Conditionally Gaussian stochastic integrals

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Abstract - We derive conditional Gaussian type identities of the form

\[ E \left[ \exp \left( i \int_0^T u_t dB_t \right) \left| \int_0^T |u_t|^2 dt \right| \right] = \exp \left( - \frac{1}{2} \int_0^T |u_t|^2 dt \right), \]

for Brownian stochastic integrals, under conditions on the process \((u_t)_{t \in [0,T]}\) specified using the Malliavin calculus. This applies in particular to the quadratic Brownian integral \(\int_0^T AB_s dB_s\) under the matrix condition \(A^\dagger A^2 = 0\), using a characterization of Yor [6].

Intégrales stochastiques conditionnellement gaussiennes

Résumé - Nous obtenons des identités gaussiennes conditionnelles de la forme

\[ E \left[ \exp \left( i \int_0^T u_t dB_t \right) \left| \int_0^T |u_t|^2 dt \right| \right] = \exp \left( - \frac{1}{2} \int_0^T |u_t|^2 dt \right), \]

pour les intégrales stochastiques browniennes, sous des conditions sur le processus \((u_t)_{t \in [0,T]}\) exprimées à l’aide du calcul de Malliavin. Ces résultats s’appliquent en particulier à l’intégrale brownienne quadratique \(\int_0^T AB_s dB_s\) sous la condition matricielle \(A^\dagger A^2 = 0\), en utilisant une caractérisation de Yor [6].

Key words: Quadratic Brownian functionals, multidimensional Brownian motion, moment identities, characteristic functions.

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1 Introduction

Let \((B_t)_{t \in [0,T]}\) be a \(d\)-dimensional Brownian motion generating the filtration \((\mathcal{F}_t)_{t \in [0,T]}\). When \(A\) is a \(d \times d\) skew-symmetric matrix, the identity

\[
E \left[ \exp \left( i \int_0^T AB_s dB_s \right) \bigg| B_t \right] = E \left[ \exp \left( -\frac{1}{2} \int_0^T |AB_s|^2 ds \right) \bigg| B_t \right],
\]

(1)

has been proved in Theorem 2.1 of [1], extending a formula of [7] for the computation of the characteristic function of Lévy’s stochastic area in case \(d = 2\).

This approach is connected to a result of Yor [6] stating that when \(A^\dagger A^2 = 0\), the filtration \((\mathcal{F}_t^k)_{t \in [0,T]}\) of \(t \mapsto \int_0^t AB_s dB_s\) is generated by \(k\) independent Brownian motions, where \(k\) is the number of distinct eigenvalues of \(A^\dagger A\).

In this Note we derive conditional versions of the identity (1) for the stochastic integral \(\int_0^T u_t dB_t\) of an \((\mathcal{F}_t)\)-adapted process \((u_t)_{t \in [0,T]}\) in Theorem 1, under conditions formulated in terms of the Malliavin calculus, using the cumulant-moment formula of [3], [4]. In particular we provide conditions for \(\int_0^T u_t dB_t\) to be Gaussian \(\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)\)-distributed given \(\int_0^T |u_t|^2 dt\), cf. Theorem 2. This holds for example when \((u_t)_{t \in [0,T]} = (AB_t)_{t \in [0,T]}\) under Yor’s condition \(A^\dagger A^2 = 0\), cf. Corollary 3.

We also consider a weakening of this condition to \(A^\dagger A\) skew-symmetric, provided that \(A^\dagger A\) is proportional to a projection, cf. Corollary 6.

2 Conditional characteristic functions

Let \(D\) denote the Malliavin gradient with domain \(\mathcal{D}_{2,1}\) on the \(d\)-dimensional Wiener space, cf. § 1.2 of [2] for definitions. Taking \(H = L^2([0,T]; \mathbb{R}^d)\) for some \(T > 0\) and \(u\) in the domain \(\mathcal{D}_{k,1}(H)\) of \(D\) in \(L^k(\Omega; H)\), we let

\[
(Du)^k u_t := \int_0^T \cdots \int_0^T (D_{tk} u_t)^1 (D_{tk-1} u_{tk})^1 \cdots (D_{t_1} u_{t_2})^1 u_{t_1} dt_1 \cdots dt_k, \quad t \in [0,T], \quad k \geq 1.
\]

**Theorem 1.** Let \(u \in \bigcap_{k \geq 1} \mathcal{D}_{k,1}(H)\) be an \((\mathcal{F}_t)\)-adapted process such that

\[
\langle u_t, (Du)^k u_t \rangle_{\mathbb{R}^d} = 0, \quad t \in [0,T], \quad k \geq 1.
\]

