Stein normal approximation for multidimensional Poisson random measures by third cumulant expansions

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Abstract

We derive normal approximation bounds by the Stein method for stochastic integrals with respect to a Poisson random measure over \( \mathbb{R}^d \), \( d \geq 2 \). This approach relies on third cumulant Edgeworth-type expansions based on derivation operators defined by the Malliavin calculus for Poisson random measures. The use of third cumulants can exhibit faster convergence rates than the standard Berry-Esseen rate for some sequences of Poisson stochastic integrals.

Key words: Stein approximation; multidimensional Poisson random measures; Poisson stochastic integrals; cumulants; Malliavin calculus; Edgeworth expansions.

Mathematics Subject Classification: 62E17; 60H07; 60H05.

1 Introduction

Normal approximation bounds for stochastic integrals with respect to a Poisson random measure have been obtained by the Stein method in [15], using finite difference operators on the Poisson space. Recent results in this direction include the proof of a fourth moment theorem [8], [9], as an extension of the result of [14] to the setting of
Poisson point processes.

In this paper we derive related bounds for compensated Poisson stochastic integrals
\[ \delta(u) := \int_{\mathbb{R}^d} u_x (\gamma(dx) - \lambda(dx)) \]
of processes \((u_x)_{x \in \mathbb{R}^d}\) with compact support in \(\mathbb{R}^d\), with respect to a Poisson random measure \(\gamma(dx)\) with intensity the Lebesgue measure \(\lambda(dx)\) on \(\mathbb{R}^d\), \(d \geq 2\). In contrast with [15], our approach is based on derivation operators and Edgeworth-type expansions that involve the third cumulant of Poisson stochastic integrals, and can result into faster convergence rates, see e.g. (1.5) below.

Edgeworth-type expansions have been obtained on the Wiener space in [11], [5], by a construction of cumulant operators based on the inverse \(L^{-1}\) of the Ornstein-Uhlenbeck operator, extending the results of [12] on Stein approximation and Berry-Esseen bounds.

In Proposition 4.1 we derive Edgeworth-type expansions of the form
\[ \mathbb{E} [\delta(u) g(\delta(u))] = \mathbb{E} \left[ \|u\|_{L^2(\mathbb{R}^d)}^2 g'(\delta(u)) \right] + \sum_{k=2}^{n} \mathbb{E} \left[ g^{(k)}(\delta(u)) \Gamma_k^u 1 \right] + \mathbb{E} \left[ g^{(n+1)}(\delta(u)) R_n^u \right] \]
when the random field \((u_x)_{x \in \mathbb{R}^d}\) is predictable with respect to a given total order on \(\mathbb{R}^d\), where \(\Gamma_k^u\) is a cumulant-type operator and \(R_n^u\) is a remainder term, defined using the derivation operators of the Malliavin calculus on the Poisson space. In comparison with the results of [15], our bounds apply to a different stochastic integral representation of random variables, and they allow for random integrands \((u_x)_{x \in \mathbb{R}^d}\). In particular, this allows us to deal with random variables \(\delta(u)\) having infinite chaos expansions.

Based on (1.1), in Corollary 5.2 we deduce Stein approximation bounds of the form
\[ d_W(\delta(u), \mathcal{N}) \leq |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[\|u\|_{L^2(\mathbb{R}^d)}^2]} \]
+ \[ \mathbb{E} \left[ \int_{\mathbb{R}^d} u_x^3 \lambda(dx) + \left< u, D \int_{\mathbb{R}^d} u_x^2 \lambda(dx) \right>_{L^2(\mathbb{R}^d)} \right] \] \[ + \mathbb{E} [\|R_1^u\|], \]
where \(D\) is a gradient operator acting on Poisson functionals, and \(\mathcal{N} \simeq \mathcal{N}(0,1)\) is a
standard Gaussian random variable, see also Proposition 5.1. Here,

\[ d_W(F, G) := \sup_{h \in \mathcal{L}} |\mathbb{E}[h(F)] - \mathbb{E}[h(G)]| \]

is the Wasserstein distance between the laws of two random variables \( F \) and \( G \), where \( \mathcal{L} \) denotes the class of 1-Lipschitz functions on \( \mathbb{R} \).

In particular, when \( f \) is a differentiable deterministic function on the closed centered ball \( B(R) := B(0; R) \) in \( \mathbb{R}^d \) with radius \( R > 0 \), vanishing on the sphere \( S(0; R) := \{x \in \mathbb{R}^d : |x| = R\} \), we obtain bounds of the form

\[
d_W\left(\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \leq \left|1 - \|f\|^2_{L^2(\mathbb{R}^d)}\right| + \int_{\mathbb{R}^d} |f^3(x)| \lambda(dx) \quad + 8(K_d v_d R)^2 \|f\|_{L^2(\mathbb{R}^d)} \|\nabla^{\mathbb{R}^d} f\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2, \tag{1.2}
\]

where \( v_d \) denotes the volume of the unit ball in \( \mathbb{R}^d \) and \( K_d > 0 \) is a constant depending only on \( d \geq 2 \). The bound (1.2) can be compared to the classical Stein bound

\[
d_W\left(\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \leq \left|1 - \|f\|^2_{L^2(\mathbb{R}^d)}\right| + \int_{\mathbb{R}^d} |f^3(x)| \lambda(dx), \tag{1.3}
\]

for compensated Poisson stochastic integrals, see Corollary 3.4 of [15], which involves the \( L^3(\mathbb{R}^d) \) norm of \( f \) instead of third cumulant \( \kappa_3^f = \int_{\mathbb{R}^d} f^3(x) \lambda(dx) \) of \( \int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)) \), and relies on the use of finite difference operators, see Theorem 3.1 of [15] and § 4.2 of [4].

For example when \( f_k, k \geq 1 \), is a radial function given on \( B(k^{1/d}R) \) by

\[ f_k(x) := \frac{1}{C \sqrt{k}} g\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right), \quad x \in B(k^{1/d}R), \]

where \( g \in C^1([0, R]) \) is continuously differentiable on \([0, R]\) with \( g(R) = 0 \), and

\[ C^2 := \int_0^R g^2(r) r^{d-1} dr < \infty, \]

so that \( \|f_k\|_{L^3(B(k^{1/d}R))} = 1 \), the bound (1.3) yields the standard Berry-Esseen convergence rate

\[
d_W\left(\int_{B(k^{1/d}R)} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \leq \frac{v_d}{C^3 \sqrt{k}} \int_0^R |g(r)|^3 r^{d-1} dr, \quad k \geq 1, \quad (1.4)
\]
as $k$ tends to infinity. While (1.2) does not improve on (1.3) when the function $f$ has constant sign, if $g$ satisfies the condition

$$
\int_0^R g^3(r)r^{d-1}dr = 0,
$$

then the third cumulant bound (1.2) yields the $O(1/k)$ convergence rate

$$
d_W \left( \int_{B(k^{1/d}R)} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}^x \right) \leq \frac{2(2K_d v_d R)^2d}{kC^2} \|g'\|_\infty^2, \quad k \geq 1, \quad (1.5)
$$

which improves on the standard Berry-Esseen rate, see Section 5 for more examples.

In Sections 2 and 3 we recall some background material on the Malliavin calculus and differential geometry on the Poisson space, by revisiting the approach of [16], [17] using the recent constructions of [1] and references therein on the solution of the divergence problem. In Section 4 we derive Edgeworth-type expansions for the compensated Poisson stochastic integral $\delta(u)$, based on a family of cumulant operators that are associated to the random field $(u_x)_{x \in \mathbb{R}^d}$. In Section 5 we obtain Stein-type approximation bounds for stochastic integrals using deterministic examples of integrands.

The $d$-dimensional setting of this paper requires $d \geq 2$ and a bounded domain in $\mathbb{R}^d$ in order to construct a gradient operator $D$ for Poisson functionals by kernel inversion of the divergence operator on $\mathbb{R}^d$ using results of [1] and references therein. Consequently it does not cover the case $d = 1$ of the standard Poisson process on the half line $\mathbb{R}_+$, which requires a significantly different treatment, see [18]. In particular, the one-dimensional case is technically easier as it does not require Laplace inversion for the construction of the gradient operator $D$, while stronger conditions on the integrands $f$ in Poisson stochastic integrals have to be imposed in the case $d \geq 2$ through the norm $\|\nabla_{\mathbb{R}^d} f\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}$.

