figure 1 shows a schematic structure of the cross section of THz QCLs. Since THz QCLs are typically measured at a cryogenic temperature in a cryostat (less than 186 K, which is the maximum operating temperature reported) under vacuum operation, a constant-temperature boundary condition is applied to the bottom of the substrate, and neglected heat conduction is applied to the surfaces exposed to the vacuum.\(^{38}\)

One-dimensional heat transport model is established due to the fast heat dissipation along the $z$-direction by the heat sink.\(^{39}\)

The top metal contact is not considered in the subsequent discussion due to its very high heat conductivity compared to the active region in our pure thermal model. The stochastic heat-conduction equation has the following expression for
\[
\frac{\partial \Delta T}{\partial t} - D(z) \frac{\partial^2 \Delta T}{\partial z^2} = \vec{V}_c \cdot F(\vec{z},t),
\]
where
\[
D(z) = D_1 = \eta_{\perp} \rho c / C, 0 \leq z < w_1,
\]
\[
D(z) = D_2 = \eta_m \rho_m c / C, w_1 \leq z < w_2,
\]
\[
D(z) = D_3 = \eta_s \rho_s c, w_2 < z \leq w_3.
\]

The boundary conditions of the previous equation are
\[
\eta_{\perp} \frac{\partial \Delta T}{\partial z} \bigg|_{z=0} = 0, \quad \Delta T \bigg|_{z=w_3} = 0,
\]
\[
\eta_{\perp} \frac{\partial \Delta T}{\partial z} \bigg|_{z=w_1} = \eta_m \frac{\partial \Delta T}{\partial z} \bigg|_{z=w_1} = 0,
\]
\[
\eta_m \frac{\partial \Delta T}{\partial z} \bigg|_{z=w_2} = \eta_s \frac{\partial \Delta T}{\partial z} \bigg|_{z=w_2} = 0.
\]

The Fourier transformation of this Langevin equation gives the spectra information
\[
i \omega \Delta Z(\omega,\omega) - D(z) \frac{\partial^2 \Delta Z(\omega,\omega)}{\partial z^2} = \vec{V}_c \cdot F(\vec{z},\omega),
\]

Let $G$ be the Green’s function of the previous equation and we have
\[
i \omega G(z,z',i \omega) - D(z) \frac{\partial^2 G(z,z',i \omega)}{\partial z^2} = \delta(z - z'),
\]
with homogeneous boundary conditions
\[
\eta_{\perp} \frac{\partial G}{\partial z} \bigg|_{z=0} = 0, \quad G \bigg|_{z=w_3} = 0,
\]
\[
\eta_{\perp} \frac{\partial G}{\partial z} \bigg|_{z=w_1} = \eta_m \frac{\partial G}{\partial z} \bigg|_{z=w_1} = 0,
\]
\[
G \bigg|_{z=w_1} = G \bigg|_{z=w_1} = 0,
\]
\[
G \bigg|_{z=w_1} = G \bigg|_{z=w_1} = 0.
\]
For the self-adjoint operator, the relation of reciprocity satisfies
\[ G^*(z, z', -i\omega) = G(z', z, i\omega). \]
(12)

The following equation is obtained by multiplying Eq. (10) by \( G \) and Eq. (11) by \( \Delta T \) and subtracting of the two
\[
\Delta T(z, \omega) = \int G(z', z, i\omega) \tilde{V}_z \bullet F(z', \omega) d\omega'
= \int \left[ G(z', z, i\omega) \nabla_z^2 \Delta T(z', \omega) \right. \\
- \Delta T(z', \omega) \nabla_z^2 G(z', z, i\omega) \left. \right] d\omega'.
\]
(13)

According to Green’s theorem and the homogeneous boundary conditions, it yields
\[
\Delta T(z, \omega) = \int G(z', z, i\omega) \tilde{V}_z \bullet F(z', \omega) d\omega' = 0.
\]
(14)

Then the correlation function can be deduced by
\[
\langle \Delta T(z, \omega) \Delta T^*(z', \omega') \rangle
= \int \int G(z, z_1, i\omega) G^*(z', z_2, i\omega') S_F(z_1, z_2, i\omega) d\omega_1 d\omega_2,
\]
(15)

where
\[
S_F(z_1, z_2, \omega) = \tilde{V}_z \bullet \tilde{V}_{z_1} (F(z_1, \omega) F^*(z_2, \omega')) = 4\pi D \Delta w j k_B T_0^2 / C_j \tilde{V}_z \bullet \tilde{V}_{z_2} \delta(z_1 - z_2) \times \delta(\omega - \omega').
\]

Equation (15) is a quadratic Green’s function. It can be transformed to be linear using the \( \Lambda \)-theorem, i.e., the Van Vliet-Fassett form.\(^{43,44} \) With the \( \Lambda \)-theorem we have the following relation
\[
\Lambda \cdot \Gamma(z, z') + \Lambda \cdot \Gamma(z', z) = S_F(z, z', \omega)/2,
\]
(16)

where \( \Lambda = -D(\nabla_z^2) \).

