INDEPENDENCE OF A CLASS OF MULTIPLE STOCHASTIC INTEGRALS

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Abstract

We show that two multiple stochastic integrals $I_n(f_n)$, $I_n(g_m)$ with respect to the solution $(M_t)_{t \in \mathbb{R}_+}$ of a deterministic structure equation are independent if and only if two contractions of $f_n$ and $g_m$, denoted as $f_n \circ_1 g_m$, $f_n \circ_1 g_m$, vanish almost everywhere.

1 Introduction

This paper aims to extend the necessary and sufficient conditions for the independence of single or multiple stochastic integrals of [12], [14], [15], [16], [17], cf. also [6], [7], proving and extending results that have been partially announced in [9]. Let $(M_t)_{t \in \mathbb{R}_+}$ be a martingale satisfying the structure equation

$$d[M,M]_t = dt + \phi_t dM_t,$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a measurable deterministic function. Such martingales are normal in the sense of [2], i.e. $d < M, M >_t = dt, t \in \mathbb{R}_+$ and they satisfy the chaos representation property, cf. [3]. Moreover, they have independent increments, and if $(B_t)_{t \in \mathbb{R}_+}$, $(N_t)_{t \in \mathbb{R}_+}$ are independent standard Brownian motion and Poisson process of intensity $ds/\phi_2$, then $(M_t)_{t \in \mathbb{R}_+}$ can be represented as

$$M_t = \int_0^t 1_{\{\phi_s = 0\}} dB_s + \int_0^t \phi_s \left( dN_s - \frac{ds}{\phi_2^2} \right), \quad t \in \mathbb{R}_+.$$  

We choose to construct the processes $(B_t)_{t \in \mathbb{R}_+}$ on the classical Wiener space $(\Omega_1, \mathcal{F}_1, P_1)$, where $\Omega_1$ is the space of cadlag functions starting at zero. We denote by $(\Omega_2, \mathcal{F}_2, P_2)$ the space

$$\Omega_2 = \left\{ \sum_{i=1}^N \delta_{t_i} : (t_i)_{i=1, \ldots, N} \in \mathbb{R}_+, N \in \mathbb{N} \cup \{\infty\} \right\},$$

with the $\sigma$-algebra and probability measure $\mathcal{F}_2$, $P_2$ under which the canonical random measure is Poisson with mean measure $\mu$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ defined as

$$\mu(A) = \int_{A \cap \{\phi \neq 0\}} \frac{1}{\phi_2^2} ds, \quad A \in \mathcal{B}(\mathbb{R}_+).$$

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With this notation, \((N_t)_{t \in \mathbb{R}_+}\) is written as \(N_t(\omega_2) = \omega_2([0, t])\), and \((B_t)_{t \in \mathbb{R}_+}\) satisfies \(B_t(\omega_1) = \omega_1(t), \ t \in \mathbb{R}_+\). For \(A \in \mathcal{B}(\mathbb{R}_+)\) we call \(\mathcal{F}_t^A\) the \(\sigma\)-algebra on \(\Omega_2\) generated by all random variables \(\omega_2 \mapsto \omega_2(A \cap B), B \in \mathcal{B}(\mathbb{R}_+)\). The martingale \(M\) is then explicitly constructed as \(M_t(\omega_1, \omega_2) = X_t(\omega_2) + B_t(\omega_1), \ t \in \mathbb{R}_+,\) on \((\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)\), where

\[
X_t = \int_0^t \phi_s(dN_s - ds/\phi_t^2), \quad t \in \mathbb{R}_+.
\]

If \(f_n \in L^2(\mathbb{R})^{\otimes n}\), the multiple stochastic integral with respect to \(M, X, B\) of \(f_n\) are respectively defined as

\[
I_n(f_n) = n! \int_0^\infty \cdots \int_0^\infty \int_0^t f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n}, \quad (3)
\]

\[
I_n^X(f_n) = n! \int_0^\infty \cdots \int_0^t \int_0^\infty f_n(t_1, \ldots, t_n) dX_{t_1} \cdots dX_{t_n}, \quad (4)
\]

\[
I_n^B(f_n) = n! \int_0^\infty \int_0^t \int_0^\infty f_n(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_n}, \quad (5)
\]

where \(\hat{f}_n\) is the symmetrization in \(n\) variables of \(f_n\). We note the relation

\[
I_n(f_n) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{k} I_{n-k}^X(I_k^B(\hat{f}_n)) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{k} I_{n-k}^B(I_k^X(\hat{f}_n)), \quad (6)
\]