**Preliminaries**

Let $d \geq 2$ and $0 < R < R' := 2R$. We let $C^\infty_0(B(R'))$ denote the space of $C^\infty$ functions on $B(R')$ which vanish on the sphere $S(0; R') = \{x \in \mathbb{R}^d : |x| = R'\}$. Given
η ∈ C^∞_0(B(R')) such that ∫_{B(R)} η(x)dx = 1, we recall the existence of a C^∞ kernel function G_η : B(R') × B(R') → ℝ^d defined as

\[ G_η(x, y) := \int_0^1 \frac{(x - y)}{s}\eta \left( y + \frac{x - y}{s} \right) \frac{ds}{s^d}, \quad x, y ∈ B(R'), \]

see [1], and satisfying the following properties:

i) The kernel G_η(x, y) satisfies the bound

\[ |G_η(x, y)|_ℝ^d ≤ \frac{K^d}{|x - y|_ℝ^d}, \quad x, y ∈ B(R'), \]

for a constant K^d > 0 depending only on d, see Lemma 2.1 of [1], by choosing K^d and the function η ∈ C^∞_0(B(R')) therein so that \( \|\eta\|_∞ ≤ (d - 1)K_d(R')^{-d}. \)

ii) For any p > 1 and g ∈ L^p(B(R')) the function

\[ f(x) := \int_{B(R')} G_η(x, y)g(y) λ(dy), \quad x ∈ B(R'), \]

satisfies the bound

\[ \|f\|_{L^p(B(R');ℝ^d)} ≤ Kdv_dR'\|g\|_{L^p(B(R'))}, \quad p > 1, \]

which follows from Young’s inequality and (1.6), cf. Theorem 2.4 in [1].

iii) For any h ∈ C^∞_0(B(R')) we have the relation

\[ h(y) - \int_{B(R') \setminus B(R)} h(x)\eta(x) λ(dx) = \int_{B(R')} \langle G_η(x, y), \nabla^ℝ^d x h(x) \rangle_{ℝ^d} λ(dx), \quad y ∈ B(R'), \]

cf. Lemma 2.2 in [1], by taking η ∈ C^∞_c(B(R') \setminus B(R)). In particular, when h ∈ C^∞_0(B(R)) we have

\[ h(y) = \int_{B(R')} \langle G_η(x, y), \nabla^ℝ^d x h(x) \rangle_{ℝ^d} λ(dx), \quad y ∈ B(R'). \]

An extension of the framework of this paper, by replacing B(R) with a compact d-dimensional Riemannian manifold M and λ(dx) with the volume element of M, would require the Laplacian \( L = \text{div}^M \nabla^M \) to be invertible on C^∞_c(M) with

\[ L^{-1}u(x) = \int_M g(x, y)u(y) λ(dy), \quad x ∈ M, \; u ∈ C^∞_c(M), \]
where $g(x, y)$ is the heat kernel on $M$. In this case we can define $G_{\eta}(x, y) \in \mathbb{R}^d$ as

$$G_{\eta}(x, y) = \nabla^M_{\mathbb{R}^d} g(x, y), \quad \lambda \otimes \lambda(dx, dy) - a.e.,$$

with the relation

$$\nabla^M_{\mathbb{R}^d} L^{-1} u(x) = \int_M u(y) G_{\eta}(x, y) \lambda(dy) \in T_x M, \quad x \in M, \ u \in \mathcal{C}^\infty_c(M),$$

from which the divergence inversion relation (1.9) holds by duality.

## 2 Gradient, divergence and covariance derivative

There exists different notions of gradient and divergence operators for functionals of Poisson random measures. The operators of [2], [19], [7], and their associated integration by parts formula rely on an $\mathbb{R}^d$-valued gradient for random functionals and a divergence operator which is associated to the non-compensated Poisson stochastic integral of the divergence of $\mathbb{R}^d$-valued random fields. This particularity, together with a lack of a suitable commutation relation between gradient and divergence operators on Poisson functionals, makes this framework difficult to use for a direct analysis of Poisson stochastic integrals, while it has found applications to statistical estimation and sensitivity analysis, see [7], [19].

In this paper we use the construction of [16], [17] which relies on real-valued tangent processes and on a divergence operator that directly extends the compensated Poisson stochastic integral. This framework also allows for simple commutation relations between gradient and divergence operators using the deterministic inner product in $L^2(\mathbb{R}^d, \lambda)$, see Proposition 2.6, and it naturally involves the Poisson cumulants, see Definition 3.2 and Relation (3.6).

### Gradient operator

In the sequel we consider a Poisson random measure $\gamma(dx)$ on $B(R)$, constructed on a probability space $(\Omega, \mathcal{F}, P)$, and we let $\{X_1, \ldots, X_n\}$ denote the configuration points of $\gamma(dx)$ when $B(R)$ contains $n$ points in the configuration $\gamma$, i.e. when $\gamma(B(R)) = n$. 

Definition 2.1 Given \( A \) a closed subset of \( B(R') \), we let \( \mathcal{S}_A \) denote the set of random functionals \( F_A \) of the form

\[
F_A = \sum_{n=0}^{\infty} 1_{\{\gamma(B(R))=n\}} f_n (X_1, \ldots, X_n),
\]

where \( f_0 \in \mathbb{R} \) and \( (f_n)_{n \geq 1} \) is a sequence of functions satisfying the following conditions:

- for all \( n \geq 1 \), \( f_n \in C^\infty (A^n) \) is a symmetric function in \( n \) variables,
- for all \( n \geq 1 \) and \( i = 1, \ldots, n \) we have the continuity condition

\[
f_n (x_1, \ldots, x_n) = f_{n-1} (x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_n),
\]

for all \( x_1, \ldots, x_n \in B(R') \) such that \( |x_i|_{R^d} \geq R \).

We also let \( \mathcal{S} \) denote the union of the sets \( \mathcal{S}_A \) over the closed subsets \( A \) of \( B(R') \).

The gradient operator \( D \) is defined on random functionals \( F \in \mathcal{S} \) of the form (2.1) as

\[
D_y F := \sum_{n=1}^{\infty} 1_{\{\gamma(B(R))=n\}} \sum_{i=1}^{n} \langle G_\gamma (X_i, y), \nabla_{x_i} f (X_1, \ldots, X_n) \rangle_{R^d},
\]

\( y \in B(R) \). For any \( F \in \mathcal{S} \), by (1.6) we have \( DF \in L^1 (\Omega \times B(R)) \) from the bound

\[
\mathbb{E} \left[ \int_{B(R)} |D_x F| \lambda(dx) \right] \leq |||\nabla_{R^d} f |||_{R^d} \mathbb{E} \left[ \int_{B(R)} \int_{B(R)} |G_\gamma (x, y)|_{R^d} \gamma(dx) \lambda(dy) \right]
\]

\[
= \|||\nabla_{R^d} f |||_{R^d} \int_{B(R)} \int_{B(R)} |G_\gamma (x, y)|_{R^d} \lambda(dx) \lambda(dy)
\]

\[
= K_d |||\nabla_{R^d} f |||_{R^d} \int_{B(R)} \int_{B(R)} \frac{1}{|x-y|_{R^d}^{d-1}} \lambda(dx) \lambda(dy)
\]

\[
\leq K_d v_d R^d |||\nabla_{R^d} f |||_{R^d} |||\nabla_{R^d} f |||_{R^d}
\]

\[
< \infty.
\]

Poisson-Skorohod integral

We let \( \mathcal{U}_0 \) denote the space of simple random fields of the form

\[
u = \sum_{i=1}^{n} g_i G_i, \quad n \geq 1,
\]

with \( G_i \in \mathcal{S}_A \) and \( g_i \in C^\infty (B(R)) \), \( i = 1, \ldots, n \).
Definition 2.2 We define the Poisson-Skorohod integral $\delta(u)$ of $u \in U_0$ of the form (2.4) as

$$\delta(u) := \sum_{i=1}^{n} \left( G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle_{L^2(B(R))} \right).$$

(2.5)

In particular, for $h \in C^\infty_0(B(R))$ we have

$$\delta(h) = \int_{B(R)} h(x)(\gamma(dx) - \lambda(dx)).$$

The proof of the next proposition, cf. Proposition 8.5.1 in [16] and Proposition 5.1 in [17], is given in the appendix.

