A LOGARITHMIC SOBOLEV INEQUALITY FOR AN INTERACTING SPIN SYSTEM UNDER A GEOMETRIC REFERENCE MEASURE

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Logarithmic Sobolev inequalities are an essential tool in the study of interacting particle systems, cf. e.g. 4, 5. In this note we show that the logarithmic Sobolev inequality proved on the configuration space $\mathbb{N}^d$ under Poisson reference measures in 1 can be extended to geometric reference measures using the results of 2. As a corollary we obtain a deviation estimate for an interacting particle system.

1. Logarithmic Sobolev inequality for the geometric distribution

Consider the forward and backward gradient operators
\[ d^+ f(k) = f(k + 1) - f(k), \quad d^- f(k) = 1_{\{k \geq 1\}}(f(k - 1) - f(k)), \quad k \in \mathbb{N}, \]
and the Laplacian
\[ \mathcal{L} = -d_\pi^* d^+ + \frac{1}{p} d^- \]
which generates a Markov process on $\mathbb{N}$ whose invariant measure is the geometric distribution $\pi$ on $\mathbb{N}$ with parameter $p \in (0, 1)$, i.e.
\[ \pi(\{k\}) = (1 - p)p^k, \quad k \in \mathbb{N}. \]
Denote by $E_\pi$ the expectation under $\pi$ and by $\text{Ent}_\pi$ the entropy under $\pi$, defined as

$$\text{Ent}_\pi[f] = E_\pi[f \log f] - E_\pi[f] \log E_\pi[f].$$

We recall the modified logarithmic Sobolev inequality proved in \cite{2} for the geometric distribution $\pi$.

**Theorem 1.1.** Let $0 < c < -\log p$ and let $f : \mathbb{N} \to \mathbb{R}$ such that $|d^+ f| \leq c$. We have

$$\text{Ent}_\pi[\exp f] \leq \frac{pe^c}{(1-p)(1-\sqrt{pe^c})} E_\pi[|d^+ f|^2 e^f]. \quad (1.1)$$

In higher dimensions the multi-dimensional gradient is defined as

$$d^+_i f(k) = f(k + e_i) - f(k), \quad i = 1, \ldots, n,$$

where $f$ is a function on $\mathbb{N}^n$, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$, and the gradient norm is

$$\|d^+ f(k)\|^2 = \sum_{i=1}^n |d^+_i f(k)|^2 = \sum_{i=1}^n |f(k + e_i) - f(k)|^2. \quad (1.2)$$

From the tensorization property of entropy, (1.1) still holds with respect to $\pi^\otimes n$ in any finite dimension $n$: 

$$\text{Ent}_{\pi^\otimes n}[\exp f] \leq \frac{pe^c}{(1-p)(1-\sqrt{pe^c})} E_{\pi^\otimes n}[||d^+ f||^2 e^f], \quad (1.3)$$

provided $|d_i f| \leq c$, $i = 1, \ldots, n$. As a consequence the following deviation inequality for functions of several variables under $\pi^\otimes n$ has been proved in \cite{2} using (1.1) and the Herbst method.

**Corollary 1.2.** Let $0 < c < -\log p$ and let $f$ such that $|d^+_i f| \leq \beta$, $i = 1, \ldots, n$, and $\|d^+ f\|^2 \leq \alpha^2$ for some $\alpha, \beta > 0$. Then for all $r > 0$,

$$\pi^\otimes n(f - E_{\pi^\otimes n}[f] \geq r) \leq \exp \left( - \min \left( \frac{c^2 r^2}{4a_{p,c} \alpha^2 \beta^2}, \frac{r \beta}{\alpha^2 a_{p,c}} \right) \right), \quad (1.4)$$

where

$$a_{p,c} = \frac{pe^c}{(1-p)(1-\sqrt{pe^c})}$$

denotes the logarithmic Sobolev constant in (1.1).