This yields
\[
\Gamma(z, z') = \pi \Delta w j k_B T_0^2 \delta(z - z') \delta(\omega - \omega')/C_j.
\]
(17)

According to Eq. (16), Eq. (15) can be written as
\[
\langle \Delta T(z, \omega) \Delta T^*(z', \omega') \rangle = 2 \int \int G(z, z_1, i\omega) G^*(z', z_2, i\omega') \times \left[ \Lambda_{z_1} + \Lambda_{z_2} \right] \Gamma(z_1, z_2) d\omega_1 d\omega_2.
\]
(18)

The right-hand side of Eq. (18) can be expressed as follows
\[
2 \int \int G(z, z_1, i\omega) G^*(z', z_2, i\omega') \left[ (\Lambda_{z_1} + i\omega) \\
+ (\Lambda_{z_1} - i\omega) \right] \Gamma(z_1, z_2) d\omega_1 d\omega_2
= 2 \int d\omega_1 G(z, z_1, i\omega) \Gamma(z_1, z_2) d\omega_1
+ 2 \int d\omega_1 G(z, z_1, i\omega) \Gamma(z_1, z_2) d\omega_1
+ 2 \int d\omega_1 G(z, z_1, i\omega) \Gamma(z, z_2, i\omega') d\omega_2.
\]
(19)

where \( \ell = \Lambda + i\omega \) and the following deduction is used
\[
\langle \Delta T(z, \omega) \Delta T^*(z', \omega') \rangle
= 2 \int d\omega_1 G(z, z_1, i\omega) \Gamma(z_1, z_2) d\omega_1
= 4\pi \Delta w j k_B T_0^2 \Re[\Gamma(z, z_1, i\omega) \Gamma(z, z_1, i\omega')]/C_j,
\]
(21)

where the notation \( \Re[\cdot] \) denotes real part.

It is noted that Eq. (16) is independent of the choice of the particular solution of the \( \Lambda \)-theorem.\(^{44} \)

The temperature fluctuations need to be averaged over the mode volume
\[
\Delta \tilde{T}(\tilde{r}, t) = \int \Delta T(\tilde{r}, t) |E(\tilde{r})|^2 d\tilde{r},
\]
(22)

where \( |E(\tilde{r})|^2 \) is the normalized optical-mode intensity distribution.

Therefore, the correlation function of \( \Delta \tilde{T}(\tilde{r}, t) \) averaged over the mode volume in the frequency domain can be written as
\[
\langle \Delta \tilde{T}(\omega) \Delta \tilde{T}^*(\omega') \rangle
= \int \int \langle \Delta T(z, \omega) \Delta T^*(z', \omega') \rangle \times |E(z)|^2 |E(z')|^2 d\omega_1 d\omega_2.
\]
(23)

From the Wiener-Khinchine relations, it is readily verified that the thermal-noise spectrum (power-spectral density) is merely the coefficient of the delta function divided by \( 2\pi \).

Hence the one-sided spectral density \( S_{\Delta T}(\omega) \) of fluctuations of temperature reads
\[
S_{\Delta T}(\omega) = 4\pi \Delta w j k_B T_0^2 / C_j \int \int \Re[\Gamma(z, z_1, i\omega)]
\times |E(z)|^2 |E(z')|^2 d\omega_1 d\omega_2.
\]
(24)

The following two additional conditions are required to determine Green’s function at \( z = z' \) by integrating Eq. (11) from \( z = z' - 0 \) to \( z = z' + 0 \):
The Green’s function $G$ can be achieved with standard methods, we thus obtain $G_1^\ast(0 \leq z < z')$, $G_1^\ast(z' < z \leq w_1)$, $G_2(w_1 \leq z \leq w_2)$, $G_3(w_2 \leq z \leq w_3)$ for the source point $z'$ lay in the active region.

\[ G_1^\ast = c_1(z') \sinh(\alpha_1 z) + c_2(z') \cosh(\alpha_1 z), \]
\[ G_2 = c_3(z') \sinh[\alpha_2(z - w_2)] + c_4(z') \cosh[\alpha_2(z - w_2)], \]
\[ G_3 = c_5(z') \sinh[\alpha_3(z - w_3)] + c_6(z') \cosh[\alpha_3(z - w_3)], \]

where

\[ \alpha_1 = \sqrt{i\omega/D_1}, \quad \alpha_2 = \sqrt{i\omega/D_2}, \quad \alpha_3 = \sqrt{i\omega/D_3}. \]

