Chapter 3
Value at Risk

This chapter deals with risk measures and financial data, including quantile risk measures and value at risk (VaR).

3.1 Financial Data with R

Package quantmod for financial data

The R package quantmod can be installed and run via the following command:

```r
install.packages("quantmod")
library(quantmod)
getSymbols("1800.HK",from="2007-01-03",to="2011-12-02",src="yahoo")
stock=Ad('1800.HK')
chartSeries(stock,up.col="blue",theme="white")
stock.rtn=diff(log(Ad('1800.HK'))) chartSeries(stock.rtn,up.col="blue",theme="white")
n = sum(!is.na(stock.rtn))
```
3.2 Risk Measures

The potential loss associated to any of the above risks will be modelled via a random variable $X$.

**Definition 3.1.** A risk measure is a mapping that assigns a value $V_X$ to a given random variable $X$.

For insurance companies, which need to hold a capital in order to meet future liabilities, the capital $C_X$ required to face the risk induced by a potential loss $X$ can be defined as

$$C_X := V_X - L_X,$$

where

a) $L_X$ represents the liabilities of the company (positive or negative, $L_X$ is deducted from $V_X$ when $L_X > 0$, while $|L_X|$ is added to $V_X$ when $L_X < 0$),

b) $V_X$ stands for a upper “reasonable” estimate of the potential loss associated to $X$. 

Estimating the liabilities of the company by $\mathbb{E}[X]$, the required capital is given by

$$C_X = V_X - \mathbb{E}[X].$$

The above code is used in the next figure to estimate liabilities by a conditional mean.

![Graph showing liability estimation](http://www.ntu.edu.sg/home/nprivault/indext.html)

Fig. 3.3: Estimating liabilities by the conditional mean $\mathbb{E}[X \mid X < 0]$ over 346 returns.

**Example: Guaranteed Maturity Benefits**

Variable annuity benefits offered by insurance companies are usually protected via different mechanisms such as Guaranteed Minimum Maturity Benefits (GMMBs) or Guaranteed Minimum Death Benefits (GMDBs). The computation of the corresponding risk measures is an important issue for the practitioner in risk management.

Given a fund value process $(F_t)_{t \in \mathbb{R}_+}$, an insurer is continuously charging annualized mortality and expense fees at the rate $m$ from the account of variable annuities, resulting into a margin offset income $M_t$ given by

$$M_t := mF_t \quad t \in \mathbb{R}_+.$$

Denoting by $\tau_x$ the future lifetime of a policyholder at the age $x$, the future payment made by the insurer at maturity $T$ is

$$(G - F_T)^+ \mathbbm{1}_{\{\tau_x > T\}}$$
where $G$ is the guarantee level expressed as a percentage of the initial fund value $F_0$, $\delta$ is a roll-up rate according to which the guarantee increases up to the payment time. In this case, the random variable $X$ is taken equal to

$$X := e^{-rT}(G - F_T) + \mathbb{1}_{\{\tau_x > T\}} - \int_0^{\min(T,\tau_x)} e^{-rs} M ds.$$

### Coherent risk measures

**Definition 3.2.** A risk measure $V$ is said to be coherent if it satisfies the following four properties, for any two random variables $X, Y$:

i) **Monotonicity:**
$$X \leq Y \implies V_X \leq V_Y,$$

ii) **(Positive) homogeneity:**
$$V_{\lambda X} = \lambda V_X, \quad \text{for constant } \lambda > 0,$$

iii) **Translation invariance:**
$$V_{X + \mu} = \mu + V_X, \quad \text{for constant } \mu > 0,$$

iv) **Subadditivity:**
$$V_{X + Y} \leq V_X + V_Y.$$

Subadditivity means that the combined risk of several portfolios is lower than the sum of risks of those portfolios, as happens usually through portfolio diversification.

The expectation of random variables

$$V_X := \mathbb{E}[X]$$

is an example of a coherent risk measure (also called pure premium risk measure) satisfying the above conditions (i)-(iv), and is additive.