Let \(L^2(\mathbb{R}_+)^{\otimes n}\) denote the subspace of \(L^2(\mathbb{R}_+)^{\otimes n}\) made of symmetric functions. Let \(f_n \otimes g_m\) denote the completed tensor product of two functions \(f_n \in L^2(\mathbb{R}_+)^{\otimes n}\) and \(g_m \in L^2(\mathbb{R}_+)^{\otimes m}\), and let \(f_n \circ g_m\) denote the symmetrization of \(f_n \otimes g_m, n, m \in \mathbb{N}\). Since \(d < M, M > 1 = dt\), we have

\[
E[I_n(f_n)I_m(g_m)] = n!(f_n, g_m)_{L^2(\mathbb{R}_+)^{\otimes n}} \mathbb{1}_{\{n=m\}}, \quad f_n \in L^2(\mathbb{R}_+)^{\otimes n}, g_m \in L^2(\mathbb{R}_+)^{\otimes m}, \quad (7)
\]

Since \((M_t)_{t \in \mathbb{R}_+}\) has the chaos representation property, any square integrable functional \(F \in L^2(\Omega, \mathcal{F}, P)\) has a chaos expansion

\[
F = \sum_{n \geq 0} I_n(f_n), \quad f_n \in L^2(\mathbb{R}_+)^{\otimes k}, k \geq 0.
\]

A linear operator \(\nabla : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbb{R}_+)^{\otimes n}\) is defined by annihilation as

\[
\nabla f_n(\omega) = nI_{n-1}(f_n(\cdot, t)), \quad t \in \mathbb{R}_+, \quad (8)
\]

\(f_n \in L^2(\mathbb{R}_+)^{\otimes n}, n \in \mathbb{N}^*,\) cf. e.g. [5]. This operator is closable, of \(L^2\)-domain \(Dom_2(\nabla)\), and its closed adjoint \(\nabla^* : L^2(\Omega) \otimes L^2(\mathbb{R}_+) \to L^2(\Omega)\) satisfies

\[
\nabla^* I_n(f_{n+1}) = I_{n+1}(\hat{f}_{n+1}),
\]
\[ f_{n+1} \in L^2(\mathbb{R}_+)^{\infty} \otimes L^2(\mathbb{R}_+). \] We denote by \( \text{Dom}_1(\nabla) \) the set of functionals \( F \in L^2(\Omega) \) such that there exists a sequence \( (F_n)_{n \in \mathbb{N}} \subset \text{Dom}_2(\nabla) \) converging to \( F \) in \( L^2(\Omega) \) and such that \( (\nabla F_n)_{n \in \mathbb{N}} \) converges in \( L^1(\Omega \times \mathbb{R}_+) \). The limit of the sequence \( (\nabla F_n)_{n \in \mathbb{N}} \) is denoted \( \nabla F \) which is well-defined, due to the relation
\[
E[|\nabla F_n|_{L^2(\mathbb{R}_+)}] = E[|\nabla F|^*(u)], \quad n \in \mathbb{N},
\]
\( u \in \text{Dom}(\nabla^*) \cap L^\infty(\Omega \times \mathbb{R}_+) \), and since \( \text{Dom}(\nabla^*) \cap L^\infty(\Omega \times \mathbb{R}_+) \) is dense in \( L^1(\Omega \times \mathbb{R}_+) \). For \( f_n \in L^2(\mathbb{R}_+)^{\infty} \) and \( g_m \in L^2(\mathbb{R}_+)^{\infty} \), we define \( f_n \otimes_k g_m \), \( 0 \leq l \leq k \), to be the function\[
(x_{i+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \mapsto \phi(x_{i+1}) \cdots \phi(x_k) \int_{\mathbb{R}^l} f_n(x_1, \ldots, x_n)g_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m)dx_1 \cdots dx_l
\]of \( n+m-k-l \) variables. We denote by \( f_n \otimes_k^l g_m \) the symmetrization in \( n+m-k-l \) variables of \( f_n \otimes_k g_m \), \( 0 \leq l \leq k \).