Coefficients $c_1(z'), \cdots, c_6(z')$ can be determined from the boundary conditions. We show here the expressions of coefficients $c_1(z')$ and $c_2(z')$, which are of interests to us.

\[ c_1(z') = 0, \quad (27a) \]
\[ c_2(z') = \frac{E_2 \eta_m \alpha_2 \sinh[\alpha_1(w_1 - z')] + E_1 \eta_m \alpha_1 \cosh[\alpha_1(w_1 - z')]}{[E_2 \eta_m \alpha_2 \cosh(\alpha_1 w_1) + E_1 \eta_m \alpha_1 \sinh(\alpha_1 w_1)]D_1 \alpha_1}, \quad (27b) \]

where

\[ E_1 = \eta_m \alpha_3 \cosh[\alpha_3(w_2 - w_3)] \sinh[\alpha_3(w_2 - w_1)] - \eta_m \alpha_2 \sinh[\alpha_3(w_2 - w_3)] \cosh[\alpha_3(w_2 - w_1)], \]
\[ E_2 = \eta_m \alpha_3 \cosh[\alpha_3(w_2 - w_3)] \cosh[\alpha_3(w_2 - w_1)] - \eta_m \alpha_2 \sinh[\alpha_3(w_2 - w_3)] \sinh[\alpha_3(w_2 - w_1)]. \]

In THz QCLs, the metal-metal waveguide has a perfect optical confinement (confinement factor $\sim 1$) in the active region, and its fundamental mode intensity is assumed to be approximately uniformly distributed across the active region. Thus, the spectrum of the averaged temperature fluctuations in the active region can be written as

\[ S_{\Delta T}(\omega) = 4w_k T_0^2/(Cw_2^2) \text{Re} \left[ \int_0^{w_1} \int_0^{w_1} dz' \int_0^{w_1} dz G_1^\ast(z, z', i\omega) + \int_0^{w_1} dz' \int_0^{w_1} dz G_1^\ast(z, z', i\omega) \right] \]
\[ = 4w_k T_0^2/(\rho c A w_2^2) \text{Re} \left[ \int_0^{w_1} \cosh(\alpha_1 z)dz \int_0^{w_1} c_2(z')dz' - \int_0^{w_1} dz' \int_0^{w_1} \frac{\sinh[\alpha_1(z - z')]}{D_1 \alpha_1} dz \right] \]
\[ = 4w_k T_0^2/(\rho c A w_2^2) \text{Re} \left[ \frac{E_2 \eta_m \alpha_2 \sinh(\alpha_1 w_1)[\cosh(\alpha_1 w_1) - 1] + E_1 \eta_m \alpha_1 \sinh^2(\alpha_1 w_1)}{[E_2 \eta_m \alpha_2 \cosh(\alpha_1 w_1) + E_1 \eta_m \alpha_1 \sinh(\alpha_1 w_1)]D_1 \alpha_1} - \frac{\sinh(\alpha_1 w_1) - \alpha_1 w_1}{D_1 \alpha_1} \right]. \quad (28) \]

**C. Frequency noise spectrum and linewidth broadening**

If we only focus on the temperature dependence of the refractive-index fluctuation and ignore the thermal expansion and energy-level broadening caused by the self-heating, the instantaneous frequency shift caused by local thermal fluctuation is given by

\[ f(t) = -v_0 \int \frac{1}{n} \frac{dn}{dT} \Delta T(\tilde{r}, t) |E(\tilde{r})|^2 d\tilde{r}, \quad (29) \]

where $v_0$ is laser frequency, $n$ is refractive index, and $dn/dT$ is the thermo-optic coefficient.

The correlation function of the frequency noise reads

\[ \langle f(t) f(t') \rangle = \left( \frac{v_0}{n} \frac{dn}{dT} \right)^2 \int \int \Delta T(\tilde{r}, t) \Delta T(\tilde{r}', t') \times |E(\tilde{r})|^2 |E(\tilde{r'})|^2 d\tilde{r} d\tilde{r}' \]. \quad (30) \]

With the Fourier transform of Eq. (30), the one-sided frequency noise spectrum caused by the temperature fluctuation yields

\[ S_f = \left( \frac{v_0}{n} \frac{dn}{dT} \right)^2 S_{\Delta T}(2\pi v). \quad (31) \]

Once we know the power-spectrum density of the frequency noise, the linewidth then can be specified by the following derivation. The autocorrelation function of the lasing field $O(t)$ can be expressed for single-mode QCLs in the case where the phase change $\Delta \phi$ is considered as a Gaussian distribution,

\[ \langle O(t) O^*(t - \tau) \rangle \sim \exp(2\pi v_0 \tau) \exp(-\Delta \phi^2/2). \quad (32) \]