More generally, any risk measure of the form

$$M_X = \mathbb{E}[X f_X(X)],$$

cf. e.g. (4.3) below, where $f_X$ is a non-decreasing distortion function satisfying

$$f_X(\lambda x) = f_X(x) \quad \text{and} \quad f_X(\mu + x) = f_X(x), \quad x \in \mathbb{R}, \quad \lambda, \mu > 0,$$

will be monotone, homogeneous, and translation invariant.
### 3.3 Quantile Risk Measures

The *Cumulative Distribution Function* (CDF) of a random variable $X$ is the function

$$F_X : \mathbb{R} \longrightarrow [0, 1]$$

defined by

$$F_X(x) := \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$

Any cumulative distribution function $F_X$ satisfies the following properties:

i) $x \longrightarrow F_X(x)$ is non-decreasing,

ii) $x \longrightarrow F_X(x)$ is right-continuous,

iii) $\lim_{x \to \infty} F_X(x) = 1$,

iv) $\lim_{x \to -\infty} F_X(x) = 0$.

In addition, any cumulative distribution function $F_X$ admits left limits in the sense that if $(x_n)_{n \geq 1}$ is a strictly increasing sequence converging to $x \in \mathbb{R}$ then we have

$$\lim_{n \to \infty} F_X(x_n) = \lim_{n \to \infty} \mathbb{P}(X \leq x_n) = \mathbb{P}(X < x). \quad (3.2)$$

The next figure shows the continuous Cumulative Distribution Function of a Gaussian $\mathcal{N}(0, 1)$ random variable.

![Gaussian CDF](Gaussian CDF)

**Fig. 3.4:** Gaussian Cumulative distribution function.

On the other hand, a random variable $X$ may have a discontinuous cumulative distribution function, as illustrated in Figure 3.5 with

```r
x <- seq(-5, 5, length=1000)
plot(x, pnorm(x, mean=0, sd=1), type="l", lwd=3, xlab = "x", ylab = "", main = "Gaussian CDF", col="blue")
```
\[ P(X = 0) = P(X \leq 0) - P(X < 0) = 0.25 > 0. \]

More generally, the discontinuity of a CDF at the point \( x \in \mathbb{R} \), if it exists, is given by

\[ P(X = x) = P(X \leq x) - P(X < x) = F_X(x) - \lim_{y \searrow x} F_X(y). \]

**Definition 3.3.** Given \( X \) a random variable with cumulative distribution function \( F_X : \mathbb{R} \rightarrow [0, 1] \) and a level \( p \in (0, 1) \), the \( p \)-quantile of \( X \) defined by

\[ q_X^p := \inf \{ x \in \mathbb{R} : P(X \leq x) \geq p \}. \]

**Performance analytics in R - quantiles of known distributions**

The quantiles of various distributions can be obtained in R. For example, the command

\[ \text{qnorm}(0.95, \text{mean}=0.5, \text{sd}=1) \]

shows that the 95%-quantile of a \( \mathcal{N}(0.5, 1) \) Gaussian random variable is 2.144854. On the other hand, the instruction

\[ \text{qt}(0.90, \text{df}=5) \]

displays the 90%-quantile of a Student \( t \)-distributed random variable with 5 degrees of freedom is 1.475884.

**Performance analytics in R - empirical CDF**

The *empirical Cumulative Distribution Function* can be estimated as

\[ * \text{ Picture taken from } \text{http://www.probabilitycourse.com/}. \]
\[ F_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{x_i \leq x\}}, \quad x \in \mathbb{R}. \]

Fig. 3.6: Empirical cumulative distribution function.