We have

\[
E \left[ \exp \left( i \int_0^T u_t dB_t \right) \bigg| (|u_t|)_{t \in [0,T]} \right] = \exp \left( -\frac{1}{2} \int_0^T |u_t|^2 dt \right),
\]

(2)

provided that \(\frac{1}{2} \int_0^T |u_t|^2 dt\) is exponentially integrable.
Proof. For any $F \in \mathcal{D}_{2,1}$ and $k \geq 1$, let

$$\Gamma_k^u F := 1_{\{k \geq 2\}} F \int_0^T \langle u_t, (Du)^{k-1} u_t \rangle_{\mathbb{R}^d} dt + \int_0^T \langle D_t F, (Du)^{k-1} u_t \rangle_{\mathbb{R}^d} dt.$$ 

Recall that for any $u \in \mathcal{D}_{2,1}(H)$ such that $\Gamma_{l_1}^u \cdots \Gamma_{l_k}^u 1$ has finite expectation for all $l_1 + \cdots + l_k \leq n$, $k = 1, \ldots, n$, by Theorem 1 of [3] or Proposition 4.3 of [4] we have

$$E \left[ F \left( \int_0^T u_t dB_t \right)^n \right] = n! \sum_{a=1}^n \sum_{l_1+\cdots+l_a=n} E \left[ \Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F \right] \frac{1}{(l_1+1) \cdots (l_a+1)} , \quad (3)$$

for $F \in \mathcal{D}_{2,1}$. Next, for any $f \in C_b^1(\mathbb{R})$ and $k \geq 1$ we have

$$\Gamma_k^u f \left( \int_a^b |u_t|^2 dt \right) = 1_{\{k=2\}} \int_0^T |u_t|^2 dt f' \left( \int_a^b |u_t|^2 dt \right) + f' \left( \int_a^b |u_t|^2 dt \right) \int_0^T \langle D_t \int_a^b |u_s|^2 ds, (Du)^{k-1} u_t \rangle_{\mathbb{R}^d} dt,$$

$$= 1_{\{k=2\}} \int_0^T |u_t|^2 dt f' \left( \int_a^b |u_t|^2 dt \right) + 2f' \left( \int_a^b |u_t|^2 dt \right) \int_a^b \langle u_s, (Du)^k u_s \rangle_{\mathbb{R}^d} ds,$$

$$= 1_{\{k=2\}} \int_0^T |u_t|^2 dt f' \left( \int_a^b |u_t|^2 dt \right), \quad 0 \leq a \leq b.$$

By induction this yields

$$\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u 1 = 1_{\{l_1 = \cdots = l_a = 2\}} \left( \int_0^T |u_t|^2 dt \right)^a, \quad 0 \leq l_1, \ldots, l_a \leq 1, \quad a \geq 1, \quad (4)$$

for any random variable $F$ of the form

$$F = f \left( \int_{a_1}^{b_1} |u_t|^2 dt, \ldots, \int_{a_m}^{b_m} |u_t|^2 dt \right), \quad 0 \leq a_i \leq b_i \leq T, \quad i = 1, \ldots, m,$$

where $f \in C_b^1(\mathbb{R}^m)$, and by (3) and (4) we find

$$E \left[ \left( \int_0^T u_t dB_t \right)^{2n} F \right] = \left( \frac{2n!}{2^n n!} \right) E \left[ \left( \int_0^T |u_t|^2 dt \right)^n F \right], \quad (5)$$

and $E \left[ \left( \int_0^T u_t dB_t \right)^{2n+1} F \right] = 0$ for all $n \in \mathbb{N}$. \hfill \Box

The following result is obtained by an argument similar to the proof of Theorem 1.
Theorem 2. Let \( u \in \bigcap_{k \geq 1} \mathcal{D}_{k,1}(H) \) be an \((\mathcal{F}_t)\)-adapted process such that
\[
\langle u, (Du)^k u \rangle_H = 0, \quad k \geq 1.
\]

We have
\[
E \left[ \exp \left( i \int_0^T u_t dB_t \right) \bigg| \int_0^T |u_t|^2 dt \right] = \exp \left( -\frac{1}{2} \int_0^T |u_t|^2 dt \right),
\]
provided that \( \frac{1}{2} \int_0^T |u_t|^2 dt \) is exponentially integrable.