In general if the spectral density of the frequency fluctuation is known, the mean square value (variance) of the phase change can be obtained as

\[ \langle \Delta \phi^2 \rangle = 8\pi \int_0^{+\infty} S_f(\omega) \frac{\sin^2(\omega \tau/2)}{\omega^2} d\omega, \quad (33) \]

where $\omega = 2\pi v$. The linewidth is then obtained from the one-sided power spectral density $S_f$ of the autocorrelation function.
TABLE I. Device parameters of THz QCLs used in numerical simulations (from Refs. 50–54).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>5.32 g cm(^{-3})</td>
<td>( w_1 )</td>
<td>10 ( \mu )m</td>
</tr>
<tr>
<td>( c )</td>
<td>0.18 J g(^{-1})K(^{-1})</td>
<td>( w_2 )</td>
<td>11 ( \mu )m</td>
</tr>
<tr>
<td>( \eta_{\perp} )</td>
<td>9.6T (-14) W m(^{-1})K(^{-1})</td>
<td>( w_3 )</td>
<td>181 ( \mu )m</td>
</tr>
<tr>
<td>( \rho_m )</td>
<td>8.96 g cm(^{-3})</td>
<td>( 1/n(dn/dT) )</td>
<td>( 6.17 \times 10^{-5} ) K(^{-1})</td>
</tr>
<tr>
<td>( c_m )</td>
<td>0.385 J g(^{-1})K(^{-1})</td>
<td>( \nu_0 )</td>
<td>3.9 THz</td>
</tr>
<tr>
<td>( \eta_m )</td>
<td>398 W m(^{-1})K(^{-1})</td>
<td>( A )</td>
<td>2.51 mm \times 170 ( \mu )m</td>
</tr>
</tbody>
</table>

of \( O(t) \), here \( S_O \) is given at \( \Delta f = \nu_0 - \nu \) by taking the Fourier transform.

\[
S_O(\Delta f) \sim \text{Re} \left[ 2 \int_{-\infty}^{+\infty} \exp(i2\pi \Delta f \tau) \exp(-\langle \Delta \phi^2 \rangle / 2) d\tau \right].
\]  

(34)

D. Results and discussions

Throughout this section the parameters and their values used for frequency noise calculations in THz QCLs are listed in Table I. The active region is made of GaAs/Al\(_{0.15}\)Ga\(_{0.85}\)As semiconductor materials, and copper is used for metal waveguide. To simplify, we neglected the effects on heat conductivity changes caused by the active region thickness, the distribution and transport of hot-injected electrons, and their interactions with lattice. The refractive-index change here is only related to current-induced device self-heating. The refractive-index variations associated with the intersub-band gain transitions are not included in the previous derivations due to the lack of experimental and theoretical data. Estimations on the linewidth broadening caused by the refractive index variations owing to the intersub-band transitions in QCLs are given in Sec. III of this paper.

Figure 2 shows the temperature-dependent frequency noise in the range of kHz and high frequency regime (1 MHz to 10 MHz), respectively. Throughout this paper the temperature refers to the lattice temperature of QCLs, where it is noted that the lattice temperature is normally 50–100 K higher than the heat-sink temperature. The temperature ranging from 100 K to 250 K is of great interest, as most of the high-performance THz QCLs are working in this range. It is noted that the highest operation temperature (heat-sink...
FIG. 4. (Color online) (a) Frequency noise as a function of the thickness of the active region at 200 K. A critical frequency of 5 kHz is found. When the frequency is less than 5 kHz, the frequency noise decreases as the thickness of the active region increases, while if the frequency is above 5 kHz, the thicker the active region, the lower the laser frequency noise. (b) Frequency noise as a function of the thickness of the substrate at 200 K. A critical frequency of \( \sim 1 \) kHz is found. The insets of the figures are partial enlarged views, respectively.

FIG. 5. (Color online) Comparison of the frequency noise caused by thermal fluctuations when gold and copper are used as metal-cladding layers at 200 K, respectively. Both gold and copper give similar performance to the frequency noise at different temperatures. The linewidth caused by thermal fluctuation can nearly be neglected in THz QCLs. However, this linewidth broadening is comparable to the value of \( \sim 3 \) Hz caused by spontaneous emission, stimulated emission, and blackbody radiation, predicted by Jirauschek\(^{19}\) in the high power regime in the THz QCLs.