Note that the empirical distribution function has a visible discontinuity, or gap, at 0, whose height 0.03967611 is given by

\[
\text{sum}(\text{is.na}(\text{stock.rtn}[\text{stock.rtn}==0]))/\text{sum}(\text{is.na}(\text{stock.rtn}))
\]

### 3.4 Value at Risk (VaR)

Value at Risk has two objectives:

i) to provide a measure for risk, and

ii) to determine an adequate level of capital reserves that matches the current level of risk.

In other words, managing risk means here determining a level \( V_X \) of provision or capital requirement that will not be “too much” exceeded by \( X \).

In this respect, the probability \( \mathbb{P}(X > V) \) that \( X \) exceeds the level \( V \) is of a capital importance. Setting \( V \) such that for example
\[ \mathbb{P}(X \leq V) \geq 0.95, \quad i.e. \quad \mathbb{P}(X > V) \leq 0.05, \]

means that insolvency will occur with probability less than 5%.

The 95%-quantile risk measure is the smallest value of \( V \) such that
\[ \mathbb{P}(X \leq V) \geq 0.95, \quad i.e. \quad \mathbb{P}(X > V) \leq 0.05. \]

More precisely, we have the following definition.

**Definition 3.4.** The Value at Risk \( V^p_X \) at the level \( p \in (0, 1) \) is the \( p \)-quantile of \( X \) is defined by
\[ V^p_X := \inf \{ x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p \}. \tag{3.3} \]

In other words, for some decreasing sequence \( (x_n)_{n \geq 1} \) such that \( \mathbb{P}(X \leq x_n) \geq p \) for all \( n \geq 1 \), we have
\[ V^p_X := \lim_{n \to \infty} x_n. \tag{3.4} \]

Note that \( V^p_X \) may also be negative, in which case we identify it to a potential profit rather than to a liability. On the other hand, the Value at Risk \( V^p_X \) does not contain any information on how large losses can be beyond \( V^p_X \).

The function \( p \to V^p_X \) is a non-decreasing function of \( p \in [0, 1] \), and it is the generalized inverse of the cumulative distribution function of \( X \), defined as
\[ x \mapsto F_X(x) := \mathbb{P}(X \leq x), \quad x \in \mathbb{R}. \]

Since the cumulative distribution function of \( X \) is non-decreasing, its generalized inverse \( p \to V^p_X \) is nondecreasing, left-continuous, and it admits limits on the right, see e.g. Proposition 2.3-(2) of [18].

In addition, if \( F_X \) is continuous and strictly increasing it admits an inverse \( F^{-1}_X \), and in this case we have
\[ V_X(p) = F^{-1}_X(p), \quad p \in (0, 01). \]

The next lemma follows from the Definition 3.4 of \( V^p_X \).

**Lemma 3.5.** For all \( x \in \mathbb{R} \) we have
\[ V^p_X \leq x \iff \mathbb{P}(X \leq x) \geq p. \tag{3.5} \]

**Proof.** If \( \mathbb{P}(X \leq x) \geq p \) then we have
\[ V^p_X = \inf \{ y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p \} \leq x. \]
On the other hand, if $V_p^X \leq x$ then there exists a strictly decreasing sequence $(x_n)_{n \geq 1}$ such that

$$\lim_{n \to \infty} x_n = V_p^X \quad \text{and} \quad \mathbb{P}(X \leq x_n) \geq p, \quad n \geq 1.$$ 

Therefore, by right continuity of the cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x)$, we have

$$\mathbb{P}(X \leq V_p^X) = \lim_{n \to \infty} \mathbb{P}(X \leq x_n) \geq p.$$ 

\[\Box\]

In particular, with probability at least $p$, the value of $X$ is always lower than $V_p^X$, i.e. we have

$$F_X(V_p^X) = \mathbb{P}(X \leq V_p^X) \geq p \quad \text{and} \quad \mathbb{P}(X > V_p^X) \leq 1 - p. \quad (3.6)$$