Our goal in the next section will be to extend these results to interacting spin systems under a geometric reference measure.
2. Logarithmic Sobolev inequality for an interacting spin system

Given a bounded finite range interaction potential $\Phi = \{ \Phi_R : R \subset \mathbb{Z}^d \}$, i.e.

$$\|\Phi\| = \sup_{k \in \mathbb{Z}^d} \sum_{R \ni k} \|\Phi_R\|_\infty < \infty,$$

let the Hamiltonian $H_\Lambda$ be defined as

$$H_\Lambda(\eta) = \sum_{R \cap \Lambda \neq \emptyset} \Phi_R(\eta_R),$$

where $\eta_R$ denotes the restriction of $\eta$ to $\mathbb{N}^R$, $R \subset \mathbb{Z}^d$. The Gibbs measure $\pi^\omega_\Lambda$ on $\mathbb{N}^\Lambda$ associated to a $\mathbb{N}$-valued spin system on a finite lattice $\Lambda \subset \mathbb{Z}^d$ with boundary condition $\omega \in \mathbb{N}^{\mathbb{Z}^d \setminus \Lambda}$ is defined by its density with respect to $\pi := \pi^\otimes \Lambda$ as:

$$\frac{d\pi^\omega_\Lambda}{d\pi_\Lambda}(\sigma) = \frac{1}{Z_\Lambda} e^{-H_\Lambda^\omega(\sigma)}, \quad \sigma \in \mathbb{N}^\Lambda,$$

where $\pi$ is the geometric reference distribution on $\mathbb{N}$, $Z_\Lambda$ is a normalization factor, and

$$H_\Lambda^\omega(\eta) = H_\Lambda(\eta_\Lambda \omega^\Lambda), \quad \eta \in \mathbb{N}^{\mathbb{Z}^d},$$

where $\eta_\Lambda \omega^\Lambda$ is defined as

$$(\eta_\Lambda \omega^\Lambda)_k = \eta_k 1_\Lambda(k) + \omega_k 1_B(k), \quad k \in \mathbb{Z}^d,$$

whenever $\eta \in \mathbb{N}^\Lambda$, $\omega \in \mathbb{N}^B$, and $A, B \subset \mathbb{Z}^d$ are such that $A \cap B = \emptyset$. Let again $d_k^+ f(\eta) = f(\eta + e_k) - f(\eta)$, and $d_k^- f(\eta) = 1_{\{\eta_k > 0\}} (f(\eta - e_k) - f(\eta))$, $\eta \in \mathbb{N}^{\mathbb{Z}^d}, k \in \mathbb{Z}^d$, for every function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$, where $(e_k)_{k \in \mathbb{Z}^d}$, denotes the canonical basis $\{e_k = 1_k : k \in \mathbb{Z}^d\}$. Consider the Markov generator

$$\mathcal{L}_\Lambda^\omega f(\eta) = \sum_{k \in \Lambda} (c_\Lambda^+(k, \eta, +) d_k^+ f(\eta) + c_\Lambda^-(k, \eta, -) d_k^- f(\eta)),$$

where $c_\Lambda^+(k, \eta, \pm)$ are rate functions such that $\mathcal{L}_\Lambda^\omega$ is self-adjoint in $L^2(\pi^\Lambda_\Lambda)$, i.e.

$c_\Lambda^+(k, \eta, +) \pi^\Lambda_\Lambda(\eta) = c_\Lambda^+(k, \eta + e_k, -) \pi^\Lambda_\Lambda(\eta + e_k), \quad k \in \Lambda, \quad \eta \in \mathbb{N}^\Lambda,$

$c_\Lambda^+(k, \eta, -) \pi^\Lambda_\Lambda(\eta) = c_\Lambda^+(k, \eta - e_k, +) \pi^\Lambda_\Lambda(\eta - e_k), \quad k \in \Lambda, \quad \eta \in \mathbb{N}^\Lambda,$
\[ \eta_k > 0, \text{cf. } 1. \] We assume that there exists a constant \( C > 0 \) depending on \( \|\Phi\| \) only, with
\[ \frac{1}{C} \leq e^\alpha(k, \eta, +) \leq C, \quad \eta \in \mathbb{N}^A, \ A \subset \mathbb{Z}^d, \ k \in \Lambda. \quad (2.1) \]
For \( f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R} \) we let:
\[ E_\omega^\Lambda(e^f) = \sum_{k \in \Lambda} \int_{\mathbb{N}^{\mathbb{Z}^d}} c_\omega^\Lambda(k, \sigma, +) e^f(\sigma) \left| d^+_k f(\sigma) \right|^2 d\pi_\omega^\Lambda(\sigma), \]
and
\[ E_\Lambda(e^f) = \sum_{k \in \Lambda} \int_{\mathbb{N}^{\mathbb{Z}^d}} e^f(\sigma) \left| d^+_k f(\sigma) \right|^2 d\pi_\Lambda(\sigma). \]
Next we consider the family of rectangles of the form
\[ R = R(k, l_1, \ldots, l_d) = k + ([1, l_1] \times \cdots \times [1, l_d]) \cap \mathbb{Z}^d, \]
where \( k \in \mathbb{Z}^d \) and \( l_1, \ldots, l_d \in \mathbb{N} \), with
\[ \text{size}(R) = \max_{k=1, \ldots, d} l_k. \]
Let \( \mathcal{R}_L \) denote the set of rectangles such that
\[ \text{size}(R) \leq L \quad \text{and} \quad \text{size}(R) \leq 10 \min_{k=1, \ldots, d} l_k. \]