**Definition 1** Let \( \mathcal{S} \) denote the vector space in \( L^2(\Omega) \) generated by
\[
\{I_n(f_1 \cdots f_n) : f_1, \ldots, f_n \in \mathcal{C}_c(\mathbb{R}_+), \ n \geq 1\}.
\]
The vector space \( \mathcal{S} \) is dense in \( L^2(\Omega) \). For \( F \in \mathcal{S} \) and \( f \in L^2(\mathbb{R}_+) \), we have from a general result in quantum stochastic calculus, cf. for example Th. II.1 of [1]:
\[
F \int_0^\infty f(s)dM_s = \int_0^\infty f(s)\nabla_s Fds + \nabla^*(JF) + \nabla^*(\phi f \nabla F).
\]
(9)
This formula is usually stated under the form
\[
\int_0^\infty f(s)dM_s = \int_0^\infty f(s)da^- + \int_0^\infty f(s)da^+ + \int_0^\infty \phi_s f(s)da^0
\]
by quantum probabilists, where \( \int_0^\infty f(s)dM_s \) is identified to a multiplication operator. The identity (9) can be easily rewritten into a multiplication formula between first and \( n \)th order stochastic integrals:

\[
I_1(h)I_n(f_n) = I_{n+1}(f_n \circ h) + n \int_0^\infty h_tI_{n-1}(f_n(\cdot, t))dt + nI_n(f_n \circ_1^0(\phi h)).
\]
(10)
We note that as a consequence of this formula, every element of \( \mathcal{S} \) has a unique expression as a polynomial in single stochastic integrals and conversely, any polynomial in stochastic integrals has a finite chaos expansion.

**Remark 1** This implies that each element of \( \mathcal{S} \) has a version which is defined for every \( \omega \) \( (\omega_1, \omega_2) \in \Omega \), since \( I_1(f) \in \mathcal{S} \) can be written as
\[
I_1(f) = -\int_0^\infty f'(s)B_s 1_{\{\phi_s = 0\}}ds + \sum_{\{t : dN_t = 1\}} \phi_t f(t) - \int_0^\infty 1_{\{\phi_s \neq 0\}}f(s)\frac{1}{\phi_s}ds.
\]
Throughout this paper, \( F \in \mathcal{S} \) will always refer to the version of \( F \) defined via the above identity.
From (10), one can prove the following result which shows that the function $\phi$ accounts for the perturbation of the usual derivation rule for the Malliavin derivative on Wiener space.

**Proposition 1** For any $F, G \in S$ we have
\[
\nabla_t(FG) = F \nabla_t G + G \nabla_t F + \phi_t \nabla_t F \nabla_t G, \quad t \in \mathbb{R}_+.
\]
(11)
If $\phi \in L^\infty(\mathbb{R}_+)$ then for any $F, G \in \text{Dom}_2(\nabla)$, we have $FG \in \text{Dom}_1(\nabla)$ and the above relation holds.