Proposition 2.3 The operators $D$ and $\delta$ satisfy the duality relation

$$\mathbb{E}[\langle u, DF \rangle_{L^2(B(R))}] = \mathbb{E}[F\delta(u)], \quad F \in S, \quad u \in U_0.$$  

(2.6)

As a consequence of Proposition 2.3 and the denseness of $S$ in $L^1(\Omega)$ and that of $U_0$ in $L^1(\Omega \times B(R))$, the gradient operator $D$ is closable in the sense that if $(F_n)_{n \in \mathbb{N}} \subset S$ tends to zero in $L^2(\Omega)$ and $(DF_n)_{n \in \mathbb{N}}$ converges to $U$ in $L^1(\Omega \times B(R))$, then $U = 0$ a.e.. Similarly, the divergence operator $\delta$ is closable in the sense that if $(u_n)_{n \in \mathbb{N}} \subset U_0$ tends to zero in $L^2(\Omega \times B(R))$ and $(\delta(u_n))_{n \in \mathbb{N}}$ converges to $G$ in $L^1(\Omega)$, then $G = 0$ a.s..

The gradient operator $D$ defines the Sobolev space $D^{1,1}$ with the Sobolev norm

$$\|F\|_{D^{1,1}} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^1(\Omega \times B(R))}, \quad F \in S.$$  

In the sequel we fix a total order $\preceq$ on $B(R)$ and consider the space $\mathcal{P}_0 \subset U_0$ of simple predictable random field of the form

$$u := \sum_{i=1}^{n} g_i F_i,$$

(2.7)

such that the supports of $g_1, \ldots, g_n$ satisfy

$$\text{Supp} \,(g_i) \preceq \cdots \preceq \text{Supp} \,(g_n) \quad \text{and} \quad F_i \in S_{A_i},$$

where $\text{Supp} \,(g_1) \cup \cdots \cup \text{Supp} \,(g_{i-1}) \subset A_i \subset B(R')$ and $A_i \preceq \text{Supp} \,(g_i), \ i = 1, \ldots, n.$
Such random fields are predictable in the sense of e.g. § 5 of [10] and references therein.

We will also assume that the order $\preceq$ is compatible with the kernel $G_\eta$ in the sense that
\[ G_\eta(x, y) = 0 \quad \text{for all} \quad x, y \in B(R) \text{ such that } x \preceq y. \tag{2.8} \]

Under the compatibility condition (2.8) we have in particular
\[ D_y F = 0, \quad y \in B(R), \quad A \preceq y, \quad F \in S_A. \]

Moreover, if $u \in \mathcal{P}_0$ is a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.8) we have
\[ D_y F_i = 0, \quad A_i \preceq y, \quad i = 1, \ldots, n, \]

hence
\[ D_y u_x = 0, \quad x \preceq y, \quad x, y \in B(R). \tag{2.9} \]

**Example.** The order $\preceq$ defined by
\[ x = (x^{(1)}, \ldots, x^{(d)}) \preceq y = (y^{(1)}, \ldots, y^{(d)}) \quad \iff \quad x^{(1)} \leq y^{(1)} \tag{2.10} \]
is compatible with the kernel $G_\eta$ provided that the support of $\eta$ is contained in
\[ \{ x = (x^{(1)}, \ldots, x^{(d)}) \in B(R') \setminus B(R) : x^{(1)} > R \}. \]

The proof of the next Proposition 2.4 is given in the appendix.

**Proposition 2.4** The Poisson-Skorohod integral of $u = (u_x)_{x \in B(R)}$ in the space $\mathcal{P}_0$ of simple predictable random fields satisfies the relation
\[ \delta(u) = \int_{B(R)} u_x (\gamma(dx) - \lambda(dx)), \tag{2.11} \]
which extends to the closure of $\mathcal{P}_0$ in $L^2(\Omega \times B(R))$ by density and the isometry relation
\[ \mathbb{E}[\delta(u)^2] = \mathbb{E} \left[ \int_{B(R)} u_x^2 \lambda(dx) \right], \quad u \in \mathcal{P}_0. \tag{2.12} \]
Covariant derivative

In addition to the gradient operator $D$, we will also need the following notion of covariant derivative operator $\tilde{\nabla}$ defined on stochastic processes that are viewed as tangent processes on the Poisson space $\Omega$, see [17].

**Definition 2.5** Let the operator $\tilde{\nabla}$ be defined on $u \in \mathcal{P}_0$ as

$$\tilde{\nabla}_y u_x := D_y u_x + (G_y(x,y), \nabla_x^R u_x)_{\mathbb{R}^d}, \quad x, y \in B(R).$$

We note that from the compatibility condition (2.8) and Relation (2.9) we also have

$$\tilde{\nabla}_y u_x = 0, \quad x \leq y, \quad x, y \in B(R). \quad (2.13)$$

From the bound

$$\mathbb{E} \left[ \int_{B(R) \times B(R)} |\tilde{\nabla}_x u_y| \lambda(dx)\lambda(dy) \right] \leq \|D u\|_{L^1(\Omega \times B(R) \times B(R))} + \mathbb{E} \left[ \int_{B(R) \times B(R)} |(G_y(x,y), \nabla_x^R u_x)_{\mathbb{R}^d}| \lambda(dx)\lambda(dy) \right]$$

$$\leq \|D u\|_{L^1(\Omega \times B(R) \times B(R))} + K_d \mathbb{E} \left[ \int_{B(R) \times B(R)} \frac{1}{|x-y|_{\mathbb{R}^d}} |\nabla_x^R u_x|_{\mathbb{R}^d} \lambda(dx)\lambda(dy) \right]$$

$$\leq \|D u\|_{L^1(\Omega \times B(R) \times B(R))} + K_d v d R' \mathbb{E} \left[ \int_{B(R)} |\nabla_x^R u_x|_{\mathbb{R}^d} \lambda(dx) \right]$$

$$= \|D u\|_{L^1(\Omega \times B(R) \times B(R))} + K_d v d R' \|\nabla_x^R u_x\|_{L^1(\Omega \times B(R); \mathbb{R}^d)},$$

we check that $\tilde{\nabla}$ extends to the Sobolev space $\tilde{D}^{1,1}_0$ of predictable random fields defined as the completion of $\mathcal{P}_0$ under the Sobolev norm

$$\|u\|_{\tilde{D}^{1,1}_0} := \|u\|_{L^2(\Omega, W^{1,1}_0(\mathbb{R}^d))} + \|D u\|_{L^1(\Omega \times B(R) \times B(R))}, \quad u \in \mathcal{P}_0,$$

where $W^{1,p}_0(B(R))$ is the first order Sobolev space completion of $C_0^\infty(B(R))$ under the norm

$$\|f\|_{W^{1,p}(B(R))} := \|f\|_{L^p(B(R))} + \|\nabla_x^R f\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)}, \quad p \geq 1.$$

Commutation relation

In the sequel, we denote by $\tilde{D}^{1,\infty}_0$ the set of predictable random fields $u$ in $\tilde{D}^{1,1}_0$ that are bounded together with their covariant derivative $\tilde{\nabla}_u$. 