The thickness of the active region and the substrate can also influence the laser frequency noise. The frequency-noise responses to the thicknesses of various layers are different in different frequency ranges. It needs to be mentioned that the effects of the active-region thickness on the heat conductivity are neglected. As shown in Fig. 4, when the frequency is less than 5 kHz, the frequency noise increases as the thickness of the active region increases. While if the frequency exceeds 5 kHz, the thicker the active region, the lower the laser frequency noise. These different noise characteristics below and above the critical frequency 5 kHz can be ascribed to the noise correlation length,\(^{33}\) which is frequency dependent. As shown in Ref. 33, the correlation length of thermal fluctuation is inverse proportional to the frequency. At the low frequencies, the fluctuations become correlated across the entire active region. Therefore, a thicker active region can induce a larger fluctuation amplitude at the low-frequency region. For the substrate this critical frequency is about 1 kHz. The frequency

FIG. 6. Buried heterostructure of a mid-IR QCL mounted epilayer down on a diamond heat sink.
noise response below and above this critical frequency in the substrate is similar to those in the active region. However, when frequency is more than 4 kHz, the thickness of the substrate nearly cannot influence the temperature fluctuation. Hence, the frequency noise of THz QCL is mainly affected by the thickness of the active region in the high frequency range (>5 kHz).

Although copper has a better waveguide performance than gold in terms of thermal conduction, the noise properties for these two waveguides are nearly the same, as shown in Fig. 5. For gold the following parameters are used: \( \rho_m = 19.3 \) g cm\(^{-3}\), \( c_m = 0.128 \) J g\(^{-1}\) K\(^{-1}\), \( \eta_m = 317 \) W m\(^{-1}\) K\(^{-1}\) (Ref. 54).

### III. ANALYSIS OF MID-IR QCLS

#### A. Fundamental thermal noise

Various device geometries have been used to improve the heat dissipation from the active region of mid-IR QCLS. In this paper we use the ridge waveguide structure with a buried heterostructure mounted epilayer down on a diamond heat sink (see Fig. 6), which demonstrated a very good heat dissipation effect. In this structure most of the heat is transferred effectively via the diamond heat sink. Thus a constant temperature condition is applied to the heat sink considering the high heat conductivity of diamond (about 2600 W m\(^{-1}\) K\(^{-1}\) at 300 K). On the top surface of the device the heat is partially transferred to the air by convection.

For this buried heterostructure, the heat dissipation can also be partially removed by the lateral heat flow. Since the heat conduction of the active region exhibits a strong anisotropy in the in-plane and cross-plane directions, the in-plane thermal noise is much smaller compared with the cross-plane thermal noise if the thermal fluctuations are independently considered (without considering the correlation/coupling of their thermal fluctuations) owing to much larger in-plane heat conductivity. This conclusion is based on the fact that the thermal frequency noise caused by temperature fluctuations is inversely proportional to the heat conductivity, as discussed in the following Sec. B (a similar conclusion is also shown in Ref. 35). Hence, the thermal fluctuations in the \( x-, y-\)directions are neglected. By considering the heat flow in the \( z-\)direction, we have

\[
\frac{\partial \Delta T}{\partial t} - D(z) \frac{\partial^2 \Delta T}{\partial z^2} = \vec{V} \cdot F(z,t), \tag{35}
\]

where

\[
D(z) = D_1 = \eta_{\text{inp}}/\rho_{\text{inp}} c_{\text{inp}}
\]

\[
0 \leq z < w_1 \quad \text{and} \quad w_2 < z \leq w_3,
\]

\[
D(z) = D_2 = \eta_\perp /\rho c \quad w_1 \leq z \leq w_2,
\]

and the boundary conditions are

\[
\eta_\perp \frac{\partial \Delta T}{\partial z} \mid_{z=w_1} + h \Delta T \mid_{z=w_3} = 0, \quad \Delta T \mid_{z=0} = 0,
\]

\[
\Delta T \mid_{z=w_1} = \Delta T \mid_{z=w_1+0},
\]

\[
\frac{\partial \Delta T}{\partial z} \mid_{z=w_1+0} = \eta_{\text{inp}} \frac{\partial \Delta T}{\partial z} \mid_{z=w_1},
\]

\[
\Delta T \mid_{z=w_2-0} = \Delta T \mid_{z=w_2+0},
\]

\[
\eta_\perp \frac{\partial \Delta T}{\partial z} \mid_{z=w_2} = \eta_{\text{inp}} \frac{\partial \Delta T}{\partial z} \mid_{z=w_2+0},
\]

where \( \eta_{\text{inp}} \) is the heat conductivity of the InP cladding and substrate, and \( \eta_\perp \) is the cross-plan heat conductivity of active region.
The Green’s function of Eq. (35) is

\[ i \omega G(z, z', i\omega) - D(z) \frac{\partial^2}{\partial z'^2} G(z, z', i\omega) = \delta(z - z'), \] (36)

with the boundary conditions

\[
\eta_{\text{inp}} \left. \frac{\partial G}{\partial z} \right|_{z=w_1} + h G|_{z=w_1} = 0, \quad G|_{z=0} = 0,
\]
\[
G|_{z=w_1+0} = G|_{z=w_1-0}, \quad \eta_{\perp} \left. \frac{\partial G}{\partial z} \right|_{z=w_1+0} = \eta_{\parallel} \left. \frac{\partial G}{\partial z} \right|_{z=w_1-0},
\]
\[
G|_{z=w_2-0} = G|_{z=w_2+0}, \quad \eta_{\perp} \left. \frac{\partial G}{\partial z} \right|_{z=w_2-0} = \eta_{\parallel} \left. \frac{\partial G}{\partial z} \right|_{z=w_2+0}.
\]

