Weitzenböck formulas on Poisson probability spaces
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Abstract
This paper surveys and compares some recent approaches to stochastic infinite-dimensional geometry on the space $\Gamma$ of configurations (i.e. locally finite subsets) of a Riemannian manifold $M$ under Poisson measures. In particular, different approaches to Bochner-Weitzenböck formulas are considered. A unitary transform is also introduced by mapping functions of $n$ configuration points to their multiple stochastic integral.

Key words: Poisson probability measure, Bochner-Weitzenböck formulas, stochastic analysis.
Classification: 60H07, 60J65, 58J65, 58A10

1 Weitzenböck formula under a measure
Let $M$ be a Riemannian manifold with volume measure $dx$, covariant derivative $\nabla$, and exterior derivative $d$. Let $\nabla^*_\mu$ and $d^*_\mu$ denote the adjoints of $\nabla$ and $d$ under a measure $\mu$ on $M$ of the form $\mu(dx) = e^{\phi(x)}dx$. The classical Weitzenböck formula under the measure $\mu$ states that

$$d^*_\mu d + dd^*_\mu = \nabla^*_\mu \nabla + R - \text{Hess } \phi,$$

where $R$ denotes the Ricci tensor on $M$. In terms of the de Rham Laplacian $H_R = d^*_\mu d + dd^*_\mu$ and of the Bochner Laplacian $H_B = d^*_\mu d + dd^*_\mu$ we have

$$H_R = H_B + R - \text{Hess } \phi.$$

In particular the term $\text{Hess } \phi$ plays the role of a curvature under the measure $\mu$.

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2 Probability: Poisson space

In this section we recall some facts on random functionals on Poisson space. The Poisson probability measure on \( \mathbb{N} \) can be introduced by considering \( N \) independent \( \{0, 1\} \)-valued Bernoulli random variables \( X_1, \ldots, X_N \), with parameter \( \lambda/N, \lambda > 0 \). Then \( X_1 + \cdots + X_N \) has a binomial law, and

\[
P(X_1 + \cdots + X_N = k) = \binom{N}{k} \left( \frac{\lambda}{N} \right)^k \left( 1 - \frac{\lambda}{N} \right)^{N-k}
\]

converges to \( \frac{\lambda^k}{k!} e^{-\lambda} \) as \( N \) goes to infinity. This defines a probability measure \( \pi_\lambda \) on \( \mathbb{N} \) as

\[
\pi_\lambda(\{k\}) = \frac{\lambda^k}{k!} e^{-\lambda}.
\]

Let \( X \) be a metric space with a \( \sigma \)-finite Borel measure \( \sigma \). The measure \( \pi_\lambda \) has the convolution property \( \pi_\lambda \ast \pi_\mu = \pi_{\lambda+\mu} \), which allows to construct the Poisson measure \( \pi_\sigma \) with intensity \( \sigma \) on

\[
\Gamma = \left\{ \gamma = \sum_{k=1}^{k=n} \delta_{x_k}; \ x_1, \ldots, x_n \in X, \ n \in \mathbb{N} \cup \{\infty\} \right\}
\]

by letting

\[
\pi_\sigma(\{\gamma \in \Gamma; \ \gamma(A_1) = k_1, \ldots, \gamma(A_n) = k_n\}) = \frac{\sigma(A_1)^{k_1}}{k_1!} e^{-\sigma(A_1)} \cdots \frac{\sigma(A_n)^{k_n}}{k_n!} e^{-\sigma(A_n)},
\]

where \( A_1, \ldots, A_n \) are disjoint compact subsets of \( X \). This measure is characterized by its Fourier transform

\[
\int_{\Gamma} e^{i \int_X f(x) d\gamma(x)} d\pi(\gamma) = \exp \left( \int_X (e^{i f(x)} - 1) d\sigma(x) \right).
\]

If \( \gamma \in \Gamma \) is finite with cardinal \( |\gamma| = n \) we write

\[
\gamma = \sum_{i=1}^{i=n} \delta_{x_i}.
\]

Given \( \Lambda \) a compact subset we consider \( F : \Gamma \to \mathbb{R} \) such that \( F(\gamma) = F(\gamma \cap \Lambda) \), and written as

\[
F(\gamma) = F(\gamma \cap \Lambda) = e^{\sigma(\Lambda)/2} \sum_{|\gamma\cap\Lambda|=n} 1 |\gamma\cap\Lambda|=n n! f_n(x_1, \ldots, x_n) = \sum_{n=0}^{\infty} J_n(f_n)
\]