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Proposition 2.6 For $u \in \tilde{D}_0^{1,\infty}$ a predictable random field, we have the commutation relation

$$D_y \delta(u) = u(y) + \delta(\tilde{\nabla}_y u), \quad y \in B(R).$$  \hspace{1cm} (2.14)

Proof. Taking $h \in \mathcal{C}_0^\infty(B(R))$, we have $\delta(h) \in \mathcal{S}$ and

$$D_y \delta(h) = D_y \int_{B(R)} h(y)(\gamma(dx) - \lambda(dx))$$
$$= \int_{B(R)} \langle \mathcal{G}_\eta(x,y), \tilde{\nabla}_x^d h(x) \rangle_{\mathbb{R}^d} \gamma(dx)$$
$$= \int_{B(R)} \langle \mathcal{G}_\eta(x,y), \tilde{\nabla}_x^d h(x) \rangle_{\mathbb{R}^d} \lambda(dx) + \delta(\tilde{\nabla}_y h)$$
$$= h(y) + \delta(\tilde{\nabla}_y h), \quad y \in B(R),$$

where we applied (1.9). Next, taking $u = hF \in \mathcal{P}_0$ a simple predictable random field, we check that $\delta(u) \in \mathcal{S}$, and by (2.5) or (6.3) we have

$$D_y \delta(Fh) = D_y (F \delta(h) - \langle h, DF \rangle_{L^2(B(R))})$$
$$= D_y (F \delta(h))$$
$$= \delta(h)D_y F + FD_y \delta(h)$$
$$= \delta(h)D_y F + F(h(y) + \delta(\tilde{\nabla}_y h))$$
$$= Fh(y) + \delta(hD_y F + \tilde{\nabla}_y h)$$
$$= Fh(y) + \delta(\tilde{\nabla}_y (Fh))$$
$$= u_y + \delta(\tilde{\nabla}_y u), \quad y \in B(R).$$

We conclude by the denseness of $\mathcal{P}_0$ in $\tilde{D}_0^{1,1}$ and by the closability of the operators $\tilde{\nabla}$, $D$ and $\delta$. \hfill \Box

3 Cumulant operators

In the sequel, given $h$ in the standard Sobolev space $W^{1,p}(B(R))$ on $B(R)$ and $f \in L^q(B(R))$ with $1 = p^{-1} + q^{-1}$, $p, q \in [1, \infty]$, we define

$$\langle \tilde{\nabla}h \rangle f_x := \int_{B(R)} f(y) \tilde{\nabla}_y h(x) \lambda(dy) = \int_{B(R)} f(y) \langle \mathcal{G}_\eta(x,y), \tilde{\nabla}_x^d h(x) \rangle_{\mathbb{R}^d} \lambda(dy).$$  \hspace{1cm} (3.1)
$x \in B(R)$. More generally, given $k \geq 1$ and $u \in \mathcal{D}^{1,1}_0$ a predictable random field, we let the operator $(\nabla_e u)^k$ be defined in the sense of matrix powers with continuous indices, as

\[(\nabla_e u)^k f_y = \int_{B(R)} \cdots \int_{B(R)} (\nabla_{x_k} u_y \nabla_{x_{k-1}} u_{x_k} \cdots \nabla_{x_1} u_{x_2}) f_{x_1} \lambda(dx_1) \cdots \lambda(dx_k),\]

$y \in B(R)$, $f \in L^2(B(R))$.

**Proposition 3.1** For any $n \in \mathbb{N}$, $p > 1$, $r \in [0, 1]$, $h \in W^{1,p/(1-r)^{n-1/r}}(B(R))$ and $f \in L^{p/(1-r)^n}(B(R))$ we have the bound

\[\| (\nabla^h_t) f \|_{L^p(B(R))} \leq (K_d v_d R')^n \| f \|_{L^{p/(1-r)^n}(B(R))} \prod_{j=1}^n \| \nabla^\mathbb{R}^d h \|_{L^{p/(1-r)^j/(1-r)}(B(R); \mathbb{R}^d)}, \]  

(3.2)

**Proof.** For $n = 1$ we have

\[\begin{aligned}
    \| (\nabla^h_t) f \|_{L^p(B(R))}^p &= \int_{B(R)} \left| \int_{B(R)} f(y) \nabla^\mathbb{R}^d h(x) \lambda(dy) \right|^p \lambda(dx) \\
    &= \int_{B(R)} \left| \int_{B(R)} f(y) \langle \nabla^\mathbb{R}^d h(x) \rangle \lambda(dy) \right|^p \lambda(dx) \\
    &= \int_{B(R)} \left| \left( \int_{B(R)} f(y) \nabla^\mathbb{R}^d h(x) \lambda(dy) \right) \right|^p \lambda(dx) \\
    &\leq \left( \int_{B(R)} \| f \|_{L^{p/(1-r)}(B(R))} \| \nabla^\mathbb{R}^d h \|_{L^{p/(1-r)}(B(R); \mathbb{R}^d)}^p \lambda(dx) \right)^{1-r} \left( \int_{B(R)} \| \nabla^\mathbb{R}^d h(x) \|_{L^{p/(1-r)}(B(R); \mathbb{R}^d)}^p \lambda(dx) \right)^r \\
    &\leq (K_d v_d R')^p \| f \|_{L^{p/(1-r)}(B(R))} \| \nabla^\mathbb{R}^d h \|_{L^{p/(1-r)}(B(R); \mathbb{R}^d)}^p, \tag{3.3}
\end{aligned}\]

where we used the bound (1.7). Next, assuming that (3.2) holds at the rank $n \geq 1$ and using (3.3), we have

\[\begin{aligned}
    \| (\nabla^h_t)^{n+1} f \|_{L^p(B(R))} &= \| (\nabla^h_t)^n (\nabla^h_t) f \|_{L^p(B(R))} \\
    &\leq (K_d v_d R')^n \| (\nabla^h_t) f \|_{L^{p/(1-r)^n}(B(R))} \prod_{j=1}^n \| \nabla^\mathbb{R}^d h \|_{L^{p/(1-r)^j/(1-r)}(B(R); \mathbb{R}^d)} \\
    &\leq (K_d v_d R')^{n+1} \| f \|_{L^{p/(1-r)^{n+1}}(B(R))} \prod_{j=1}^{n+1} \| \nabla^\mathbb{R}^d h \|_{L^{p/(1-r)^j/(1-r)}(B(R); \mathbb{R}^d)},
\end{aligned}\]

and we conclude to (3.2) by induction on $n \geq 1$. \[\square\]
In particular, for \( r = 0 \), \( f \in L^p(B(R)) \), \( p > 1 \), and \( h \in W^{1,1}(B(R)) \) the argument of Proposition 3.1 shows that
\[
\|(\tilde{\nabla}h)^n f\|_{L^p(B(R))} \leq (K_d v_d R^p)^n \|f\|_{L^p(B(R))} \|\nabla^{R^d} h\|_{L^\infty(B(R); \mathbb{R}^d)}^n, \quad n \in \mathbb{N}.
\]
We note that for \( u \in \tilde{\mathbb{D}}_0^{1,\infty} \) a predictable random field, the random field \((\tilde{\nabla} u)u \in \tilde{\mathbb{D}}_0^{1,\infty}\) is also predictable from (2.13) and (3.1).

In the next definition we construct a family of cumulant operators which differs from the one introduced in [13] on the Wiener space.

**Definition 3.2** Given \( k \geq 2 \) and \( u \in \tilde{\mathbb{D}}_0^{1,\infty} \) a predictable random field we define the operators \( \Gamma_k^u : \mathbb{D}_{1,1} \rightarrow L^1(\Omega) \) by
\[
\Gamma_k^u F := F\langle (\tilde{\nabla} u)^{k-2} u, u \rangle_{L^2(B(R))} + \langle (\tilde{\nabla} u)^{k-1} u, DF \rangle_{L^2(B(R))}, \quad F \in \mathbb{D}_{1,1}.
\]
We note that for \( h \) in the space \( W^{1,\infty}(B(R)) \) of bounded functions in \( W^{1,1}(B(R)) \), and \( f \in L^p(B(R)) \), \( p > 1 \), \( m \geq 1 \), we have
\[
\langle h^m, (\tilde{\nabla} h f) \rangle_{L^2(B(R))} = \int_{B(R)} h^m(x) \int_{B(R)} f(y) \langle G_{\eta}(x, y), \nabla_x^{R^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx)
\]
\[
= \frac{1}{m+1} \int_{B(R)} \int_{B(R)} f(y) \langle G_{\eta}(x, y), \nabla_x^{R^d} h^{m+1}(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx)
\]
\[
= \frac{1}{m+1} \int_{B(R)} f(x) h^{m+1}(x) \lambda(dx),
\]
where we applied (1.8), hence
\[
\langle h^m, (\tilde{\nabla} h)^n f \rangle_{L^2(B(R))} = \frac{1}{m+1} \int_{B(R)} h^{m+1}(x) (\tilde{\nabla} h)^n f(x) \lambda(dx),
\]
which implies by induction
\[
\langle (\tilde{\nabla} h)^n f, h^m \rangle_{L^2(B(R))} = \frac{m!}{(m+n)!} \int_{B(R)} h^{m+n}(x) f(x) \lambda(dx).
\]
In Lemma 3.3 we generalize this identity to \( h \) a random field.