**Proof.** We first notice that for $F = I_1(h)$ and $G = I_n(f_n)$, this formula is a consequence of the multiplication formula (10), since
\[
\nabla_t(I_1(h)I_n(f_n)) \\
= \nabla_t \left( I_{n+1}(f_n \circ h) + n \int_0^\infty h_s I_{n-1}(f_n(\cdot, s) ds + n I_n(f_n \circ h) \right)
\]
\[
= I_n(f_n) \nabla I_1(h) + n I_n(f_n(\cdot, t) \circ h) + n(n-1) \int_0^\infty h_s I_{n-2}(f_n(\cdot, t, s)) ds \\
+ n(n-1) I_n(f_n(\cdot, t) \circ h) + \phi_t \nabla I_1(h) I_n(f_n) \\
= I_n(f_n) \nabla I_1(h) + I_1 \nabla I_n(f_n) + \phi_t \nabla I_1(h) \nabla I_n(f_n).
\]
Next, we prove by induction on $k \geq 1$ that
\[
\nabla_t(I_n(f_n)I_1(h)^k) = I_1(h) \nabla I_n(f_n) + I_n(f_n) \nabla I_1(h)^k + \phi_t \nabla I_1(h)^k \nabla I_n(f_n).
\]
We have
\[
\nabla_t(I_n(f_n)I_1(h)^{k+1}) \\
= I_1(h)^k \nabla I_n(f_n)I_1(h) + I_n(f_n)I_1(h) \nabla I_1(h)^k \\
+ \phi_t \nabla I_1(h)^k \nabla I_n(f_n)I_1(h) \\
= I_1(h)^k \nabla I_n(f_n)I_1(h) + I_n(f_n)I_1(h) \nabla I_1(h)^k + I_n(f_n)I_1(h)^k \nabla I_n(f_n)I_1(h) \\
+ \phi_t I_n(f_n)I_1(h) \nabla I_1(h)^k + \phi_t I_n(f_n)I_1(h)^k \nabla I_n(f_n)I_1(h) \\
+ \phi_t I_n(f_n)I_1(h)^k \nabla I_n(f_n)I_1(h) \\
= I_1(h)^k \nabla I_n(f_n) + I_n(f_n) \nabla I_1(h)^{k+1} + \phi_t \nabla I_1(h)^{k+1} \nabla I_n(f_n).
\]
Consequently, (11) holds for any polynomial in single stochastic integrals, hence it holds for any $F, G \in S$. In order to prove the second part of the proposition, we assume that $F, G \in \text{Dom}_2(\nabla)$ and choose two sequences $(F_n)_{n \in \mathbb{N}}$ and $(G_n)_{n \in \mathbb{N}}$ contained in $S$, converging respectively to $F$ and $G$ in $L^2(\Omega)$ and such that $(\nabla F_n)_{n \in \mathbb{N}}$ and $(\nabla G_n)_{n \in \mathbb{N}}$ converge to $\nabla F$ and $\nabla G$ in $L^1(\Omega \times \mathbb{R}_+)$. Then $(\phi \nabla F_n \nabla G_n)_{n \in \mathbb{N}}$ converges in $L^1(\Omega \times \mathbb{R}_+)$ to $\phi \nabla F \nabla G$, hence $(\nabla F_n G_n)_{n \in \mathbb{N}}$ converges in $L^1(\Omega \times \mathbb{R}_+)$ to $F \nabla G + G \nabla F + \phi \nabla F \nabla G$, and $FG \in \text{Dom}_1(\nabla)$. □

The product rule for $\nabla$ unifies the chain rule of derivation of the Wiener space Malliavin derivative and the finite difference rule of the Poisson space gradient of [8].
Proposition 2 For any \( F \in S \) we have

\[
\nabla_t F = \lim_{\varepsilon \to 0} \frac{F(M + (\varepsilon + \phi_t)1_{[t, \infty[}(\cdot)) - F(M)}{\varepsilon + \phi_t}, \quad t \in \mathbb{R}_+.
\]

(12)

Proof. The statement (12) can be more precisely formulated as

\[
\nabla_t F(\omega_1, \omega_2) = \lim_{\varepsilon \to 0} \frac{F(\omega_1 + \varepsilon 1_{[t, \infty[}, \omega_2 + \phi_t(\delta_t)) - F(\omega_1, \omega_2)}{\varepsilon + \phi_t},
\]

where the notation \( F \) refers to the version defined in Remark 1. We first show that (12) holds for \( F = I_1(f) \):

\[
\lim_{\varepsilon \to 0} \frac{F(\omega_1 + \varepsilon 1_{[t, \infty[}, \omega_2 + \phi_t(\delta_t)) - F(\omega_1, \omega_2)}{\varepsilon + \phi_t} = \left(1_{\{\phi_t \neq 0\}} \frac{1}{\phi_t} \left( \sum_{\{s : dN_s = 1\}} \phi_s f(s) - \int_0^\infty 1_{\{\phi_s \neq 0\}} f(s) \frac{1}{\phi_s} ds + \phi_t f(t) \right) \right.
\]