2
where \( f_n \) is a symmetric function with support in \( \Lambda^n \), with

\[
J_n(f_n)(\gamma) = J_n(f_n)(\gamma \cap \Lambda) = n!1_{\{\gamma \cap \Lambda = n\}}e^{\sigma(\Lambda)/2}f_n(x_1, \ldots, x_n), \quad n \geq 1.
\]

The multiple Poisson stochastic integral of \( f_n \) is defined as

\[
I_n(f_n) = \int_{\{x_1, \ldots, x_n\} \in \mathcal{X}^n: \ x_i \neq x_j, \ i \neq j}. f_n(x_1, \ldots, x_n)(\gamma - \sigma)(dx_1) \cdots (\gamma - \sigma)(dx_n),
\]
and extends to \( f_n \in L^2_{\sigma}(X)^\otimes n \) via the well-known isometry

\[
\int_{\Gamma} I_n(f_n)I_m(g_m)d\pi = n!1_{\{n=m\}}\langle f_n, g_m \rangle_{L^2_{\sigma}(X)^\otimes n}, \quad f_n \in L^2_{\sigma}(X)^\otimes n, \ g_m \in L^2_{\sigma}(X)^\otimes m.
\]

We introduce a combinatorial transform \( \tilde{K} \) which has some similarities with the \( K \)-transform, cf. [6] and references therein. The transform \( \tilde{K} \) identifies the functional \( J_n(f_n) \), which makes sense only in finite volume, to \( I_n(f_n) \) which is defined for all square-integrable \( f_n \).

**Proposition 2.1** The operator \( \tilde{K} \) defined by

\[
\tilde{K}J_n(f_n) = I_n(f_n), \quad f_n \text{ symmetric in } C_c(\Lambda^n), \ n \in \mathbb{N},
\]

is unitary on \( L^2_{\pi^*}(\Gamma) \). Moreover, \( \tilde{K} \) satisfies

\[
\tilde{K}F(\gamma) = \sum_{\eta \subset \gamma} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{X^k} F(\eta \cup \{y_1, \ldots, y_k\})\sigma(dy_1) \cdots \sigma(dy_k).
\]

**Proof.** We have

\[
\int_{\Gamma} J_n(f_n)J_m(g_m)d\pi = n!1_{\{n=m\}}e^{\sigma(\Lambda)}\int_{\Gamma} 1_{\{\gamma \cap \Lambda = n\}}f_n(x_1, \ldots, x_n)g_n(x_1, \ldots, x_n)d\pi(\gamma)
= \frac{n!2^n}{\sigma(\Lambda)^n}e^{\sigma(\Lambda)}\pi_\sigma(\gamma \cap \Lambda = n)\langle f_n, g_n \rangle_{L^2_{\sigma}(X)^\otimes n}
= n!1_{\{n=m\}}\langle f_n, g_n \rangle_{L^2_{\sigma}(X)^\otimes n},
\]

which shows the first statement. On the other hand we have

\[
\tilde{K}J_n(f_n)(\gamma)
= \sum_{\eta \subset \gamma \cap \Lambda} \sum_{k=0}^{\infty} \frac{(-1)^k n!}{k!} \int_{X} 1_{\{\eta \cup \{y_1, \ldots, y_k\} = n\}}f_n(\eta \cup \{y_1, \ldots, y_k\})\sigma(dy_1) \cdots \sigma(dy_k)
\]

3
\[
= \sum_{k=0}^{k=n} (-1)^k \frac{n!}{k!} \sum_{\eta \cap \Lambda \ni \{y_1, \ldots , y_k\}} \int_X f_n(\eta \cup \{y_1, \ldots , y_k\}) \sigma(dy_1) \cdots \sigma(dy_k)
\]

\[
= \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \sum_{x_1, \ldots , x_{n-k}, y_1, \ldots , y_k \in \cap \Lambda} \int_X f_n(\{x_1, \ldots , x_{n-k}, y_1, \ldots , y_k\}) \sigma(dy_1) \cdots \sigma(dy_k)
\]

\[
= I_n(f_n)(\gamma),
\]

the last relation follows from e.g. Prop. 4.1 of [9].