**Lemma 3.3** For \( n \in \mathbb{N} \), \( m \geq 1 \), \( u \in \tilde{\mathbb{D}}_0^{1,\infty} \) a predictable random field and \( f \in L^p(B(R)) \), \( p > 1 \), we have
\[
\langle (\tilde{\nabla} u)^n f, u^m \rangle_{L^2(B(R))} = \frac{m!}{(m+n)!} \int_{B(R)} u_x^{m+n} f(x) \lambda(dx).
\]

(3.4)
with the duality relation

\[ \langle v, (\tilde{\nabla}^* u) h \rangle_{L^2(B(R))} = \langle (\tilde{\nabla} u) v, h \rangle_{L^2(B(R))}, \quad h, v \in L^2(B(R)), \]

we will show by induction on \( k = 0, 1, \ldots, n \) that

\[
(\tilde{\nabla}^* u)^n u_{x_0}^m = \int_{B(R)} \cdots \int_{B(R)} u_{x_0}^m \tilde{\nabla} x_0 u_{x_1} \tilde{\nabla} x_1 u_{x_2} \cdots \tilde{\nabla} x_{n-1} u_{x_n} \lambda(dx_1) \cdots \lambda(dx_n)
\]

\[
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-i-1} u_{x_{n-i}} D_x u_{x_{n-i}}^{m+i} \lambda(dx_1) \cdots \lambda(dx_{n-i})
\]

\[
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^{m+k} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-k-1} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k}). \tag{3.5}
\]

By (3.1), this relation holds for \( k = 0 \). Next, assuming that the identity (3.5) holds for some \( k \in \{0, 1, \ldots, n-1\} \), and using the relation

\[ \tilde{\nabla} x_{n-k-1} u_{x_{n-k}} = D x_{n-k-1} u_{x_{n-k}} + \langle G_\eta(x_{n-k, x_{n-k-1}}, \tilde{\nabla} x_{n-k} u_{x_{n-k}}) \rangle_{R^d}, \quad x_{n-k}, x_{n-k} \in B(R), \]

we have

\[
(\tilde{\nabla}^* u)^n u_{x_0}^m
\]

\[
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-i-1} u_{x_{n-i}} D_x u_{x_{n-i}}^{m+i} \lambda(dx_1) \cdots \lambda(dx_{n-i})
\]

\[
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^{m+k} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-k-1} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k})
\]

\[
= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-i-1} u_{x_{n-i}} D_x u_{x_{n-i}}^{m+i} \lambda(dx_1) \cdots \lambda(dx_{n-i})
\]

\[
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} u_{x_{n-k}}^{m+k} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-k-2} u_{x_{n-k-1}} D_{x_{n-k-1}} u_{x_{n-k}} \lambda(dx_1) \cdots \lambda(dx_{n-k})
\]

\[
+ \frac{m!}{(m+k)!} \int_{B(R)} \cdots \int_{B(R)} \langle G_\eta(x_{n-k, x_{n-k-1}}, \tilde{\nabla} x_{n-k} u_{x_{n-k}}) \rangle_{R^d}.
\]
\begin{equation}
\times u_{x_n-k}^{m+k-2} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-2-k} u_{x_{n-k-1}} \lambda(dx_1) \cdots \lambda(dx_{n-k})
\end{equation}

\begin{align*}
&= \sum_{i=1}^{k} \frac{m!}{(m+i)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-i} u_{x_{n-i}} D_{x_{n-i}} u_{x_{n+i-1}}^{m+i} \lambda(dx_1) \cdots \lambda(dx_{n+1-i}) \\
&+ \frac{m!}{(m+k+1)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-k} u_{x_{n-k}} D_{x_{n-k}} u_{x_{n-k}}^{m+k+1} \lambda(dx_1) \cdots \lambda(dx_{n-k}) \\
&+ \frac{m!}{(m+k+1)!} \int_{B(R)} \cdots \int_{B(R)} \tilde{\nabla} x_0 u_{x_1} \cdots \tilde{\nabla} x_{n-k-2} u_{x_{n-k-1}} \lambda(dx_1) \cdots \lambda(dx_{n-k-1}) \\
&= \sum_{i=1}^{k} \frac{m!}{(m+i)!} (\tilde{\nabla} u)^{n-i} D_{x_0} \int_{B(R)} u_{x_1}^{m+i} \lambda(ds) + \frac{m!}{(m+k+1)!} (\tilde{\nabla} u)^{n-k-1} u_{x_0}^{m+k+1},
\end{align*}

which shows by induction that (3.5) holds at the rank \( k = n \), in particular we have

\begin{align*}
(\tilde{\nabla} u)^n u_x &= \frac{m!}{(m+k)!} u_x^{m+n} + \sum_{i=2}^{n+1} \frac{m!}{(m+i-1)!} (\tilde{\nabla} u)^{n+1-i} D_x \int_{B(R)} u_x^{m+i-1} \lambda(dy),
\end{align*}

\( x \in B(R) \), which yields (3.4) by integration with respect to \( x \in B(R) \) and duality.

\[ \square \]

As a consequence of Lemma 3.3 we have

\begin{equation}
\Gamma_k^h 1 = \int_{B(R)} \frac{u_x^k}{(k-1)!} \lambda(dx) + \sum_{i=2}^{k-1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{k-1-i} u, D \int_{B(R)} u_x^i \lambda(dx) \right\rangle_{L^2(B(R))},
\end{equation}

\( k \geq 2 \). Hence when \( h \in W^{1,p}(B(R)) \), \( p > 1 \), is a deterministic function such that \( \|\nabla^4 h\|_{\infty} < \infty \), we find the relation

\begin{equation}
\Gamma_k^h 1 = \frac{1}{(k-1)!} \int_{B(R)} h^k(x) \lambda(dx) = \frac{1}{(k-1)!} \kappa_k^h, \quad k \geq 2,
\end{equation}

which shows that \( \Gamma_k^h 1 \) coincides with the cumulant \( \kappa_k^h = \int_{B(R)} h^k(x) \lambda(dx) \) of order \( k \geq 2 \) of the Poisson stochastic integral \( \int_{B(R)} h(x)(\gamma(dx) - \lambda(dx)) \).
4 Edgeworth-type expansions

Classical Edgeworth series provide expansion of the cumulative distribution function $P(F \leq x)$ of a centered random variable $F$ with $\mathbb{E}[F^2] = 1$ around the Gaussian cumulative distribution function $\Phi(x)$, using the cumulants $(\kappa_n)_{n \geq 1}$ of a random variable $F$ and Hermite polynomials. Edgeworth-type expansions of the form

$$\mathbb{E}[F g(F)] = \sum_{l=1}^{n} \frac{\kappa_{l+1}}{l!} \mathbb{E}[g^{(l)}(F)] + \mathbb{E}[g^{(n+1)}(F) \Gamma_{n+1} F], \quad n \geq 1,$$

for $F$ a centered random variable, have been obtained by the Malliavin calculus in [11], where $\Gamma_{n+1}$ is a cumulant-type operator on the Wiener space such that $n! \mathbb{E}[\Gamma_n F]$ coincides with the cumulant $\kappa_{n+1}$ of order $n+1$ of $F$, $n \in \mathbb{N}$, cf. [13], extending the results of [3] to the Wiener space.

In this section we establish an Edgeworth-type expansion of any finite order with an explicit remainder term for the compensated Poisson stochastic integral $\delta(u)$ of a predictable random field $(u_x)_{x \in B(R)}$. In the sequel we let $\langle \cdot, \cdot \rangle$ denote $\langle \cdot, \cdot \rangle_{L^2(B(R))}$.