**Definition 2.1.** We say that \( \pi_\Lambda^\omega \) satisfies the mixing condition if there exists constants \( C_1 \) and \( C_2 \), depending on \( d \) and \( \|\Phi\| \) only, such that:
\[ \sup_{\sigma, \omega} \left| \frac{\pi_\Lambda^\omega(\{\eta : \eta_A = \sigma_A\}) \pi_\Lambda^\omega(\{\eta : \eta_B = \sigma_B\})}{\pi_\Lambda^\omega(\{\eta : \eta_{A \cup B} = \sigma_{A \cup B}\})} - 1 \right| \leq C_1 e^{-C_2 d(A, B)}, \quad (2.2) \]
for all \( L \geq 1, \ A \in \mathcal{R}_L \) and \( B \subset \Lambda \) such that \( A, B \in \mathcal{R}_L \) with \( A \cap B = \emptyset \).
We refer to \(^1\) and \(^4\) for conditions on \( \Phi \) under which \((2.2)\) holds under a geometric reference measure.

Our goal is to prove the following logarithmic Sobolev inequality under the Gibbs measure \( \pi_\Lambda^\omega \).

**Theorem 2.2.** Assume that the mixing condition \((2.2)\) holds, and let \( 0 < c < -\log p \). Then there exists a constant \( \gamma_c > 0 \), independent of \( \Lambda \) and \( \omega \), such that
\[ \text{Ent}_{\pi_\Lambda^\omega}(e^f) \leq \gamma_c e^{E_\Lambda^\omega(e^f)}, \quad (2.3) \]
for every \( f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R} \) such that \( \|d^+ f\|_{l^\infty(\Lambda)} \leq c, \ \pi_\Lambda^\omega \text{-a.e.} \).
In particular we have
\[
\text{Ent}_{\pi_{\Lambda}}[e^f] \leq \gamma_c \left\| \sum_{k \in \Lambda} c^\omega(k, \cdot, +)d^+_k f(\cdot)^2 \right\|_{L^\infty(\pi_{\Lambda})} \times \mathbb{E}_{\pi_{\Lambda}}[e^f],
\]
which implies, as in Corollary 1.2, a deviation inequality under Gibbs measures.

**Corollary 2.3.** Assume that the mixing condition (2.2) holds, and let 0 < c < −\log p. Let f be such that
\[
\left\| d^+_k f(\eta) \right\|_{L^\infty(\pi_{\Lambda})} \leq \beta \quad \text{and} \quad \sum_{k \in \Lambda} c^\omega(k, \eta, +)\left| d^+_k f(\eta) \right|^2 \leq \alpha^2,
\]
for some α, β > 0. Then for all r > 0,
\[
\pi_{\Lambda}(f - \mathbb{E}_{\pi_{\Lambda}}[f] \geq r) \leq \exp \left( -\min \left( \frac{c^2 r^2}{4 \gamma_c \alpha^2 \beta^2}, \frac{r c}{\beta} - \alpha^2 \gamma_c \right) \right).
\]

Due to Hypothesis (2.1), condition (2.4) can be replaced by
\[
\left\| d^+_f(\eta) \right\|_{\mathbb{P}(\Lambda)}^2 \leq C^{-1} \alpha^2, \quad \pi_{\Lambda}(d\eta) - a.e.
\]
Denoting by \(\Pi\) denote the infinite volume Gibbs measure associated to \(\pi_{\Lambda}\), for some \(r_0 > 0\) we get the Ruelle type bound:

\[
\Pi(\{\eta \in \mathbb{N}^Z : \|\eta\| \geq r\}) \leq \exp (-r c - C \gamma_c) + c \mathbb{E}_{\Pi}(\|\eta\|), \quad r > r_0,
\]
for all finite subset \(\Lambda\) of \(\mathbb{Z}^d\), under the mixing condition (2.2). Indeed, it suffices to apply the uniform bound (2.5) with \(f(\eta) = |\eta|\), \(\alpha^2 = C|\Lambda|\), \(\beta = 1\), and the compatibility condition