\[
- \sum_{\{s : dN_s = 1\}} \phi_s f(s) - \int_0^\infty 1_{\{\phi_s \neq 0\}} f(s) \frac{1}{\phi_s} ds + 1_{\{\phi_t \neq 0\}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( - \int_0^\infty f'(s)(B_s + \varepsilon)1_{[t, \infty[}(s)1_{\{\phi_s \neq 0\}} ds + \int_0^\infty f'(s)B_s 1_{\{\phi_s \neq 0\}} ds \right)
\]

\[
= 1_{\{\phi_t \neq 0\}} f(t) + 1_{\{\phi_t \neq 0\}} f(t) = f(t), \quad t \in \mathbb{R}_+.
\]

Moreover, the limit (12) satisfies the product rule (11), hence if \( F, G \in S \) are of the form \( F = I_1(f) \) and \( G = I_1(g) \), we have

\[
\lim_{\varepsilon \to 0} \frac{(FG) \left(M + (\varepsilon + \phi_t)1_{[t, \infty[}(\cdot)) - (FG)(M)\right)}{\varepsilon + \phi_t} = F \nabla_t G + G \nabla_t F + \phi_t \nabla_t (FG)
\]

\[
= \nabla_t (FG), \quad t \in \mathbb{R}_+.
\]

Thus by induction, (12) holds for any polynomial in single stochastic integrals, and for any element of \( S \). \( \square \)

With help of Prop. 11, the following multiplication formula has been proved in [9], as a generalization of (10). We refer to p. 216 of [2], and to [4], [13], [14], for different versions of this formula in the Poisson case. In [11] a more general result is proven, allowing to represent the product \( I_n(f_n)I_m(g_m) \) as a sum of \( n \wedge m \) terms that are not necessarily linear combinations of multiple stochastic integrals with respect to \( (M_t)_{t \in \mathbb{R}_+} \), except if \( d[M, M]_t \) is a linear deterministic combination of \( dt \) and \( dM_t \), cf. [10].
Proposition 3 The product $I_n(f_n)I_m(g_m) \in L^2(\Omega)$ is in $L^2(B)$ if and only if the function

$$h_{n,m,s} = \sum_{i \leq 2! \leq 2(s \wedge m)} i! \left( \begin{array}{c} n \\ i \\ \end{array} \right) \left( \begin{array}{c} m \\ i \\ \end{array} \right) (s-i)_i^\alpha g_m$$

is in $L^2(\mathbb{R}_+)^{s+2} \cap \cap^m$, $0 \leq s \leq 2(n \wedge m)$, and in this case the chaotic expansion of $I_n(f_n)I_m(g_m)$ is

$$I_n(f_n)I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}).$$ (13)

The fact that $I_n(f_n)I_m(g_m)$ can be expanded as a sum of multiple stochastic integrals with respect to $(M_t)_{t \in \mathbb{R}_+}$ is essential in the proof of independence, cf. Th. 1.

2 Independence of multiple stochastic integrals

In the case of single stochastic integrals, the following proposition extends the result of [15] to a process that does not have stationary increments. In the case of multiple stochastic integrals, it extends the result of [17] since it includes a Poisson component in the martingale $(M_t)_{t \in \mathbb{R}_+}$.

Theorem 1 Let $f_n \in L^2(\mathbb{R}_+)^{s_2} \cap \cap^m$ and $g_m \in L^2(\mathbb{R}_+)^{s_3} \cap \cap^m$. The random variables $I_n(f_n)$ and $I_m(g_m)$ are independent and if and only if $f_n \circ \phi g_m = 0$ and $f_n \circ \phi g_m = 0$ a.e., i.e.

$$\int_0^\infty f_n(t,x_1,x_2,\ldots,x_n)g_m(t,x_{n+1},\ldots,x_{n+m-2})dt = 0, \quad dx_2 \cdots dx_{n+m-2} \text{ a.e. (14)}$$

and

$$f_n(x_1,x_2,\ldots,x_n)g_m(x_1,x_{n+1},\ldots,x_{n+m-1}) = 0, \quad |\phi_x| \cdot dx_1 dx_2 \cdots dx_{n+m-1} \text{ a.e. (15)}$$