**Proposition 3.6.** If $\mathbb{P}(X = V_p^X) = 0$ then we have $p = \mathbb{P}(X \leq V_p^X)$.

**Proof.** Assume that $V_p^X = x$. By (3.6) we have $\mathbb{P}(X \leq x) \geq p$. If $\mathbb{P}(X \leq V_p^X) > p$, we choose a strictly increasing sequence $(x_n)_{n \geq 1}$ such that

$$\lim_{n \to \infty} x_n = V_p^X \quad \text{and} \quad \mathbb{P}(X \leq x_n) \leq p, \quad n \geq 1.$$ 

Since the cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x)$ admits left limits by (3.2), we have

$$\mathbb{P}(X < V_p^X) = \lim_{n \to \infty} \mathbb{P}(X \leq x_n) \leq p < \mathbb{P}(X \leq V_p^X),$$

which contradicts $\mathbb{P}(X = V_p^X) = \mathbb{P}(X \leq V_p^X) - \mathbb{P}(X < V_p^X) = 0$, and therefore we have $\mathbb{P}(X \leq V_p^X) = p$. \[\Box\]

In particular, if $\mathbb{P}(X = V_p^X) = 0$ then $1 - p = \mathbb{P}(X > V_p^X)$, and if

$$p < \mathbb{P}(X \leq V_p^X) \quad \text{or} \quad 1 - p > \mathbb{P}(X > V_p^X)$$

then we have $\mathbb{P}(X = V_p^X) > 0$.

**Proposition 3.7.** Assume that the cumulative distribution function $F_X$ is continuous and strictly increasing. Then we have $V_{-X}^p = -V_X^{1-p}$.

**Proof.** We have

$$F_{-X}(x) = \mathbb{P}(-X \leq x)$$

$$= \mathbb{P}(X \geq -x)$$

$$= 1 - \mathbb{P}(X < -x)$$

$$= 1 - \mathbb{P}(X \leq -x)$$

$$= 1 - F_X(-x),$$
hence
\[ p = F_X(F_X^{-1}(p)) = 1 - F_X(-F_X^{-1}(p)), \]
which yields
\[ V_X^p = F_X^{-1}(p) = -F_X^{-1}(1 - p) = -V_X^{1-p}, \quad p \in (0, 1). \]

Next, we check the properties of Value at Risk.

a) **Monotonicity.** Value at Risk satisfies the monotonicity property of coherent risk measures.

   **Proof.** If \( X \leq Y \) then
   \[ \mathbb{P}(Y \leq x) = \mathbb{P}(X \leq Y \leq x) \leq \mathbb{P}(X \leq x), \quad x \in \mathbb{R}, \]
   hence
   \[ \mathbb{P}(Y \leq x) \geq p \implies \mathbb{P}(X \leq x) \geq p, \quad x \in \mathbb{R}, \]
   which shows that
   \[ V_X^p \leq V_Y^p \]
   by (3.3).

b) **Positive homogeneity and translation invariance.** Value at Risk also satisfies the positive homogeneity and translation invariance properties of coherent risk measures, in addition to monotonicity.

   **Proof.** For all \( a > 0 \) and \( b \in \mathbb{R} \) we have
   \[
   V_{a+bX}^p = \inf\{x \in \mathbb{R} : \mathbb{P}(a + bX \leq x) \geq p\} \\
   = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq (x - a)/b) \geq p\} \\
   = \inf\{a + by \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \\
   = a + b \inf\{y \in \mathbb{R} : \mathbb{P}(X \leq y) \geq p\} \\
   = a + bV_X^p. \]

c) **Subadditivity and coherence.** Although Value at Risk satisfies monotonicity, positive homogeneity and translation invariance, it is not subadditive in general. Namely, the Value at Risk \( V_{X+Y}^p \) of \( X + Y \) may be larger than the sum \( V_X^p + V_Y^p \). Therefore, Value at Risk is not a coherent risk measure.