\[
\Pi(E) = \int_{\mathbb{N}^Z} \pi_{\Lambda}(E_\Lambda) d\omega,
\]
to \(E = \{\eta \in \mathbb{N}^Z : |\eta| \geq r|\Lambda|\}\), with

\[
E_\Lambda := \{\eta \in \mathbb{N}^\Lambda : \eta_\Lambda \omega_\Lambda \in E\} = \{\eta \in \mathbb{N}^\Lambda : |\eta_\Lambda| \geq r|\Lambda|\}.
\]
This shows in particular that \(\Pi\) satisfies the \((RPB)^1\) condition in 3.

### 3. Proof of Theorem 2.2
Recall that for 0 < c < −\log p, by tensorization, Theorem 1.1 yields as in (1.3) the logarithmic Sobolev inequality
\[
\text{Ent}_{\pi_{\Lambda}}[e^f] \leq s_c \mathcal{E}(e^f),
\]
for some \(s_c > 0\).
for all $f : \mathbb{N}^d \to \mathbb{R}$ such that $\|d^+ f\|_{l^\infty(\Lambda)} \leq c$, $\pi_{\Lambda}$-a.e., with an optimal constant $s_c \leq a_{p,c}$ which is independent of $\Lambda \subset \mathbb{Z}^d$. Let now $s_{\Lambda,\omega,c}$ denote the optimal constant in the inequality

$$\text{Ent}_{\pi_{\Lambda}}[e^f] \leq s_{\Lambda,\omega,c} \delta_{\Lambda}^c(e^f), \quad \|d^+ f\|_{l^\infty(\Lambda)} \leq c.$$  

**Lemma 3.1.** For every $\Lambda \subset \mathbb{Z}^d$, there exists a constant $A := C e^{4|\Lambda||\Phi|} < \infty$ depending only on $|\Lambda|$, $c$ and independent of $\omega \in \mathbb{N}^{\mathbb{Z}^d \setminus \Lambda}$, such that

$$\frac{s_c}{A} \leq s_{\Lambda,\omega,c} \leq A s_c.$$

**Proof.** We follow the proof of Proposition 3.1 in 1. From (2.1) we obtain:

$$C - e^{-2|\Lambda||\Phi|} \delta_{\Lambda}^c(e^f) \leq e^{2|\Lambda||\Phi|} \delta_{\Lambda}^c(e^f).$$

(3.2)

From the relation

$$\text{Ent}_{\pi_{\Lambda}}[e^f] = \min_{t > 0} \mathbb{E}_{\pi_{\Lambda}}[f \log f - f \log t - f + t]$$

and the bound

$$e^{-2|\Lambda||\Phi|} \leq \frac{d\pi_{\Lambda}}{d\pi_{\Lambda}} \leq e^{2|\Lambda||\Phi|},$$

we have

$$e^{-2|\Lambda||\Phi|} \text{Ent}_{\pi_{\Lambda}}[e^f] \leq \text{Ent}_{\pi_{\Lambda}}[e^f] \leq e^{2|\Lambda||\Phi|} \text{Ent}_{\pi_{\Lambda}}[e^f],$$

from which the conclusion follows using (3.1) and (3.2). \hfill \Box

Let for $L \geq 1$:

$$S_{L,c} := \sup_{R \in \mathcal{R}_L} \sup_{\omega \in \mathbb{N}^{\mathbb{Z}^d \setminus \Lambda}} s_{R,\omega,c} \leq C s_c e^{4|\Lambda||\Phi|} < \infty,$$

which is finite by Lemma 3.1.

**Proposition 3.2.** Assume the mixing condition (2.2) is satisfied. Then there exists a constant $\kappa$ depending on $\||\Phi||$, such that

$$S_{2L,c} \leq \left(1 - \frac{\kappa}{\sqrt{L}}\right)^{-1} S_{L,c}$$

(3.3)

for $L$ large enough.

**Proof.** The proof of this proposition is identical to that of Proposition 4.1, pp. 1970-1972 and Proposition 5.1, p. 1975 in 1, replacing the Dirichlet form used in 1 with $\delta_{\Lambda}^c$. \hfill \Box

Finally, Theorem 2.2 is proved by taking $\gamma_c = \sup_L S_{L,c}$, which is finite from Proposition 3.2.
References