Furthermore, the continuity of \( G \) and the jumping condition on \( \partial G/\partial z \) yield, respectively,

\[
G(z = z' - 0, z', i\omega) = G(z = z' + 0, z', i\omega), \] (37a)
\[
\left. \frac{\partial G(z, z', i\omega)}{\partial z} \right|_{z=z'+0} - \left. \frac{\partial G(z, z', i\omega)}{\partial z} \right|_{z=z'-0} = -\frac{1}{D(z)} \] (37b)

With the Green’s theorem and the \( \Lambda \)-theorem, the correlation function of \( \Delta T(\tilde{r}, t) \) in the frequency domain yields

\[
\langle \Delta T(z, \omega) \Delta T^*(z', \omega') \rangle = 4\pi \Delta w_{j} k_{B} T_{0}^{2} \text{Re} \[ G(z, z', i\omega) \delta(\omega - \omega') \]/C_{j}. \] (38)

The Green’s functions \( G_1(0 \leq z \leq w_1), G_2^>(w_1 \leq z < z'), G_2^<(z' < z \leq w_2), G_3(w_2 \leq z \leq w_3) \) for the source point \( z' \) lay in active region are obtained

\[
G_1 = c_1(z') \sinh(\alpha_1 z) + c_2(z') \cosh(\alpha_1 z),
\]
\[
G_2^> = c_3(z') \sinh[\alpha_3(z - w_2)] + c_4(z') \cosh[\alpha_3(z - w_2)] \]
\[
+ \sinh[\alpha_3(z - z')] / D_{2} \alpha_2,
\]
\[
G_2^< = c_3(z') \sinh[\alpha_2(z - w_2)] + c_4(z') \cosh[\alpha_2(z - w_2)],
\]
\[
G_3 = c_5(z') \sinh[\alpha_1(z - w_1)] + c_6(z') \cosh[\alpha_1(z - w_1)],
\] (39)
where

\[ \alpha_1 = \sqrt{i\omega/D_1}, \quad \alpha_2 = \sqrt{i\omega/D_2}. \]

Coefficients \( c_1(z'), \cdots, c_5(z') \) can be determined from the boundary conditions. The expressions of coefficients \( c_3(z') \) and \( c_4(z') \) of interest to us read

\[
c_3(z') = \frac{E_2\eta_{\text{imp}}\alpha_1}{E_3E_2\eta_{\text{imp}}\alpha_1 + E_4E_2\eta_{\text{imp}}\alpha_1} [E_1\alpha_1\alpha_2 \sinh(\alpha_1 w_1) \cosh(\alpha_2 w_1 - z') - \alpha_1\eta_{\text{imp}} \cosh(\alpha_1 w_1) \sinh(\alpha_2 (w_1 - z'))] \]

\[
c_4(z') = \frac{E_1\eta_{\text{imp}}\alpha_1}{E_3E_2\eta_{\text{imp}}\alpha_1 + E_4E_2\eta_{\text{imp}}\alpha_1} [E_1\alpha_1\alpha_2 \sinh(\alpha_1 w_1) \cosh(\alpha_2 w_1 - z') - \alpha_1\eta_{\text{imp}} \cosh(\alpha_1 w_1) \sinh(\alpha_2 (w_1 - z'))] \]

where

\[
E_1 = -h \sinh[\alpha_1 (w_2 - w_1)] + \eta_{\text{imp}} \alpha_1 \cosh[\alpha_1 (w_2 - w_1)],
E_2 = -h \cosh[\alpha_1 (w_2 - w_1)] + \eta_{\text{imp}} \alpha_1 \sinh[\alpha_1 (w_2 - w_1)],
E_3 = \alpha_1 \eta_{\text{imp}} \cosh(\alpha_1 w_1) \sinh(\alpha_2 (w_1 - w_2)) - \alpha_2 \eta_{\text{imp}} \sinh(\alpha_1 w_1) \cosh(\alpha_2 w_1 - w_2),
E_4 = \alpha_1 \eta_{\text{imp}} \cosh(\alpha_1 w_1) \cosh(\alpha_2 (w_1 - w_2)) - \alpha_2 \eta_{\text{imp}} \sinh(\alpha_1 w_1) \sinh(\alpha_2 (w_1 - w_2)).
\]

The normalized fundamental-mode intensity is assumed to be Gaussian:

\[
|E(z)|^2 = \exp[-(z - (w_1 + w_2)/2)^2/m_0^2]/\sqrt{\pi} m_0,
\]

where \( m_0 \) is the mode-field radius.