If \( \Lambda \) is compact and \( F(\gamma) = F(\gamma \cap \Lambda) \) we have

\[
\int_{\Gamma} F(\gamma)d\pi(\gamma) = e^{\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \ldots , x_n) \sigma(dx_1) \cdots \sigma(dx_n).
\]

In the particular case \( X = \mathbb{R}_+ \) with \( \sigma \) the Lebesgue measure, the standard Poisson process is defined as

\[
N_t(\gamma) = \gamma([0, t]) = \sum_{k=1}^{\infty} \mathbb{1}_{[r_k, \infty]}(t), \quad t > 0,
\]

i.e. every configuration \( \gamma \in \Gamma \) can be viewed as the ordered sequence \( \gamma = (T_k)_{k \geq 1} \) of jump times of \( (N_t)_{t \in \mathbb{R}_+} \) on \( \mathbb{R}_+ \). Let \( f_n \in C_c([0, \lambda]^n) \) be symmetric. Then

\[
\int_{\Gamma} f_n(T_1, \ldots , T_n)d\pi(\gamma) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^\lambda \cdots \int_0^\lambda f_n(t_1, \ldots , t_n) \sigma(dt_1) \cdots \sigma(dt_k).
\]

\[
= \sum_{k=n}^{\infty} e^{-\lambda} \int_0^\lambda \cdots \int_0^{t_{k-1}} f_n(t_1, \ldots , t_n) \sigma(dt_1) \cdots \sigma(dt_k).
\]

\[
= \sum_{k=n}^{\infty} e^{-\lambda} \int_0^\lambda \frac{(\lambda - t_n)(k-n)!}{(k-n)!} \int_0^{t_n} \cdots \int_0^{t_{k-1}} f_n(t_1, \ldots , t_n) \sigma(dt_1) \cdots \sigma(dt_n).
\]

\[
= \int_0^\infty e^{-\lambda} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots , t_n)dt_1 \cdots dt_n.
\]

This formula extends to \( f \) bounded and measurable.

### 3 Geometry

We recall the construction of [1], [2] in the case of 1-forms, see also [3] for the case of \( n \)-forms. We assume that \( X \) is a Riemannian manifold. The tangent space at \( \gamma \in \Gamma \)
is taken to be

$$L^2(X; TX, \gamma) = \bigoplus_{x \in \gamma} T_x X.$$ 

A differential form of order $n$ maps $\gamma \in \Gamma$ into the antisymmetric tensor product

$$\wedge^n(T\gamma \Gamma) = \bigwedge^n \left( \bigoplus_{x \in \gamma} T_x X \right).$$

Bochner and de Rham Laplacians on differential forms over configuration spaces are then constructed from their counterparts at the level of the manifold $X$. Let $d^X_x$ be the exterior differential on $X$, let $\nabla^X_x$, $\Delta^X_x$ be the natural covariant derivative and Bochner Laplacian on the bundle $T_{\gamma \setminus \{x\}} \Gamma \to y \in O_{\gamma,x}$, where $O_{\gamma,x}$ is an open set in $X$ such that $\partial_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. The covariant derivative of the smooth differential 1-form $W$ is defined as

$$(\nabla^X_x W_x(\gamma, x))_{x \in \gamma} \in T_{\gamma \Gamma} \otimes T_{\gamma \Gamma},$$  

where $W_{x}(\gamma, y) = W((\gamma \setminus \{x\}) \cup \{y\})$, $x, y \in X$. The Bochner Laplacian $H^B$ on $\Gamma$ is defined as

$$H^B W(\gamma) = - \sum_{x \in \gamma} \Delta^X_x W_x(\gamma, x).$$

The exterior derivative $d^\Gamma$ is defined as

$$d^\Gamma W = \sum_{x \in \gamma} \sum_{y \in \gamma} d^X_x W_x(\gamma, x)_y,$$

where $W_x(\gamma, x)_y$ is the component of $W_x(\gamma, x)$ of index $y \in \gamma$, with adjoint

$$d^{*\Gamma} W = \sum_{x \in \gamma} \sum_{y \in \gamma} d^{X*}_x W_x(\gamma, x)_{x,y},$$

where $W_x(\gamma, x)_{x,y}$ is the component of $W_x(\gamma, x)$ of index $(x, y)$ and $d^{X*}_x$ is the adjoint of $d^X_x$ under the volume element $\sigma$ on $X$. A Weitzenböck formula is stated in [1], [3] as