Before proceeding to the statement of general expansions in Proposition 4.1, we illustrate the method with the derivation of an expansion of order one for a deterministic integrand $f$. By the duality relation (2.6) between $D$ and $\delta$, the chain rule of derivation for $D$ and the commutation relation (2.14) we get, for $g \in C^2_b(\mathbb{R})$ and $f \in W^{1,1}_0(B(R))$ such that $\|\nabla^{rd} f\|_{\infty} < \infty$,

$$\mathbb{E}[\delta(f) g(\delta(f))] = \mathbb{E}[\langle f, D\delta(f) \rangle g'(\delta(f))]$$

$$= \mathbb{E}[\langle f, f \rangle g'(\delta(f))] + \mathbb{E}[\langle f, \delta(\nabla^* f) \rangle g'(\delta(f))].$$

$$= \mathbb{E}[\langle f, f \rangle g'(\delta(f))] + \mathbb{E}[\langle \nabla^* f, D(g'(\delta(f)) f) \rangle]$$

$$= \mathbb{E}[\langle f, f \rangle g'(\delta(f))] + \mathbb{E}[\langle (\nabla f) f, D\delta(f) \rangle g''(\delta(f))]$$

$$= \mathbb{E}[\langle f, f \rangle g'(\delta(f))] + \frac{1}{2} \int_{B(R)} f^3(x) \lambda(dx) \mathbb{E}[g''(\delta(f))] + \mathbb{E}[\langle (\nabla f) f, \delta(\nabla^* f) \rangle g''(\delta(f))]$$

$$= \kappa_2 f \mathbb{E}[g'(\delta(f))] + \frac{1}{2} \kappa_3 \mathbb{E}[g''(\delta(f))],$$
since by Lemma 3.3 we have
\[
\langle (\bar{\nabla} f) f, f \rangle = \frac{1}{2} \int_{B(R)} f^3(x) \lambda(dx) = \frac{1}{2} k_3^f.
\]

In the next proposition we derive general Edgeworth-type expansions for predictable integrand processes \((u_x)_{x \in \mathbb{R}^d}\).

**Proposition 4.1** Let \(u \in \mathbb{D}^{1,\infty}_0\) and \(n \geq 0\). For all \(g \in C^{n+1}_b(\mathbb{R})\) and bounded \(G \in \mathbb{D}_{1,1}\) we have

\[
\mathbb{E}[G\delta(u)g(\delta(u))] = \mathbb{E}[(u, DG)g(\delta(u))] + \sum_{k=1}^{n} \mathbb{E}[g^{(k)}(\delta(u))\Gamma_k G]
\]

\[
+ \mathbb{E}\left[Gg^{(n+1)}(\delta(u)) \left( \int_{B(R)} \frac{v_{x+2}^{n+2}}{(n+1)!} \lambda(dx) + \sum_{k=2}^{n+1} \left( (\bar{\nabla} u)^{n+1-k} u, D \int_{B(R)} \frac{u^k}{k!} \lambda(dx) \right) \right) \right]
\]

\[
+ \mathbb{E}\left[Gg^{(n+1)}(\delta(u))(\bar{\nabla} u)^n u, \delta(\bar{\nabla} u) \right].
\]

**Proof.** By the duality relation (2.6) between \(D\) and \(\delta\), the chain rule of derivation for \(D\) and the commutation relation (2.14), we get

\[
\mathbb{E}[G(\bar{\nabla} u)^k u, D\delta(u))g(\delta(u))] = \mathbb{E}[G(\bar{\nabla} u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= \mathbb{E}[G(\bar{\nabla} u)^k u, g(\delta(u))] + \mathbb{E}[G(\bar{\nabla} u)^k u, \delta(\bar{\nabla} u)g(\delta(u))] - \mathbb{E}[G(\bar{\nabla} u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= \mathbb{E}[G(\bar{\nabla} u)^k u, g(\delta(u))] + \mathbb{E}[\delta(\bar{\nabla} u), D(\delta(\bar{\nabla} u)(\bar{\nabla} u)^k u)) - \mathbb{E}[G(\bar{\nabla} u)^{k+1} u, D\delta(u))g'(\delta(u))]
\]

\[
= \mathbb{E}[G(\bar{\nabla} u)^k u, g(\delta(u))] + \mathbb{E}[\delta(\bar{\nabla} u), D(\bar{\nabla} u)^k u))g(\delta(u))] + \mathbb{E}[G(\bar{\nabla} u)^k u, D((\bar{\nabla} u)^k u))g(\delta(u))]
\]

\[
= \mathbb{E}[g(\delta(u))\Gamma_{k+2} G],
\]

where we used (2.9) and (2.13). Therefore, we have

\[
\mathbb{E}[G\delta(u)g(\delta(u))] = \mathbb{E}[\langle u, D(Gg(\delta(u))) \rangle]
\]

\[
= \mathbb{E}[\langle u, DG \delta(u) g(\delta(u)) \rangle] + \mathbb{E}[\langle u, DG \rangle g(\delta(u))]
\]

\[
= \mathbb{E}[\langle u, DG \rangle g(\delta(u)) + \mathbb{E}[Gg^{(n+1)}(\delta(u))(\bar{\nabla} u)^n u, D\delta(u))]
\]

\[
+ \sum_{k=0}^{n-1} \mathbb{E}[Gg^{(k+1)}(\delta(u))(\bar{\nabla} u)^k u, D\delta(u)) - \mathbb{E}[Gg^{(k+2)}(\delta(u))(\bar{\nabla} u)^{k+1} u, D\delta(u))]
\]

\[
= \mathbb{E}[\langle u, DG \rangle g(\delta(u))] + \sum_{k=1}^{n} \mathbb{E}[g^{(k)}(\delta(u))\Gamma_k G] + \mathbb{E}[Gg^{(n+1)}(\delta(u))(\bar{\nabla} u)^n u, D\delta(u))]
\]

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\[ I_E [g(\delta(u))] + \sum_{k=1}^{n} \mathbb{E} \left[ g^{(k)}(\delta(u)) \Gamma_k G \right] + \mathbb{E} \left[ Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n, u \rangle + \mathbb{E} \left[ Gg^{(n+1)}(\delta(u)) \langle (\nabla u)^n, \delta(\nabla^* u) \rangle \right] \right], \]

and we conclude by Lemma 3.3.

When \( f \in W_{0}^{1,1}(B(R)) \) is a deterministic function such that \( \|\nabla^d f\|_{\infty} < \infty \), and \( g \in \mathcal{C}_b^\infty(\mathbb{R}) \), Proposition 4.1 shows that

\[ \mathbb{E} \left[ \delta(f)g(\delta(f)) \right] = \sum_{k=1}^{n+1} \frac{1}{k!} \int_{B(R)} f^{n+1}(x) \lambda(dx) \mathbb{E} \left[ g^{(k)}(\delta(f)) \right] + \mathbb{E} \left[ g^{(n+1)}(\delta(f)) \langle (\nabla f)^n, \delta(\nabla^* f) \rangle \right]. \]

In addition, as \( n \) tends to \(+\infty\) we have

\[ \mathbb{E} \left[ \delta((\nabla f)^n f) \right] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{B(R)} f^{k+1}(x) \lambda(dx) \mathbb{E} \left[ g^{(k)}(\delta(f)) \right] \]

5 Stein approximation

Applying Proposition 4.1 with \( n = 0 \) and \( G = 1 \) to the solution \( g_x \) of the Stein equation

\[ 1_{(-\infty,x]}(z) - \Phi(z) = g_x'(z) - zg_x(z), \quad z \in \mathbb{R}, \]
and letting \( u \in \tilde{D}^{1,1}_0 \) be a predictable random field, this gives the expansion

\[
P(\delta(u) \leq x) - \Phi(x) = \mathbb{E}[g'_x(\delta(u)) \langle u, u \rangle - \delta(u)g_x(\delta(u))]
\]

\[
= \mathbb{E}[(1 - \langle u, u \rangle)g'_x(\delta(u))] + \mathbb{E} \left[ \langle \delta(\vec{N}u) \rangle g'_x(\delta(u)) \right],
\]

around the Gaussian cumulative distribution function \( \Phi(x) \), with \( \|g_x\|_\infty \leq \sqrt{2\pi}/4 \) and \( \|g'_x\|_\infty \leq 1 \), \( x \in \mathbb{R} \), by Lemma 2.2-(v) of [6]. The next result follows from the application of Proposition 4.1 with \( n = 1 \) and \( G = 1 \).