Therefore, for the spectrum of the averaged temperature fluctuations, we find

\[
S_{\Delta T} = 4k_B T_0^2/(\rho c A) \text{Re} \int_{w_1}^{w_2} dz' \int_{w_1}^{w_2} dz G_2(z, z', i\omega)|E(z)|^2 |E(z')|^2 dz' dz + \int_{w_1}^{w_2} dz' \int_{w_1}^{w_2} dz G_2(z, z', i\omega)|E(z)|^2 |E(z')|^2 dz' dz
\]

\[
= 4k_B T_0^2/(\rho c A) \text{Re} \int_{w_1}^{w_2} \sinh[\alpha_2 (z - w_2)] |E(z)|^2 dz \int_{w_1}^{w_2} c_3(z') |E(z')|^2 dz' + \int_{w_1}^{w_2} \cosh[\alpha_2 (z - w_2)] |E(z)|^2 dz
\times \int_{w_1}^{w_2} c_4(z') |E(z')|^2 dz' + \int_{w_1}^{w_2} dz' \int_{w_1}^{w_2} dz \sinh[\alpha_2 (z - z')] |D_2 \alpha_2 | E(z')^2 |E(z')|^2.
\]

For mid-IR QCLs, the expressions of frequency noise and linewidth broadening caused by the temperature fluctuation are the same as Eq. (31) and Eq. (34), respectively. The main difference of the thermal noise between mid-IR and THz QCLs is in \( S_{\Delta T} \).

B. Results and discussions

For mid-IR QCLs, material system of In\(_{0.20}\)Al\(_{0.71}\)As/In\(_{0.73}\)Ga\(_{0.27}\)As/InP is considered. The parameters used for frequency noise calculations are listed in Table II. As indicated in Ref. 56, we also assume that the cross-plane heat conductivity of the active region is independent of temperature. Similar to the analysis in THz QCLs, the effects of active region thickness, the distribution and transport of hot-injected electrons, and their interactions with lattice on the heat conductivity are neglected. The refractive-index change discussed here is related to current-induced device self-heating; the refractive-index variations associated with the intersub-band gain transition are neglected due to the lack of experimental and theoretical data in the literature.

As shown in Fig. 7, the same characteristics of temperature-dependent laser frequency noise for mid-IR QCLs are observed, as for THz QCLs. The frequency noises can be almost neglected when the frequency exceeds 400 kHz. It is noted that the frequency noise does not show a 1/f trend in the whole frequency region. This characteristic is not reflected in the experimental measurements using an ultralow-technical-noise laser-current source, where a 1/f frequency noise exits in mid-IR QCLs at the low frequency region from 10 Hz to 100 kHz. The origin of 1/f noise in low frequency range is not clear in QCLs at the moment. In Ref. 17, due to the use of an ultralow-technical-noise laser-current source, the parasitic noise from the characterization equipment is minimized. In addition the spontaneous emission noise and carrier shot noise have a white noise spectrum. Thus, the origin of 1/f noise characteristic could probably come from the thermal noise owing to the non-Langevin heat source associated with the emissive and absorptive processes of photons and carriers, e.g., random spontaneous emission events as shown in fiber laser. It could also come from the noise due to the optical feedback in cold cavities or from the non-Markovian...
processes. Further investigations are required to understand this phenomenon.

Figure 8(a) shows the temperature-dependent mean square-phase change ($\Delta \phi^2$). Through curve fitting we find the variance ($\langle \Delta \phi^2 \rangle \propto a + b T^{1.11}$, where $a$ and $b$ are constants). Hence, the resultant line shape of mid-IR QCLs is neither Lorentzian nor Gaussian shape [see Fig. 8(b)], which is in contrast to that of diode lasers. The main reason is due to the geometric Gaussian shape, which is in contrast to that of mid-IR QCLs. Figure 8(c) shows the linewidth broadening at different temperatures considering only the temperature dependence of the refractive-index fluctuations caused by current-induced device self-heating. The linewidth increases from 14.74 Hz to 62.02 Hz as the temperature increases from 200 K to 400 K.