$$H^R = H^B + R,$$

where $H^R$ is the de Rham Laplacian $H^R = d^\Gamma d^{*\Gamma} + d^{*\Gamma} d^\Gamma$ and the curvature term

$$R(\gamma) = \sum_{x \in \gamma} R(\gamma, x)$$

5
has the explicit expression

\[ R(\gamma, x)(V(\gamma)_y) = 1_{\{x=y\}} \sum_{i,j=1}^{d} \text{Ric}_{ij}(x)e_i(V(\gamma)_x, e_j)_x, \]

where \((e_j)_{j=1}^d\) is an orthonormal basis of \(T_xX\). Formula (3.1) can be viewed as the lifting to \(\Gamma\) of the Weitzenböck formula on \(X\).

Note that in the above construction the curvature term in (3.1) is essentially due to the curvature of \(X\), in particular it vanishes if \(X = \mathbb{R}^d\) and no curvature term is induced from the Poisson measure itself.

In this paper we present a different geometry on the infinite-dimensional space \(\Gamma\), in which the Ricci curvature tensor under the Poisson measure appears to be the identity operator when \(X = \mathbb{R}_+\), see [8] when \(X\) is a more general Riemannian manifold.

**Lifting of differential structure**

Let \(\mathcal{S}\) denote the space of cylindrical functionals of the form

\[ F(\gamma) = f(T_1, \ldots, T_n), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n). \] (3.2)

Let \(\mathcal{U}\) denote the space of smooth processes of the form

\[ u(\gamma, x) = \sum_{i=1}^{i=n} F_i(\gamma)h_i(x), \quad (\gamma, x) \in \Gamma \times \mathbb{R}_+, \quad h_i \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad F_i \in \mathcal{S}, \quad i = 1, \ldots, n. \] (3.3)

The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.
<table>
<thead>
<tr>
<th>Geometry</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>element of $\Gamma$</td>
</tr>
<tr>
<td>$C_c^\infty(\mathbb{R}_+)$</td>
<td>tangent vectors to $\Gamma$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Riemannian metric on $\Gamma$</td>
</tr>
<tr>
<td>$d$</td>
<td>gradient on $\Gamma$</td>
</tr>
<tr>
<td>$U$</td>
<td>vector fields on $\Gamma$</td>
</tr>
<tr>
<td>$du$</td>
<td>exterior derivative of $u \in U$</td>
</tr>
<tr>
<td>${,\cdot}$</td>
<td>bracket of vector fields</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>curvature tensor on $\Gamma$</td>
</tr>
<tr>
<td>$d^*$</td>
<td>divergence on $\Gamma$</td>
</tr>
</tbody>
</table>

### Divergence operator

The definition of the following gradient operator goes back to [4].

**Definition 3.1** Given $F \in \mathcal{S}$, $F = f(T_1, \ldots, T_d)$, let

$$d_tF(\gamma) = -\sum_{k=1}^{d} 1_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_d), \quad t \geq 0.$$ 

The following is a finite-dimensional integration by parts formula for $d$.

**Lemma 3.1** We have for $F = f(T_1, \ldots, T_d)$ and $h \in C_c(\mathbb{R}_+)$:

$$\int_\Gamma \langle dF, h \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) = \int_\Gamma F(\gamma) \left( \sum_{k=1}^{d} h(T_k) - \int_0^{T_d} h(t) dt \right) d\pi(\gamma).$$

**Proof.** All $C^\infty$ functions on $\Delta_d = \{ (t_1, \ldots, t_d) ; \ 0 \leq t_1 < \cdots < t_d \}$ are extended by continuity to the closure of $\Delta_d$. We have