In the particular case where \( u_t = R_t h, \ t \in [0,T], \ h \in H, \) where \( R \) is a random, adapted (or quasi-nilpotent) isometry of \( H \), we find that \( \int_0^T |u_t|^2 dt = \int_0^T |h(t)|^2 dt \) is deterministic, hence
\[
\langle u, (Du)^k u \rangle_H = \frac{1}{2} \langle (Du)^{k-1} u, D\langle u, u \rangle_H \rangle_H = 0, \quad k \geq 1,
\]
and Theorem 2 shows that \( \int_0^T (R_t h) dB_t \) has a centered Gaussian distribution with variance \( \int_0^T |h(t)|^2 dt \), as in Theorem 2.1-(b) of [5].

Theorems 1 and 2 also apply when \( \int_0^T |u_t|^2 dt \) is random, for example when \( (u_t)_{t \in [0,T]} \) takes the form \( u_t = g(B_t), \ t \in [0,T], \) where \( g \in C^1_b(\mathbb{R}^d, \mathbb{R}^d) \) satisfies the condition
\[
\langle g(x), ((\nabla g(x))^\dagger)^k g(x) \rangle_{\mathbb{R}^d} = 0, \quad x \in \mathbb{R}^d, \ k \geq 1.
\]
Next, we check that this condition is satisfied on concrete examples based on [6], when \( g \) is a linear mapping of the form \( g(x) = Ax, \ x \in \mathbb{R}^d \).

**Vanishing of \( A^\dagger A^2 \)**

Applying Theorem 1 to the adapted process \( (u_t)_{t \in [0,T]} := (AB_t)_{t \in [0,T]} \) under Yor’s [6] condition \( A^\dagger A^2 = 0 \), by the relation \( D_t B_s = 1_{[0,a]}(t)I_{\mathbb{R}^d} \) we obtain the vanishing
\[
\langle u_t, (Du)^k u_t \rangle_{\mathbb{R}^d} = \int_0^T \cdots \int_0^T \langle u_t, (D_{t_k} u_t)^\dagger (D_{t_{k-1}} u_t)^\dagger \cdots (D_{t_1} u_{t_1})^\dagger u_{t_1} \rangle_{\mathbb{R}^d} dt_1 \cdots dt_k
\]
\[
= \int_0^T \int_0^t \cdots \int_0^{t_{k-2}} \langle AB_t, (A^\dagger)^k AB_t \rangle_{\mathbb{R}^d} dt_1 \cdots dt_k
\]
\[
= 0, \quad t \in [0,T], \ k \geq 1.
\]

This yields the next corollary of Theorem 1, in which the condition \( A^\dagger A^2 = 0 \) includes 2-nilpotent matrices as a particular case.
Corollary 3. Assume that $A^\dagger A^2 = 0$. We have
\[
E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \right| (|AB_t|)_{t \in [0,T]} ] = \exp \left( -\frac{1}{2} \int_0^T |AB_t|^2 \, dt \right). \tag{6}
\]
Note that the filtration of $(|AB_t|)_{t \in [0,T]}$ coincides with the filtration $(\mathcal{F}_t^k)_{t \in [0,T]}$ generated by $k$ independent Brownian motions where $k$ is the number of nonzero eigenvalues of $A^\dagger A$, cf. Corollary 2 of [6].

3 Skew-symmetric $A^\dagger A^2$

When $A^\dagger A$ has only one nonzero eigenvalue, i.e. $A^\dagger A$ is proportional to a projection, the condition $A^\dagger A^2 = 0$ can be relaxed using stochastic calculus, by only assuming that $A^\dagger A^2$ is skew-symmetric. We start with the following variation of Corollary 2 of [6].

Lemma 4. Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue $\lambda_1$. Then the processes
\[
Y_1^t := \frac{1}{\sqrt{\lambda_1}} \int_0^t \frac{AB_s}{|AB_s|} dAB_s, \quad \text{and} \quad Y_2^t := \int_0^t \frac{AB_s}{|AB_s|} dB_s, \quad t \in [0,T], \tag{7}
\]
are independent standard Brownian motions.