In the previous analysis we only consider the macroscopic physics of the linewidth broadening. The microscopic physics of the gain medium of QCLs is ignored. We note that the refractive index can be affected by not only the current-induced device self-heating but also the laser transitions near the lasing wavelength, which can have a strong temperature dependence associated with electron populations and scattering rates, as discussed in Ref. 14. The thermal expansion and energy-level broadening caused by current-induced device self-heating can also cause a significant linewidth broadening. If all of these factors are considered, the frequency noise spectrum caused by the temperature fluctuation becomes

$$S_f = \left[ \frac{1}{v_0} \left( \frac{1}{n} \frac{dn}{dT} + \frac{1}{n_e} \frac{d n_e}{dT} + \frac{1}{l} \frac{dl}{dT} + \frac{2 \pi}{\hbar \nu} \frac{d \Delta \epsilon}{dT} \right) \right]^2 S_{\Delta T}(2 \pi \nu).$$

(43)

The second term (on the right side of the previous equation) stands for the refractive-index change due to the laser transition near the lasing wavelength; the third term is due to the thermal expansion at various temperatures; and the fourth is owing to the temperature-dependent energy-level fluctuation of the laser emission. From the measurement results in Ref. 14, we can deduce that the third term has the same order of amplitude with the first term, and the second term is negligible. The fourth term is also believed to have an essential effect on the linewidth broadening of QCLs as is the case in quantum dots lasers, while the value of this parameter was not reported in QCLs in the literature, to the best of our knowledge. Therefore, even if we assume both the second and the fourth items are negligible, we can conclude that at least the first and the third items are comparable. In this case the linewidth broadening is at least four times larger than previous calculations, which achieves from 64.3 Hz to 266.4 Hz as the temperature of 400 K. However, it needs to be mentioned that the exact values of these three parameters are not available in the literature. Exact calculation of the linewidth broadening needs further theoretical and experimental investigations.

The fundamental thermal frequency noise is strongly dependent on the heat conductivity of active region, as shown in Fig. 9(a). A higher heat conductivity of the active region induces a smaller thermal fluctuation, hence reducing the linewidth broadening [see Fig. 9(b)].

Figure 10 shows the influence of the thicknesses of the active region and the metal cladding on the laser frequency noise caused by temperature fluctuations. Unlike the case in THz QCLs, the active region can greatly influence the temperature fluctuation in mid-IR QCLs. The frequency noise first increases with the increase of the thickness of the active region, then decreases. The effects of the thicknesses of the InP cladding layers are more complicated. Decreasing the thickness of the upper-cladding layer can reduce the frequency noise, while decreasing the thickness of the lower-cladding layer can increase the temperature fluctuations. Therefore the overall effect is that reducing simultaneously the thicknesses of both cladding layers can reduce the frequency noise at the beginning but increase it at around 5 kHz, as shown in Fig. 10(d). With the optimized-cladding layers the temperature fluctuation can be effectively minimized.

The effects of the lower- and upper-cladding layers on the linewidth broadening can be directly predicted from Fig. 10(b) and (c). The frequency noise, as well as the linewidth broadening, decreases with the decrease of the thickness of the upper-cladding layer and the increase of the thickness of the lower-cladding layer. In addition influence of the thickness variation of the active region on the linewidth broadening is complicated...
Lower-layer InP cladding (c)

FIG. 10. (Color online) Frequency noise as a function of the thickness of mid-IR QCL structure at 300 K. (a) Active region. (b) Upper-layer InP cladding. (c) Lower-layer InP cladding. (d) The thickness of the two-layer InP cladding changes simultaneously.

The calculations indicate that the frequency noise strongly increases with the decrease of the mode-field radius, hence the increasing linewidth broadening. Therefore, by optimizing the...
geometry of the waveguide structure of mid-IR QCLs, one can change the frequency noise effectively.

**IV. CONCLUSION**

A theoretical model based on Green function analysis and Van Vliet-Fassett theory has been presented for the analysis of the fundamental frequency noise and linewidth broadening of QCLs caused by intrinsic thermal fluctuations. It is found that the intrinsic frequency noise of QCLs due to thermal fluctuation is a low frequency noise, which shows strong temperature dependence in both THz and mid-IR QCLs. For fluctuation is a low frequency noise, which shows strong that the intrinsic frequency noise of QCLs due to thermal fluctuations in QCLs caused by intrinsic thermal fluctuations. It is found of the fundamental frequency noise and linewidth broadening of QCLs the linewidth increases from 14.74 Hz to 62.02 Hz as the temperature changes from 200 K to 400 K. The resulting lineshape is neither Lorentzian nor Gaussian shape. We also find that if the refractive-index change associated with the intersub-band gain transition, the thermal expansion and the energy-level broadening caused by self-heating are considered, the linewidth broadening should increase at least from 64.3 Hz to 266.4 Hz at 400 K. These microscopic effects should be further determined theoretically and experimentally. The effects of heat and structure parameters, such as heat conductivity of the active region, the thickness of the active region/substrate, and the optical-mode intensity distribution on the frequency noise and the linewidth broadening of mid-IR QCLs have also been investigated. The results show that the fundamental frequency noise and the intrinsic linewidth broadening decrease when the heat conductivity, the thickness of the lower-cladding layer and the mode-field radius increase. They also increase with the decrease of the thicknesses of the active region and the upper-cladding layer. Through the optimization of these heat and structure parameters, the effects on the temperature fluctuations can be effectively reduced.

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