\[
\int_\Gamma \langle dF(\gamma), h \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) = -\sum_{k=1}^{d} \int_0^{\infty} e^{-t_k} \int_0^{t_k} \cdots \int_0^{t_2} \int_0^{t_1} h(s) ds \partial_k f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
= \int_0^{\infty} e^{-t_k} \int_0^{t_k} \cdots \int_0^{t_2} \int_0^{t_1} h(t_1) f(t_1, \ldots, t_d) dt_1 \cdots dt_d \\
- \int_0^{\infty} e^{-t_k} \int_0^{t_k} \cdots \int_0^{t_2} \int_0^{t_3} \int_0^{t_1} h(s) ds f(t_2, t_1, \ldots, t_d) dt_2 \cdots dt_d
\]
\[
\begin{align*}
+ \sum_{k=d}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_k)f(t_1, \ldots, t_d)dt_1 \cdots dt_d \\
- \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)dt_1 \cdots dt_d \\
- \sum_{k=2}^{k=d-1} \int_0^{t_k} e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k-1}} \int_0^{t_1} h(s)ds f(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d)dt_1 \cdots dt_{k+1}dt_{k-1} \cdots dt_d \\
+ \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k-2}} \int_0^{t_1} h(s)ds f(t_1, \ldots, t_{k-2}, t_k, \ldots, t_d)dt_1 \cdots dt_d \\
= \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_k)f(t_1, \ldots, t_d)dt_1 \cdots dt_d \\
- \int_0^\infty e^{-t_d} \int_0^{t_d} h(s)ds \int_0^{t_2} f(t_1, \ldots, t_d)dt_1 \cdots dt_d \\
= \int_\Gamma F(\gamma) \left( \sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)dt \right) d\pi(\gamma).
\end{align*}
\]

The following definition of the divergence coincides with the compensated Poisson stochastic integral with respect to \((N_t - t)_{t \in \mathbb{R}_+}\) on the adapted square-integrable processes.

**Definition 3.2** We define \(d^*_\pi\) on \(\mathcal{U}\) by

\[
d^*_\pi(hG) = \int_0^\infty h(t)(\gamma(dt) - dt) - \langle h, dG \rangle_{L^2(\mathbb{R}_+)} , \quad G \in \mathcal{S}, \; h \in L^2(\mathbb{R}_+).\]

Using this definition, an integration by parts formula can be obtained independently of the dimension.

**Proposition 3.1** The divergence operator \(d^*_\pi : L^2(\Gamma \times \mathbb{R}_+) \rightarrow L^2(\Gamma)\) is the closable adjoint of \(d\), i.e.

\[
\int_\Gamma Fd^*_\pi u \ d\pi(\gamma) = \int_\Gamma \langle dF, u \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma), \quad F \in \mathcal{S}, \ u \in \mathcal{U}. \quad (3.4)
\]

8
Proof. Given Lemma 3.1 it suffices to notice that if \( k > d \),
\[
\int_{\Gamma} F(\gamma) h(T_k) d\pi(\gamma) = \int_{0}^{\infty} e^{-t_k} h(t_k) \int_{0}^{t_k} \cdots \int_{0}^{t_k} f(t_1, \ldots, t_d) dt_1 \cdots dt_k
\]
\[
= \int_{0}^{\infty} e^{-t_k} \int_{0}^{t_k} h(s) ds \int_{0}^{t_k} \cdots \int_{0}^{t_k} f(t_1, \ldots, t_d) dt_1 \cdots dt_k
\]
\[
- \int_{0}^{\infty} e^{-t_{k-1}} \int_{0}^{t_{k-1}} h(s) ds \int_{0}^{t_{k-1}} \cdots \int_{0}^{t_{k-1}} f(t_1, \ldots, t_d) dt_1 \cdots dt_{k-1}
\]
\[
= \int_{\Gamma} F(\gamma) \int_{x_{k-1}}^{T_k} h(t) dt d\pi(\gamma),
\]
in other terms the discrete-time process
\[
\left( \sum_{k=1}^{k=n} h(T_k) - \int_{0}^{T_k} h(t) dt \right)_{k \geq 1} = \left( \int_{0}^{T_k} h(t) d(N_t - t) \right)_{k \geq 1}
\]
is a martingale. Hence Relation (3.4) also implies that for \( F, G \in \mathcal{S} \),
\[
\int_{\Gamma} \langle dF, hG \rangle_{L^2(\mathbb{R}^+)} d\pi(\gamma) = \int_{\Gamma} \langle d(FG), h \rangle_{L^2(\mathbb{R}^+)} - F \langle dG, h \rangle_{L^2(\mathbb{R}^+)} d\pi(\gamma) \quad (3.5)
\]
\[
= \int_{\Gamma} F \left( G \int_{0}^{\infty} h(t)(\gamma dt) - \langle h, dG \rangle_{L^2(\mathbb{R}^+)} \right) d\pi(\gamma)
\]
\[
= \int_{\Gamma} F d^*_n(hG) d\pi(\gamma). \quad (3.6)
\]

\[\blacksquare\]

