Notes on Financial Risk and Analytics
Preface

The topics covered in these notes include an introduction to stochastic modeling with discrete-valued stochastic processes, a basic coverage of Value at Risk and expected shortfall, as well as structures of random dependence. Various types of risk [22] can be classified into market risk, liquidity risk, credit risk, counterparty risk, model risk, estimation risk. For insurance businesses, a more detailed classification can be set as follows.

a) Financial risk

- Investment risk
  - Credit risk
  - Market risk (e.g. depreciation)

- Liability risk
  - Catastrophe risk
  - Non-catastrophe risk (e.g. claim volatility)

b) Operational risk

- Business risk (e.g. lower production)
- Event risk (e.g. system failure).

Financial, investment, market and non-catastrophe risks are covered in Chapters 1, 3 and 4 on time series, Value at Risk and expected shortfall. Credit risk is covered in Chapters 6, 7 and 8 on the structural and reduced-form approaches to credit risk and valuation, and in Chapter 9 on credit scoring. Credit default is treated via defaultable bonds, Credit Default Swaps (CDS) and collateralized debt obligations (CDOs), based on stochastic calculus. Liability, catastrophe and operational risks such as business or event risk are covered in Chapter 2 on risk theory. Basic risk Theory and credit scoring are presented with illustrative examples in R.

This material has been used for teaching in the Masters of Science in Financial Engineering (MFE) and in Analytics (MSA) at the Nanyang Technological University in Singapore.
This pdf file also contains external links and 69 figures, including animated figures, and embedded videos, that may require using Acrobat Reader for viewing on the complete pdf file. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on the problem number sends the reader to the corresponding solution, however this feature should not be misused. This text also includes 35 exercises with solutions.
# Contents

1 Time Series ................................................................. 1
   1.1 Autoregressive Moving Average .......................... 1
   1.2 Autoregressive Integrated Moving Average .............. 6
   1.3 Stationarity of Time Series ................................ 9
   1.4 Fitting Time Series to Financial Data ................. 14
   1.5 Application: Pair Trading .............................. 17
Exercises ................................................................. 22

2 Risk Theory ............................................................ 25
   2.1 The Poisson Process ......................................... 25
   2.2 Compound Poisson Process .............................. 32
   2.3 Claim and Reserve Processes ............................ 37
   2.4 Ruin Probabilities ........................................ 38
Exercises ................................................................. 43

3 Value at Risk .......................................................... 45
   3.1 Financial Data with R ................................ 45
   3.2 Risk Measures ................................................ 46
   3.3 Quantile Risk Measures .................................... 49
   3.4 Value at Risk (VaR) ......................................... 51
Exercises ................................................................. 58

4 Expected Shortfall ...................................................... 61
   4.1 Tail Value at Risk (TVaR) .......................... 61
   4.2 Conditional tail expectation (CTE) .................... 62
   4.3 Expected Shortfall (ES) ................................. 66
   4.4 Gaussian Measures of Risk and Market Data .......... 71
Exercises ................................................................. 74
5 Correlation and Dependence ........................................ 77
  5.1 Bernoulli Random Variables ............................. 77
  5.2 Joint Gaussian Distributions ............................. 78
  5.3 Copulas and Dependence Structures ...................... 80
Exercises .................................................. 88

6 Structural Approach to Credit Risk ...................... 91
  6.1 Merton Model ......................................... 91
  6.2 Black-Cox Model ....................................... 95
  6.3 Correlated Default Times ................................ 97
Exercises .................................................. 103

7 Reduced-Form Approach to Credit Risk .................. 105
  7.1 Survival Probabilities .................................. 105
  7.2 Stochastic Default ...................................... 107
  7.3 Defaultable Bonds ...................................... 110
Exercises .................................................. 114

8 Credit Valuation .......................................... 117
  8.1 Credit Default Swaps (CDS) ............................. 117
  8.2 Collateralized Debt Obligations (CDO) .................... 121
  8.3 Credit Valuation Adjustment (CVA) ...................... 127
Exercises .................................................. 130

9 Credit Scoring ............................................ 133
  9.1 Discriminant Analysis ................................... 133
  9.2 Decision rule ........................................... 136
  9.3 Implementation ........................................ 139
  9.4 Receiver Operating Characteristic (ROC) Curves ....... 141
Exercises .................................................. 142