Proof. If $I_n(f_n)$ and $I_m(g_m)$ are independent, then $I_n(f_n)I_m(g_m) \in L^2(\Omega, \mathcal{F}, P)$ and following [16],

$$|f_n \circ \phi g_m|_{L^2(\mathbb{R}_+)^{s_2+s_3} \cap \cap^m} = (n+m)! |f_n \circ \phi g_m|_{L^2(\mathbb{R}_+)^{s_2+s_3} \cap \cap^m} \geq n! m! |f_n|_{L^2(\mathbb{R}_+)^{s_2} \cap \cap^m} \cdot |g_m|_{L^2(\mathbb{R}_+)^{s_3} \cap \cap^m} = E[I_n(f_n)^2] \cdot E[I_m(g_m)^2] = E[(I_n(f_n)I_m(g_m))^2]$$

$$= \sum_{r=0}^{2(n \wedge m)} (n+m-r)! |h_{n,m,r}|_{L^2(\mathbb{R}_+)^{s_2+s_3} \cap \cap^m}.$$
We obtain \( f_n \circ_0 g_m = 0 \) a.e., and \( f_n \circ_1 g_m = 0 \) a.e.

Conversely, if (14) is satisfied, then \( dP_2(\omega_2) \) almost surely, \( I_n(f_n)(\cdot, \omega_2) \) and \( I_m(g_m)(\cdot, \omega_2) \) are Wiener integrals of square-integrable functions that also satisfy (14), hence \( I_n(f_n)(\cdot, \omega_2) \) is independent of \( I_m(g_m)(\cdot, \omega_2) \) under \( P_1 \) from [16], and for any \( u, v \in \mathbb{C}_b(\mathbb{R}), \)

\[
\int_{\Omega} u(I_n(f_n))v(I_m(g_m))dP_1 = \int_{\Omega_1} u(I_n(f_n))dP_1 \int_{\Omega_1} v(I_m(g_m))dP_1, \quad dP_2(\omega_2) - \text{a.s.}
\]

If further (15) is satisfied, we choose two version \( \tilde{f}_n \) and \( \tilde{g}_m \) of \( f_n, g_m \) and let

\[
A = \{ s : \| f_n(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} \neq 0 \text{ and } \phi_s \neq 0 \},
\]

and

\[
B = \{ s : \| g_m(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} \neq 0 \text{ and } \phi_s \neq 0 \}
\]

Then \( \int_{\Omega_1} u(I_n(f_n))dP_1 \) and \( \int_{\Omega_1} v(I_m(g_m))dP_1 \) are respectively \( \mathcal{F}^A_1 \)-measurable and \( \mathcal{F}^B_1 \)-measurable. Moreover,

\[
0 = \int_0^\infty \| \tilde{f}_n(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} \| \tilde{g}_m(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} | \phi_s | ds
\]

\[
= \int_{A \cap B} \| \tilde{f}_n(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} \| \tilde{g}_m(s, \cdot) \|_{L^2(\mathbb{R}^+_1)^{(n-1)}} | \phi_s | ds,
\]

hence \( \mu(A \cap B) = 0 \) and \( \mathcal{F}^A_1, \mathcal{F}^B_2 \) are independent \( \sigma \)-algebras because \( (N_t)_{t \in \mathbb{R}^+_1} \) has independent increments, and

\[
\int_{\Omega} u(I_n(f_n))v(I_m(g_m))dP = \int_{\Omega} u(I_n(f_n))dP \int_{\Omega} v(I_m(g_m))dP, \quad u, v \in \mathbb{C}_b(\mathbb{R}),
\]

proving the independence of \( I_n(f_n) \) and \( I_m(g_m) \). \( \square \)

The following corollaries, cf. [16], [17], can be extended from the Wiener case to the martingale \( (M_t)_{t \in \mathbb{R}^+_1} \).