Covariant derivative

Given \( u \in \mathcal{U} \) we define the covariant derivative \( \nabla_u v \) in the direction \( u \in L^2(\mathbb{R}^+) \) of the vector field \( v = \sum_{i=1}^{i=n} F_i h_i \in \mathcal{U} \) as
\[
\nabla_u v(t) = \sum_{i=1}^{i=n} h_i(t)d_u F_i - F_i h_i(t) \int_{0}^{t} u(s) ds, \quad t \in \mathbb{R}^+, \quad (3.7)
\]
where
\[
d_u F = \langle dF, u \rangle_{L^2(\mathbb{R}^+)}; \quad F \in \mathcal{S}.
\]

We have
\[
\nabla_u F(vG) = Fvd_u G + FG \nabla_u v, \quad u, v \in C_c^\infty(\mathbb{R}_+), \quad F, G \in \mathcal{S}. \quad (3.8)
\]

9
We also let
\[ \nabla_s v(t) = \sum_{i=1}^{i=n} h_i(t) d_x F_i - F_i \dot{h}_i(t) 1_{[0,t]}(s), \quad s, t \in \mathbb{R}_+, \]
in order to write
\[ \nabla_u v(t) = \int_0^\infty u(s) \nabla_s v(t) ds, \quad t \in \mathbb{R}_+, \quad u, v \in \mathcal{U}. \]

**Lie-Poisson bracket**

**Definition 3.3** The Lie bracket \( \{ u, v \} \) of \( u, v \in C^\infty_c(\mathbb{R}_+) \), is defined to be the unique element of \( C^\infty_c(\mathbb{R}_+) \) satisfying \( (d_u d_v - d_v d_u) F = d_w F, \ F \in \mathcal{S} \).

The bracket \( \{ u, v \} \) is defined for \( u, v \in \mathcal{U} \) with
\[ \{ F u, G v \}(x) = FG \{ u, v \}(x) + v(x) F d_u G - u(x) G d_v F, \quad x \in \mathbb{R}_+, \]
\( u, v \in C^\infty_c(\mathbb{R}_+) \), \( F, G \in \mathcal{S} \).

**Vanishing of torsion**

**Proposition 3.2** The Lie bracket \( \{ u, v \} \) of \( u, v \in \mathcal{U} \) satisfies
\[ \{ u, v \} = \nabla_u v - \nabla_v u, \]
i.e. the connection defined by \( \nabla \) has a vanishing torsion.

**Proof.** We have \( F(\gamma) = T_n \). If \( u, v \in C^\infty_c(\mathbb{R}_+) \) we have
\[
(d_u d_v - d_v d_u) T_n = -d_u \int_0^{T_n} v(s) ds + d_v \int_0^{T_n} u(s) ds \\
= v(T_n) \int_0^{T_n} u(s) ds - u(T_n) \int_0^{T_n} v(s) ds \\
= \int_0^{T_n} \left( \dot{v}(t) \int_0^t u(s) ds - \dot{u}(t) \int_0^t v(s) ds \right) dt \\
= d_{\nabla_u v - \nabla_v u} T_n.
\]

Since \( d \) is a derivation, this shows that
\[ d_u d_v - d_v d_u = d_{\nabla_u v - \nabla_v u}, \quad u, v \in \mathcal{U}. \]

The extension to \( u, v \in \mathcal{U} \) follows from (3.8). \( \Box \)
Vanishing of curvature

**Proposition 3.3** The curvature tensor $\Omega$ of $\nabla$ vanishes on $\mathcal{U}$, i.e.

$$\Omega(u,v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u,v\}}h = 0, \quad u,v,h \in \mathcal{U},$$

and $\mathcal{U}$ is a Lie algebra under the bracket $\{\cdot,\cdot\}$.