Proof. Since $A^\dagger A$ is symmetric it can be written as $A^\dagger A = R^\dagger CR$, where $R$ is orthogonal and $C$ is diagonal, therefore since $(RB_t)_{t \in [0,T]}$ is also a standard Brownian motion we can assume that $A^\dagger A$ has the form $A^\dagger A = (\lambda_1 1_{1 \leq k = l \leq r})_{1 \leq k, l \leq d}$ with $\lambda_i > 0$, $1 \leq i \leq r$. Clearly $(Y_2^t)_{t \in [0,T]}$ is a standard Brownian motion, and
\[
d\langle Y_1^t, Y_2^t \rangle_t = \frac{(A^\dagger A^2 B_t, B_t)}{|AB_t|^2 \sqrt{\lambda_1}} dt = 0.
\]
In addition we have $dY_1^t = \frac{\lambda_1^{-1/2}}{|AB_t|} \sum_{i=1}^r \lambda_i B_i^t dB_i^t$ and
\[
d\langle Y_1^t, Y_1^t \rangle_t = \frac{(\lambda_1 B_1^t)^2 + \cdots + (\lambda_r B_r^t)^2}{\lambda_1 (\lambda_1 (B_1^t)^2 + \cdots + \lambda_r (B_r^t)^2)} dt,
\]
hence $(Y_1^t)_{t \in [0,T]}$ is also a standard Brownian motion when $\lambda_1 = \cdots = \lambda_r$. \qed

The following result relaxes the vanishing hypothesis of Corollary 3.

Corollary 5. Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue $\lambda_1$. Then we have
\[
E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \right| (|AB_t|)_{t \in [0,T]} ] = \exp \left( -\frac{1}{2} \int_0^T |AB_t|^2 \, dt \right). \tag{8}
\]
Proof. We let $S_t := |AB_t|^2, t \in [0, T]$, and note that by Corollary 2 of [6], the filtration generated by $(|AB_t|)_{t \in [0,T]}$ coincides with the filtration $(\mathcal{F}_t^1)_{t \in [0,T]}$ of $(Y^1_t)_{t \in [0,T]}$. Next, Itô’s formula shows that

$$S_t = 2 \int_0^t AB_s dAB_s + \text{Tr} (A^\dagger A) t = 2 \int_0^t \sqrt{\lambda_1 S_s} dY^1_s + r \lambda_1 t, \quad t \in [0, T],$$

hence $(|AB_t|)_{t \in [0,T]}$ is $(\mathcal{F}^1_t)_{t \in [0,T]}$-adapted and therefore independent of $(Y^2_t)_{t \in [0,T]}$, hence

$$\int_0^T AB_t dB_t = \int_0^T |AB_t| dY^2_t$$

is centered Gaussian with variance $\int_0^T |AB_t|^2 dt$ given $\mathcal{F}_T$, which yields (8). □

Commutation with orthogonal matrices

Under the assumptions of Corollaries 3 or 5 it follows that

$$E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| |AB_t| \right] = E \left[ \exp \left( -\frac{1}{2} \int_0^T |AB_t|^2 dt \right) \bigg| |AB_t| \right], \quad (9)$$

since $(|AB_t|)_{t \in [0,T]}$ and $(Y^1_t)_{t \in [0,T]}$ generate the same filtration on $(\mathcal{F}^1_t)_{t \in [0,T]}$.

Corollary 6. Assume that either $A^\dagger A^2 = 0$, or $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue. If in addition $A$ commutes with orthogonal matrices, then we have

$$E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| AB_t \right] = E \left[ \exp \left( -\frac{1}{2} \int_0^T |AB_t|^2 ds \right) \bigg| AB_t \right], \quad (10)$$

$0 \leq t \leq T.$

Proof. We check that for any $d \times d$ orthogonal matrix $R$ we have

$$E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| AB_t = Rx \right] = E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| AB_t = x \right],$$

$x \in \mathbb{R}^d$, which shows that

$$E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| AB_t \right] = E \left[ \exp \left( i \int_0^T AB_t dB_t \right) \bigg| |AB_t| \right]$$

and similarly for the right hand side, and we conclude by (9). □

6
Skew-symmetric orthogonal $A$

We note that when $A$ is skew-symmetric and orthogonal the condition $A^\dagger A^2$ skew-symmetric is satisfied as in this case we have $(A^\dagger A^2)^\dagger = A^\dagger A^\dagger A = A^\dagger = -A = -A^\dagger A^2$, and (10) can be written as

$$E \left[ \exp \left( i \int_0^T A B_s dB_s \right) \left| B_t \right. \right] = E \left[ \exp \left( -\frac{1}{2} \int_0^T |A B_s|^2 ds \right) \left| B_t \right. \right], \quad (11)$$

$0 \leq t \leq T$. This holds in particular when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, in which case $A^\dagger A = I_{\mathbb{R}^2}$ has the unique eigenvalue $\lambda_1 = 1$ and $A^\dagger A^2 = A$ is skew-symmetric, in which case we recover the result of [7] which has been used to show that (11) holds when $A$ is skew-symmetric and not necessarily orthogonal in Theorem 2.1 of [1].

References