**Proposition 4** Two arbitrary families \( \{ I_n(f_{n_k}) : k \in I \} \) and \( \{ I_m(g_{m_l}) : l \in J \} \) of Poisson multiple stochastic integrals are independent if and only if \( I_n(f_{n_k}) \) is independent of \( I_m(g_{m_l}) \) for any \( k \in I, l \in J \).
Proof. We start by considering families of the form \{I_n(f_n)\}, \{I_k(g_k), I_m(h_m)\}. If \(I_n(f_n)\) is independent of \(I_k(g_k)\) and \(I_n(f_n)\) is independent of \(I_m(h_m)\), then \((14)\) is satisfied for \(f_n, g_k\) and for \(f_n, g_m\). Moreover, \(dP_2(\omega_2)\) almost surely, \(I_n(f_n)(\cdot, \omega_2)\), \(I_k(g_k)(\cdot, \omega_2)\) and \(I_m(h_m)(\cdot, \omega_2)\) are multiple Wiener integrals of square-integrable functions that also satisfy \((14)\), hence for \(u \in \mathcal{C}_0(\mathbb{R})\) and \(v \in \mathcal{C}_0(\mathbb{R}^2)\), \(u(I_n(f_n))(\cdot, \omega_2)\) is independent of \(v(I_k(g_k), I_m(h_m))(\cdot, \omega_2)\) under \(P_1\) from the analog of this proposition in \([16]\), and

\[
\int_{\Omega_1} u(I_n(f_n))v(I_k(g_k), I_m(h_m))dP_1 = \int_{\Omega_1} u(I_n(f_n))dP_1 \int_{\Omega_1} v(I_k(g_k), I_m(h_m))dP_1,
\]

\(dP_2(\omega_2)\)-a.s.

We choose three versions \(f_n, g_k, h_m\) of \(f_n, g_k, h_m\), and let

\[
A = \{ s : \| f_n(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \neq 0 \text{ and } \phi_s \neq 0 \},
\]

\[
B = \{ s : \| g_k(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \neq 0 \text{ and } \phi_s \neq 0 \},
\]

and

\[
C = \{ s : \| f_m(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \neq 0 \text{ and } \phi_s \neq 0 \}.
\]

Since \(I_n(f_n)\) is independent of \(I_k(g_k)\) and \(I_n(f_n)\) is independent of \(I_m(h_m)\), \((15)\) holds for \(f_n, g_k\) and \(f_n, h_m\). This implies

\[
0 = \int_0^\infty \| f_n(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \| g_k(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \| \phi_s \| ds
\]

\[
= \int_{A \cap B} \| f_n(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \| g_k(s, \cdot) \|_{L^2(\mathbb{R}^+)}^{(s- \cdot)} \| \phi_s \| ds,
\]

hence \(\mu(A \cap B) = 0\) and in the same way we get \(\mu(A \cap C) = 0\), hence \(\mu(A \cap (B \cup C)) = 0\). Consequently, \(F_2^A\) is independent of \(F_2^{B \cup C}\) since \((N_t)_{t \in \mathbb{R}^+}\) has independent increments. Moreover, \(\int_{\Omega_1} u(I_n(f_n))dP_1\) and \(\int_{\Omega_1} v(I_k(g_k), I_m(h_m))dP_1\) are respectively \(F_2^A\) and \(F_2^{B \cup C}\)-measurable, hence

\[
\int_{\Omega_1} u(I_n(f_n))v(I_k(g_k), I_m(h_m))dP = \int_{\Omega_1} u(I_n(f_n))dP \int_{\Omega_1} v(I_k(g_k), I_m(h_m))dP,
\]

\(u \in \mathcal{C}_0(\mathbb{R}), v \in \mathcal{C}_0(\mathbb{R}^2)\), and \(u(I_n(f_n))\) is independent of \(v(I_k(g_k), I_m(h_m))\). The above proof generalizes to arbitrary families of multiple stochastic integrals. \(\square\)

**Corollary 1** Let \(f_n \in L^2(\mathbb{R}^+)^{\alpha_n}, g_m \in L^2(\mathbb{R}^+)^{\alpha_m}\), and

\[
S_{f_n} = \{ f_n \circ_{\alpha_n} h : h \in L^2(\mathbb{R}^+) \}, \quad S_{g_m} = \{ g_m \circ_{\alpha_m} h : h \in L^2(\mathbb{R}^+) \}.
\]

The following statements are equivalent.

(i) \(I_n(f_n)\) is independent of \(I_m(g_m)\).