Appendix: Background on Probability Theory .............. 145
  10.1 Probability Spaces and Events ........................... 145
  10.2 Probability Measures ................................... 149
  10.3 Conditional Probabilities and Independence ............ 151
  10.4 Random Variables ...................................... 153
  10.5 Probability Distributions ............................... 155
  10.6 Expectation of Random Variables ........................ 162
Exercises .................................................. 177

Exercise Solutions ............................................ 181
  Chapter 1 .................................................. 181
  Chapter 2 .................................................. 184
  Chapter 3 .................................................. 185
  Chapter 4 .................................................. 188
  Chapter 5 .................................................. 192
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>MA(2) samples</td>
<td>3</td>
</tr>
<tr>
<td>1.2</td>
<td>AR(2) samples</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>ARMA(2) samples</td>
<td>6</td>
</tr>
<tr>
<td>1.4</td>
<td>ARIMA(1, 2, 3) samples</td>
<td>9</td>
</tr>
<tr>
<td>1.5</td>
<td>Autocovariances of AR(2) samples</td>
<td>11</td>
</tr>
<tr>
<td>1.6</td>
<td>Hypothesis testing</td>
<td>13</td>
</tr>
<tr>
<td>1.7</td>
<td>Stock returns</td>
<td>14</td>
</tr>
<tr>
<td>1.8</td>
<td>ARIMA(2, 0, 2) samples</td>
<td>15</td>
</tr>
<tr>
<td>1.9</td>
<td>ARIMA(0, 0, 3) samples</td>
<td>16</td>
</tr>
<tr>
<td>1.10</td>
<td>Cumulative stock returns</td>
<td>16</td>
</tr>
<tr>
<td>1.11</td>
<td>ARIMA(2, 1, 2) samples</td>
<td>17</td>
</tr>
<tr>
<td>1.12</td>
<td>MSFT vs APPL graphs</td>
<td>18</td>
</tr>
<tr>
<td>1.13</td>
<td>Comparison graph after linear regression</td>
<td>19</td>
</tr>
<tr>
<td>1.14</td>
<td>Spread graph</td>
<td>19</td>
</tr>
<tr>
<td>1.15</td>
<td>Pair trading signal</td>
<td>21</td>
</tr>
<tr>
<td>1.16</td>
<td>Pair trading returns</td>
<td>22</td>
</tr>
<tr>
<td>1.17</td>
<td>Performance of pair trading</td>
<td>22</td>
</tr>
<tr>
<td>2.1</td>
<td>Sample path of a Poisson process $N_t_{t \in \mathbb{R}_+}$</td>
<td>25</td>
</tr>
<tr>
<td>2.2</td>
<td>Sample path of a compound Poisson process $Y_t_{t \in \mathbb{R}_+}$</td>
<td>33</td>
</tr>
<tr>
<td>2.3</td>
<td>Sample path (without ruin) of a risk process $R_x(t)<em>{t \in \mathbb{R}</em>+}$</td>
<td>37</td>
</tr>
<tr>
<td>2.4</td>
<td>Sample path (with ruin) of a risk process $R_x(t)<em>{t \in \mathbb{R}</em>+}$</td>
<td>38</td>
</tr>
<tr>
<td>2.5</td>
<td>Sample paths of reserve process $^*$</td>
<td>41</td>
</tr>
<tr>
<td>3.1</td>
<td>Cumulative stock returns</td>
<td>46</td>
</tr>
<tr>
<td>3.2</td>
<td>Stock returns</td>
<td>46</td>
</tr>
<tr>
<td>3.3</td>
<td>Estimating liabilities by a conditional mean over 346 returns</td>
<td>47</td>
</tr>
<tr>
<td>3.4</td>
<td>Gaussian Cumulative distribution function $^*$</td>
<td>49</td>
</tr>
<tr>
<td>3.5</td>
<td>Cumulative distribution function with jumps $^*$</td>
<td>50</td>
</tr>
<tr>
<td>3.6</td>
<td>Empirical cumulative distribution function</td>
<td>51</td>
</tr>
<tr>
<td>3.7</td>
<td>Cumulative distribution function</td>
<td>55</td>
</tr>
</tbody>
</table>
List of Tables

5.1 Gaussian, Student, and Gumbel copulas ........................ 86
5.2 Exponential marginals with Gaussian, Student, and Gumbel copulas 88

7.1 Mortality table ........................................... 106
7.2 Cumulative historic default rates .............................. 113

11.1 CDS data .................................................. 204
11.2 Spread and survival probabilities .............................. 206