   **Proof.** We show that Value at Risk is not subadditive by considering two independent Bernoulli random variables \( X, Y \in \{0, 1\} \) with the distribution
   \[
   \begin{cases} 
   \mathbb{P}(X = 1) = \mathbb{P}(Y = 1) = 2\%, \\
   \mathbb{P}(X = 0) = \mathbb{P}(Y = 0) = 98\%. 
   \end{cases} \]
hence $V_{X}^{0.975} = V_{Y}^{0.975} = 0$.

Fig. 3.7: Cumulative distribution function of $X$ and $Y$.

On the other hand, we have

\[
\begin{align*}
\mathbb{P}(X + Y = 2) &= (0.02)^2 = 0.04\%, \\
\mathbb{P}(X + Y = 1) &= 2 \times 0.02 \times 0.98 = 3.92\%, \\
\mathbb{P}(X + Y = 0) &= (0.98)^2 = 96.04\%,
\end{align*}
\]

hence

$V_{X+Y}^{0.975} = 1 > V_{X}^{0.975} + V_{Y}^{0.975} = 0$.

Fig. 3.8: Cumulative distribution function of $X + Y$.

Proposition 3.8. Gaussian Value at Risk. Given $X \sim \mathcal{N}(\mu, \sigma^2)$ we have

\[
V_{X}^{p} = \mu + \sigma q_{Z}^{p} = \mu + \sigma V_{Z}^{p}
\]

(3.7)

where the normal quantile $q_{Z}^{p}$ at level $p$ satisfies

\[
\mathbb{P}(Z \leq q_{Z}^{p}) = p \quad \text{for} \quad Z \sim \mathcal{N}(0,1).
\]

Proof. We write $X \sim \mathcal{N}(\mu, \sigma^2)$ as

\[
X = \mu + \sigma Z
\]
where $Z \sim \mathcal{N}(0, 1)$ is a standard normal random variable, and use the relation

\[
p = \mathbb{P}(X \leq V_p^X) = \mathbb{P}(\mu + \sigma Z \leq V_p^X) = \mathbb{P}(Z \leq (V_p^X - \mu)/\sigma) = \mathbb{P}(Z \leq q_p^Z).
\]

We note that Value at Risk is sub-additive (and hence coherent) on (not necessarily independent) Gaussian random variables. Indeed, by (3.7), for any two random variables $X$ and $Y$ we have

\[
\sigma_{X+Y}^2 = \text{Var}[X + Y] = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - \mathbb{E}[X]\mathbb{E}[Y] = \text{Var}[X] + \text{Var}[Y] + 2\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \leq \text{Var}[X] + \text{Var}[Y] + 2\sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2] \mathbb{E}[(Y - \mathbb{E}[Y])^2]} = \left(\sqrt{\text{Var}[X]} + \sqrt{\text{Var}[Y]}\right)^2, \tag{3.8}
\]

where, from (3.8) to (3.9) we applied the Cauchy-Schwarz inequality.

Hence in particular when $X$ and $Y$ are Gaussian, by (3.7) we get

\[
V_{X+Y}^p = \mu_{X+Y} + \sigma_{X+Y}q_p^Z \leq \mu_X + \mu_Y + \sigma_{X+Y}q_p^Z \leq \mu_X + \mu_Y + (\sigma_X + \sigma_Y)q_p^Z = V_X^p + V_Y^p.
\]

**Performance analytics in R - Value at Risk**

We are using the PerformanceAnalytics R package, which can be installed via the commands
Value at Risk

```
install.packages("PerformanceAnalytics")
library(PerformanceAnalytics)
getSymbols("0700.HK",from="2010-01-03",to="2018-02-01",src="yahoo")
stock=Ad(0700.HK)
chartSeries(stock,up.col="blue",theme="white")
stock.rtn=diff(log(Ad(0700.HK)))
chart.CumReturns(stock.rtn,main="Cumulative Returns")
var=VaR(stock.rtn, p=.95, method="historical")
sum(!is.na(stock.rtn[stock.rtn<var[1]]))/sum(!is.na(stock.rtn))
```