**Proposition 5.1** For any random field \( u \in \tilde{D}^{1,\infty}_0 \) we have

\[
d_W(\delta(u), \mathcal{N}) \leq \mathbb{E} \left[ |1 - \langle u, u \rangle - \langle \vec{\nabla}^* u, Du \rangle| \right] + \mathbb{E} \left[ \left| \int_{B(R)} u_3^2 \lambda(dx) + \left\langle u, D \int_{B(R)} u_2^2 \lambda(dx) \right\rangle \right| \right]
\]

\[
+ 2 \mathbb{E} \left[ |\langle (\vec{\nabla} u) u, \delta(\vec{\nabla}^* u) \rangle| \right].
\]

**Proof.** For \( n = 1 \) and \( G = 1 \), Proposition 4.1 shows that

\[
\mathbb{E}[\delta(u)g(\delta(u))] = \mathbb{E}[g'(\delta(u))(\langle u, u \rangle + \langle \vec{\nabla}^* u, Du \rangle)]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ g''(\delta(u)) \left( \int_{B(R)} u_3^2 \lambda(dx) + \left\langle u, D \int_{B(R)} u_2^2 \lambda(dx) \right\rangle \right) \right]
\]

\[
+ \mathbb{E}[g''(\delta(u))(\langle \delta(\vec{\nabla} u) u, \delta(\vec{\nabla}^* u) \rangle)].
\]

Let \( h : \mathbb{R} \to [0, 1] \) be a continuous function with bounded derivative. Using the solution \( g_h \in C_b^1(\mathbb{R}) \) of the Stein equation

\[
h(z) - \mathbb{E}[h(\mathcal{N})] = g'(z) - zg(z), \quad z \in \mathbb{R},
\]

with the bounds \( \|g'_h\|_\infty \leq \|h'\|_\infty \) and \( \|g''_h\|_\infty \leq 2\|h''\|_\infty \), \( x \in \mathbb{R} \), cf. Lemma 1.2-(v) of [12] and references therein, we have

\[
\mathbb{E}[h(\delta(u))] - \mathbb{E}[h(\mathcal{N})] = \mathbb{E}[\delta(u)g_h(\delta(u)) - g'_h(\delta(u))]
\]

\[
= \mathbb{E}[g'_h(\delta(u))(\langle u, u \rangle + \langle \vec{\nabla}^* u, Du \rangle - 1)]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ g''(\delta(u)) \left( \int_{B(R)} u_3^2 \lambda(dx) + \left\langle u, D \int_{B(R)} u_2^2 \lambda(dx) \right\rangle \right) \right]
\]

\[
+ 2 \mathbb{E}[g''_h(\delta(u))(\langle \vec{\nabla} u) u, \delta(\vec{\nabla}^* u) \rangle].
\]
hence
\[
|E[\delta(u)h(\delta(u))] - E[h(\mathcal{N})]| \leq \|h\|_\infty E\left[1 - \langle u, u \rangle - \langle \nabla^* u, Du \rangle \right] \\
+ \|h\|_\infty E\left[\left\lVert \int_{\mathbb{R}} u_x^2 \lambda(dx) + \left\langle u, D \int_{\mathbb{R}} u_x^2 \lambda(dx) \right\rangle \right\rVert^2 \right] \\
+ 2\|h\|_\infty E\left[\left\lVert \int_{\mathbb{R}} u \gamma(dx) \lambda(dx) \right\rVert^2 \right] \\
+ 2\|h\|_\infty E\left[\left\lVert \int_{\mathbb{R}} u^2 \lambda(dx) \right\rVert^2 \right] + 2 E\left[\left\lVert \nabla^* u, Du \right\rVert^2 \right].
\]
which yields (5.2).

As a consequence of Proposition 5.1 and the Itô isometry (2.12) we have the following corollary.

**Corollary 5.2** For \( u \in \tilde{D}_{1,\infty}^0 \) we have
\[
d_W(\delta(u), \mathcal{N}) \leq |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[\|u\|_{L^2(\mathbb{R})}^2]} \\
+ E\left[\left\lVert \int_{\mathbb{R}} u_x^2 \lambda(dx) + \left\langle u, D \int_{\mathbb{R}} u_x^2 \lambda(dx) \right\rangle \right\rVert \right] \\
+ E[\left\lVert \int_{\mathbb{R}} u \gamma(dx) \lambda(dx) \right\rVert^2] + 2 E\left[\left\lVert \nabla^* u, Du \right\rVert^2 \right].
\]

**Proof.** By the Itô isometry (2.12) we have
\[
\text{Var}[\delta(u)] = E\left[\left(\int_{\mathbb{R}} u_x \gamma(dx) - \lambda(dx)\right)^2 \right] = E[\langle u, u \rangle],
\]
hence
\[
E\left[\left\lVert \langle u, u \rangle - \langle \nabla^* u, Du \rangle \right\rVert^2 \right] \\
\leq E\left[\left\lVert \langle u, u \rangle - E[\langle u, u \rangle] \right\rVert + E[\langle u, u \rangle - E[\langle u, u \rangle] \right\rVert^2 \right] + E\left[\left\lVert \nabla^* u, Du \right\rVert^2 \right] \\
= |1 - \text{Var}[\delta(u)]| + \sqrt{E[\langle u, u \rangle - E[\langle u, u \rangle]^2]} + E[\left\lVert \nabla^* u, Du \right\rVert^2] \\
= |1 - \text{Var}[\delta(u)]| + \sqrt{\text{Var}[\|u\|_{L^2(\mathbb{R})}^2]} + E[\left\lVert \nabla^* u, Du \right\rVert^2].
\]

In particular, when \( \text{Var}[\delta(u)] = 1 \), Corollary 5.2 shows that
\[
d_W(\delta(u), \mathcal{N}) \leq \sqrt{\text{Var}[\|u\|_{L^2(\mathbb{R})}^2]} + E\left[\left\lVert \int_{\mathbb{R}} u_x^2 \lambda(dx) + \left\langle u, D \int_{\mathbb{R}} u_x^2 \lambda(dx) \right\rangle \right\rVert \right] \\
+ E[\left\lVert \nabla^* u, Du \right\rVert^2] + 2 E\left[\left\lVert \langle \nabla^* u, \delta(\nabla^* u) \rangle \right\rVert^2 \right].
\]

□
When \( f \in W^1_0(B(R)) \) is a deterministic function we have

\[
\text{Var}[\delta(f)] = \mathbb{E} \left[ \left( \int_{B(R)} f(x) (\gamma(dx) - \lambda(dx)) \right)^2 \right] = \int_{B(R)} f^2(x) \lambda(dx),
\]

and Corollary 5.1 shows that

\[
d_W(\delta(f)) \leq \left| 1 - \int_{B(R)} f^2(x) \lambda(dx) \right| + \left| \int_{B(R)} f^3(x) \lambda(dx) \right| + 2 \mathbb{E} \left[ \delta((\nabla f)^2 f) \right].
\]

Given the bound

\[
\mathbb{E} \left[ |\delta((\nabla f)^2 f)| \right] \leq \sqrt{\mathbb{E} \left[ |\delta((\nabla f)^2 f)|^2 \right]}
\]

\[
= \|(\nabla f)^2 f\|_{L^2(B(R))}
\]

\[
\leq (K_d v_d R)^2 \|f\|_{L^2(B(R))} \|\nabla f\|_{L^\infty(B(R);\mathbb{R}^d)}^2
\]

obtained from Proposition 3.1 with \( p = 2 \) and \( r = 0 \), \( f \in W^1_0(B(R)) \), we also have the following corollary.

**Corollary 5.3** For \( f \in W^1_0(B(R)) \) we have

\[
d_W \left( \int_{B(R)} f(x) (\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \left| 1 - \|f\|_{L^2(B(R))}^2 \right| + \left| \int_{B(R)} f^3(x) \lambda(dx) \right|
\]

\[
+ 2 (K_d v_d R)^2 \|f\|_{L^2(B(R))} \|\nabla f\|_{L^\infty(B(R);\mathbb{R}^d)}^2.
\]

In particular, if \( \|f\|_{L^2(B(R))} = 1 \) we find

\[
d_W \left( \int_{B(R)} f(x) (\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \left| \int_{B(R)} f^3(x) \lambda(dx) \right| + 2 (K_d v_d R)^2 \|\nabla f\|_{L^\infty(B(R);\mathbb{R}^d)}^2.
\]

As an example, consider \( f_k \) given on \( B(k^{1/d} R) \) by

\[
f_k(x) := \frac{1}{C \sqrt{k}} g \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right), \quad x \in B(k^{1/d} R),
\]

where \( g \in C^1([0, R]) \) is such that \( g(R) = 0 \), and

\[
C^2 := v_d \int_0^R g^2(r) r^{d-1} dr,
\]

so that \( f_k \in L^2(B(k^{1/d} R)) \) with

\[
\|f\|_{L^2(B(k^{1/d} R))}^2 = \frac{v_d}{C^2 k} \int_0^{k^{1/d} R} g^2 \left( \frac{r}{k^{1/d}} \right) r^{d-1} dr = \frac{v_d}{C^2} \int_0^R g^2(r) r^{d-1} dr = 1,
\]