**Proof.** We have, letting $\tilde{u}(t) = -\int_0^t u(s)ds$:

$$[\nabla_u, \nabla_v]h = \tilde{u}\nabla_v h - \tilde{v}\nabla_u h = \tilde{u}\tilde{v}h = -\tilde{v}\tilde{u}h,$$

and

$$\nabla_{\{u,v\}}h = \nabla_{\tilde{u} - \tilde{v}}h = (\tilde{u} - \tilde{v})h = (u - v)\tilde{h},$$

hence $\Omega(u,v)h = 0$, $h,u,v \in C^\infty_c(\mathbb{R}_+)$. The extension of the result to $\mathcal{U}$ follows again from (3.8). The Lie algebra property follows from the vanishing of $\Omega$. \qed

**Exterior derivative**

The exterior derivative $du$ of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle du, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)} = \langle \nabla h_1 u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla h_2 u, h_1 \rangle_{L^2(\mathbb{R}_+)} ,$$

$h_1, h_2 \in \mathcal{U}$. We have

$$\|du\|_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)}^2 = 2 \int_0^\infty \int_0^\infty (du(s,t))^2 ds dt , \quad (3.9)$$

where

$$du(s,t) = \frac{1}{2}(\nabla_s u(t) - \nabla_t u(s)), \quad s,t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

**Isometry formula**

**Lemma 3.2** We have for $u \in \mathcal{U}$:

$$\int_\Gamma (d^*_u)^2 d\pi(\gamma) = \int_\Gamma \|u\|_{L^2(\mathbb{R}_+)}^2 d\pi(\gamma) + \int_\Gamma \int_0^\infty \int_0^\infty \nabla_s u(t) \nabla_t u(s) ds dt d\pi(\gamma). \quad (3.10)$$

11
\[ u = \sum_{i=1}^{n} h_i F_i \in \mathcal{U} \] 
we have
\[
\int_{\Gamma} d^*_\pi(h_i F_i) d^*_\pi(h_j F_j) d\pi(\gamma) = \int_{\Gamma} F_{d_{h_j}} d^*_\pi(h_j F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} F_{d_{h_j}} (F_j d^*_\pi(h_j) - d_{h_j} F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} (F_i F_j d_{h_i} d^*_\pi(h_j) + F_i d^*_\pi(h_j) d_{h_i} F_j - F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j d^*_\pi(\nabla h_i h_j) + F_i d^*_\pi(h_j) d_{h_i} F_j - F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + d_{h_i} d_{h_j} (F_i F_j) + d_{h_j} (F_i d_{h_i} F_j) - d_{h_i} d_{h_j} F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + d_{h_i} d_{h_j} (F_i F_j) + d_{h_j} (F_i d_{h_i} F_j) + F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma)
\]
\[
= \int_{\Gamma} \left( F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^\infty d_s F_i \int_0^\infty \nabla h_j (s) h_i(t) dt ds \\
+ F_i \int_0^\infty d_s F_j \int_0^\infty \nabla h_i (t) h_j(s) ds dt + \int_0^\infty h_i(t) d_s F_j \int_0^\infty h_j(s) d_s F_i ds dt \right) d\pi(\gamma),
\]
where we used the commutation relation satisfied by the gradient d:
\[
d_u d^*_\pi v = d^*_\pi \nabla u v + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \quad u, v \in C_c^\infty(\mathbb{R}_+),
\]
which can be proved as follows:
\[
d_u d^*_\pi v = - \sum_{k=1}^\infty \dot{v}(T_k) \int_0^{T_k} u(s) ds = -d^*_\pi \left( v(\cdot) \int_0^{\infty} u(s) ds \right) - \int_0^{\infty} \dot{v}(t) \int_0^{t} u(s) ds dt = d^*_\pi (\nabla u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}.\]
\]
Finally we state a Weitzenb"{o}ck type identity on configuration space, that can be read as
\[
\dd d^*_\pi + d^* d = \nabla^* \nabla + \text{Id}_{L^2(\mathbb{R}_+)},
\]
i.e. the Ricci tensor under the Poisson measure is the identity \( \text{Id}_{L^2(\mathbb{R}_+)} \) on \( L^2(\mathbb{R}_+) \).
Theorem 3.1 We have for $u \in U$:

$$
\int_{\Gamma} (d^* u)^2 d\pi(\gamma) + \int_{\Gamma} \|d u\|^2_{L^2(N)} d\pi(\gamma) \tag{3.12}
$$

$$
= \int_{\Gamma} \|u\|^2_{L^2(N)} d\pi(\gamma) + \int_{\Gamma} \|\nabla u\|^2_{L^2(N)} d\pi(\gamma).
$$

Proof. Relation (3.12) for $u = \sum_{i=1}^n h_i F_i \in U$ follows from (3.9) and Lemma 3.2.

\[ \square \]

References