The historical 95%-Value at Risk over $N$ samples $(x_i)_{i=1,...,N}$ can be estimated using by inverting the empirical cumulative distribution function $F_N(x)$, $x \in \mathbb{R}$. It is found equal to $V_{X}^{95\%} = -0.03157435$.

```
VaR(stock.rtn, p=.95, method="gaussian")
```

The Gaussian 95%-Value at Risk is estimated from (3.7) by

$$V_{X}^{95\%} = \mu + \sigma q_{Z}^p,$$

where $\mu = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}[X]$, and is found equal to $V_{X}^{95\%} = -0.03115105$. It can be recovered up to approximation as

```
m=mean(stock.rtn,na.rm=TRUE)
s=sd(stock.rtn,na.rm=TRUE)
q=qnorm(.95, mean=0, sd=1)
m-s*q
```

which yields $-0.0311592$. Note that here we are concerned about large negative returns, which explains the negative sign in $m-s*q$.

The next lemma is useful for random simulation purposes, and it will also be used in the proof of Proposition 4.4 below.

**Lemma 3.9.** Any random variable $X$ can be represented as $X = V^U_X$ where $U$ is uniformly distributed on $[0, 1]$.

**Proof.** It suffices to note that by (3.5) we have

$$\mathbb{P}(V^U_X \leq x) = \mathbb{P}(U \leq \mathbb{P}(X \leq x)) = \mathbb{P}(X \leq x) = F_X(x), \quad x \in \mathbb{R}.$$
Exercises

Exercise 3.1  Consider a random variable $X$ having the Pareto distribution with probability density function

$$f_X(x) = \frac{\gamma \theta^\gamma}{(\theta + x)^{\gamma+1}}, \quad x \in \mathbb{R}_+.$$ 

a) Compute the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \quad x \in \mathbb{R}_+.$$ 

b) Compute the value at risk $V_p^X$ at the level $p$ for any $\theta$ and $\gamma$, and then for $p = 99\%$, $\theta = 40$ and $\gamma = 2$.

Exercise 3.2  Consider a random variable $X$ with the cumulative distribution function

![Cumulative distribution function of X.](http://www.ntu.edu.sg/home/nprivault/indext.html)

a) Give the value of $P(X = 100)$.

b) Give the value of $V_q^X$ for all $q$ in the interval $[0.97, 0.99]$.

c) Compute the value of $V_q^X$ for all $q$ in the interval $[0.99, 1]$.

*Hint:* We have

$$F_X(x) = P(X \leq x) = 0.99 + 0.01 \times (x - 100)/50, \quad x \in [100, 150].$$

Exercise 3.3  Discrete distribution. Consider $X \in \{10, 100, 110\}$ with the distribution

$$P(X = 10) = 90\%, \quad P(X = 100) = 9.5\%, \quad P(X = 110) = 0.5\%.$$ 

Compute $V_{99\%}^X$. 

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http://www.ntu.edu.sg/home/nprivault/indext.html
Exercise 3.4 Exponential distribution. Assume that $X$ has an exponential distribution with parameter $\lambda > 0$ and mean $1/\lambda$, i.e.

$$P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$ 

a) Compute

$$V_X^p := \inf \{ x \in \mathbb{R} : P(X \leq x) \geq p \}$$

and $V_X^{95\%}$.

b) Assuming that the liabilities of a company are estimated by $E[X]$, compute the amount of required capital $C_X$ from (3.1).

Exercise 3.5 Estimating risk probabilities from moments.

a) Using the Chebyshev inequality, show that for every $r > 0$

$$V_X^p \leq \left( \frac{\mathbb{E}[X^r]}{1 - p} \right)^{1/r} = \frac{\|X\|_{L^r(\Omega)}}{(1 - p)^{1/r}}.$$ 

b) Give an upper bound for $V_X^{95\%}$ when $p = 95\%$ and $r = 1$. 