(ii) For any \(f \in S_{f_n}\) and \(g \in S_{g_m}\) we have \(fg = 0\), \(| \phi_t | dt\)-a.e. and \((f, g)_{L^2(\mathbb{R}^+)} = 0\).

(iii) The \(\sigma\)-algebras \(\sigma(I_1(f) : f \in S_{f_n})\) and \(\sigma(I_1(g) : g \in S_{g_m})\) are independent.
INDEPENDENCE OF STOCHASTIC INTEGRALS

Proof. (i) $\Leftrightarrow$ (ii) relies on the fact that any $f \in S_f$ and $g \in S_g$ can be written as $f = f_n \circ^{m-1}_{n-1} h$, $g = g_m \circ^{m-1}_{m-1} k$ with $h \in L^2(\mathbb{R}_+)^{m-1}$, $k \in L^2(\mathbb{R}_+)^{m-1}$, and that $\phi_f(t)g(t) = (f_n \otimes^i g_m(t), h \otimes k)_{L^2(\mathbb{R}_+)^{m+n-1}}$, $t \in \mathbb{R}_+$, and $(f,g)_{L^2(\mathbb{R}_+)} = (f_n \circ^i_{n-1} g_m, h \otimes k)_{L^2(\mathbb{R}_+)^{m+n-1}}$. (ii) $\Leftrightarrow$ (iii) is a consequence of Prop. 4. □

Let $(h_k)_{k \in \mathbb{N}_*}$ be an orthonormal basis of $L^2(\mathbb{R}_+)$. For simplicity, we denote by

$$\sigma(I_n(f_n), \nabla I_n(f_n), \ldots, \nabla^{n-1} I_n(f_n))$$

the $\sigma$-algebra

$$\sigma \left( I_n(f_n), \left( \nabla I_n(f_n), h_k^j \right)_{L^2(\mathbb{R}_+)} \right) \right.$$

\[ \left( \nabla^{n-1} I_n(f_n), h_{k_{n-1}^i} \circ \ldots \circ h_{k_{n-1}^n} \right)_{L^2(\mathbb{R}_+)^{m+n-1}}, \quad k_i^j \in \mathbb{N}_*, \quad 1 \leq i \leq j \]

Corollary 2 The multiple stochastic integrals $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if the $\sigma$-algebras

$$\sigma(I_n(f_n), \nabla I_n(f_n), \ldots, \nabla^{n-1} I_n(f_n))$$

and

$$\sigma(I_m(g_m), \nabla I_m(g_m), \ldots, \nabla^{m-1} I_m(g_m))$$

are independent.

Proof. This is a consequence of Th. 1, Prop. 4, and the definition (8) of $\nabla$. □

Let $\lambda$ denote the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$.

Corollary 3 If $F \in Dom_2(\nabla)$ and $G \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $G = \sum_{m \geq 0} I_m(g_m)$, then $F$ is independent of $G$ if for any $m \geq 1$,

$$g_m \circ^1_1 \nabla F = 0 \quad \lambda^{\otimes (m-1)} \otimes \mathbb{P} - a.e. \quad \text{and} \quad g_m \circ^0_1 \nabla F = 0, \quad \lambda^{\otimes m} \otimes \mathbb{P} - a.e. \quad (16)$$

Proof. Assume that $F = \sum_{n \geq 0} I_n(f_n)$. Condition (16) is equivalent to $g_m \circ^1_1 f_n = 0$ and $g_m \circ^0_1 f_n = 0$ a.e. for any $n, m \in \mathbb{N}$, since the decomposition $\nabla F = \sum_{n \geq 0} nI_{n-1}(f_n)$ is orthogonal in $L^2(\Omega) \otimes L^2(\mathbb{R}_+)$. The result follows then from Th. 1 and Prop. 4. □

Remarks. a) In the Poisson case, the results of this paper can also be obtained for a Poisson measure on a metric space with a $\sigma$-finite diffuse measure.

b) The independence criterion also means that $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if their Wick product coincides with their ordinary product:

$$I_n(f_n)I_m(g_m) = I_{n+m}(f_n \circ g_m) = I_n(f_n) : I_m(g_m).$$
References


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