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and
\[ \int_{B(k^{1/d}R)} f_k^3(x) \, dx = \frac{1}{N^{3/2}} \int_0^{k^{1/d}R} g^3(\sqrt{r} \gamma(x)) \, dx = \frac{1}{N^{3/2}} \int_0^R g^3(\sqrt{r}) \, dr, \]
for \( k \geq 1 \). We have
\[ \|\nabla^d f_k\|_{L^\infty(B(R);\mathbb{R}^d)}^2 \leq \frac{\|g\|_{L^\infty}^2}{C^2k^{1+2/d}}, \]
hence
\[ d_W \left( \int_{B(R)} f_k(x) \gamma(dx) - \lambda(dx), \mathcal{N} \right) \leq \left| \int_{B(R)} \left( f_k^3(x) \gamma(dx) - \lambda(dx) \right) \right| + \frac{2(K_d x d R^d)^{2d}}{k^{1+2/d}C^2} \|g\|_{L^\infty}^2 \]
\[ \leq \frac{\nu_d}{C^3k} \left( \int_0^R g^3(r) r^{d-1} \, dr \right)^{1/2} + \frac{2(K_d x d R^d)^{2d}}{k^{1+2/d}C^2} \|g\|_{L^\infty}^2. \]
In particular, if \( g \) satisfies the condition
\[ \int_0^R g^3(r) r^{d-1} \, dr = 0, \]
then we find the \( O(1/k) \) convergence rate
\[ d_W \left( \int_{B(R)} f_k(x) \gamma(dx) - \lambda(dx), \mathcal{N} \right) \leq \frac{2(K_d x d R^d)^{2d}}{kC^2} \|g\|_{L^\infty}^2, \quad k \geq 1. \]
For example, taking
\[ f_k(x) := \frac{1}{C\sqrt{k}^{1/d}} g \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right) = \frac{1}{C\sqrt{k}} \left( h_1 \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right) - a h_2 \left( \frac{|x|_{\mathbb{R}^d}}{k^{1/d}} \right) \right), \quad x \in B(k^{1/d}R), \]
with \( a \in \mathbb{R}, h_1, h_2 \in C^1([0, R]) \) such that \( h_1(R) = h_2(R) = 0 \), and
\[ C^2 := \int_0^R (h_1(r) - a h_2(r))^2 r^{d-1} \, dr > 0, \]
we can choose \( a \in \mathbb{R} \) satisfying the cubic equation
\[ \int_{B(R)} g^3(r) r^{d-1} \, dr \]
\[ = a^3 \int_0^R h_2^3(r) r^{d-1} \, dr + 3a^2 \int_0^R h_1(r) h_2^2(r) r^{d-1} \, dr - 3a \int_0^R h_1^2(r) h_2(r) r^{d-1} \, dr + \int_0^R h_1^3(r) r^{d-1} \, dr \]
\[ = 0, \]
which yields the bound
\[ d_W \left( \int_{B(k^{1/d}R)} f_k(x) \gamma(dx) - \lambda(dx), \mathcal{N} \right) \leq \frac{c(a, d, h_1, h_2)}{k}, \quad k \geq 1, \]
from (1.5), where \( c(a, d, h_1, h_2) \) depends only on \( a \in \mathbb{R}, d \geq 2 \) and \( h_1, h_2 \in C^1([0, R]) \), whereas (1.3) can only yield the standard Berry-Esseen convergence rate (1.4) as \( \int_0^R |g(r)|^3 r^{d-1} \, dr > 0. \)
as a consequence of (1.8) and (2.2) we have

\[
\begin{align*}
&f_n(x_1, \ldots, x_i-1, y, x_{i+1}, \ldots, x_n) - f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\
&= f_n(x_1, \ldots, x_i-1, y, x_{i+1}, \ldots, x_n) - f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \int_{B(R') \setminus B(R)} \eta(x) \lambda(dx) \\
&= f_n(x_1, \ldots, x_i-1, y, x_{i+1}, \ldots, x_n) - \int_{B(R') \setminus B(R)} \eta(x) f_n(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \lambda(dx) \\
&= \int_{B(R')} (G(x, y), \nabla^y f_n(x_1, \ldots, x_n)) \mathbb{R}^d \lambda(dx) \\
&= \int_{B(R')} (G(x, y), \nabla^y f_n(x_1, \ldots, x_n)) \mathbb{R}^d \lambda(dx), \tag{6.1}
\end{align*}
\]

6 Appendix

Proof of Proposition 2.3.

As a consequence of (1.8) and (2.2) we have

\[
\mathbb{E}[F] = e^{-\lambda(B(R))} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B(R)} \cdots \int_{B(R)} f_n(x_1, \ldots, x_n) \lambda(dx_1) \cdots \lambda(dx_n).
\]

Hence, using (6.1), for \( g \in C_0^1(B(R)) \) and \( F \) of the form (2.1) we have

\[
\begin{align*}
&\mathbb{E} \left[ \int_{B(R)} g(y) D_y F \lambda(dy) \right] \\
&= \mathbb{E} \left[ \sum_{n=1}^{\infty} \frac{1}{n!} \int_{B(R)} \cdots \int_{B(R)} \int_{B(R)} \cdots \int_{B(R)} g(y) \langle G(x, y), \nabla^y f_n(x_1, \ldots, x_n) \rangle \mathbb{R}^d \lambda(dy) \lambda(dx_1) \cdots \lambda(dx_n) \right] \tag{6.2}
\end{align*}
\]
= \mathbb{E} \left[ F \left( \int_{B(R)} g(x)(\gamma(dx) - \lambda(dx)) \right) \right].

Next, for \( u \) of the form (2.4), we check by a standard argument that

\[
\mathbb{E}[\langle u, DF \rangle] = \sum_{i=1}^{n} \mathbb{E}[G_i \langle g_i, DF \rangle]
\]

\[
= \sum_{i=1}^{n} \left( \mathbb{E}[\langle g_i, D(FG_i) \rangle] - F\langle g_i, DG_i \rangle] \right)
\]

\[
= \mathbb{E} \left[ F \sum_{i=1}^{n} \left( G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle \right) \right]
\]

\[
= \mathbb{E}[F\delta(u)].
\]

\[\square\]

**Proof of Proposition 2.4.** Taking \( u \in \mathcal{P}_0 \) a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.10) we have

\[ g_i(y)D_y F_i = 0, \quad y \in B(R), \quad i = 1, \ldots, n, \]

hence by (2.5) we have

\[ \delta(u) = \delta \left( \sum_{i=1}^{n} F_i g_i \right) = \sum_{i=1}^{n} F_i \delta(g_i) \]

(6.3)

\[
= \sum_{i=1}^{n} F_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx))
\]

\[
= \int_{B(R)} u_x(\gamma(dx) - \lambda(dx)),
\]

showing that \( \delta(u) \) coincides with the Poisson stochastic integral of \( (u_x)_{x \in B(R)} \). Regarding the isometry relation (2.12), we have

\[
\mathbb{E}[\delta(u)^2] = \mathbb{E} \left[ \left( \sum_{i=1}^{n} F_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \right)^2 \right]
\]

\[
= \mathbb{E} \left[ \sum_{i,j=1}^{n} F_i F_j \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \int_{B(R)} g_j(x)(\gamma(dx) - \lambda(dx)) \right]
\]

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\[ = 2 \mathbb{E} \left[ \sum_{1 \leq i < j \leq n} F_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) F_j \int_{B(R)} g_j(x)(\gamma(dx) - \lambda(dx)) \right] \]
\[ + \mathbb{E} \left[ \sum_{i=1}^{n} F_i^2 \left( \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \right)^2 \right] \]
\[ = \mathbb{E} \left[ \sum_{i=1}^{n} F_i^2 \int_{B(R)} g_i^2(x) \lambda(dx) \right] \]
\[ = \mathbb{E} \left[ \int_{B(R)} u^2(x) \lambda(dx) \right], \]

which shows that (2.11) extends to the closure of \( \mathcal{P}_0 \) in \( L^2(\Omega \times B(R)) \) by density and a Cauchy sequence argument. \( \Box \)

References


