This chapter deals with the structural approach to credit risk, in which default occurs when the assets of a firm drop below a certain pre-defined level. We also consider the possibility to correlate multiple default times in this model.

6.1 Merton Model

The Merton [43] credit risk model reframes corporate debt as an option on a firm’s underlying value. Precisely the value $S_t$ of a firm’s asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

under the historical measure $\mathbb{P}$. Recall that $S_t$ is modeled as

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$$

under the risk-neutral probability measure $\mathbb{P}^*$. The company debt is represented by an amount $K$ in bonds to be paid at maturity $T$, cf. § 4.1 of [23].

Default occurs if $S_T < K$ with probability $\mathbb{P}(S_T < K)$, the bond holder will receive the recovery value $S_T$. Otherwise, if $S_T \geq K$ the bond holder receives $K$ and the equity holder is entitled to receive $S_T - K$, which can be represented as $(S_T - K)^+$ in general. The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of $S_T$ as
\[ \mathbb{P}(S_T < K | \mathcal{F}_t) = \mathbb{P}(S_0 e^{\sigma B_T + (\mu - \sigma^2/2)T} < K | \mathcal{F}_t) \]
\[ = \mathbb{P}(B_T < (-\mu - \sigma^2/2)T + \log(K/S_0)/\sigma | \mathcal{F}_t) \]
\[ = \mathbb{P}(B_T - B_t + y < (-\mu - \sigma^2/2)T + \log(K/S_0)/\sigma)_{y=B_t} \]
\[ = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{(-\mu - \sigma^2/2)(T-t) + \log(K/S_0)/\sigma} e^{-x^2/(2(T-t))} dx \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{(-\mu - \sigma^2/2)(T-t) + \log(K/S_0)/\sigma\sqrt{T-t}} e^{-x^2/2} dx \]
\[ = 1 - \Phi \left( \frac{(-\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \right) \]
\[ = 1 - \Phi(d_{\mu}^-) \]
\[ = \Phi(-d_{\mu}^-) \]
\[ = \Phi \left( -\frac{(-\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \right), \]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution, and

\[ d_{\mu}^- := \frac{(-\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}}. \]

Note that under the risk-neutral probability measure \( \mathbb{P}^* \) we have, replacing \( \mu \) with \( r \),

\[ \mathbb{P}^*(S_T < K | \mathcal{F}_t) = \Phi(-d_{\mu}^-), \]

with

\[ d_{\mu}^- = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}}, \]

which implies the relation

\[ d^r_- = d_{\mu}^- - \frac{\mu - r}{\sigma} \sqrt{T-t} \]

or

\[ \Phi^{-1}(\mathbb{P}(S_T < K | \mathcal{F}_t)) = -\frac{\mu - r}{\sigma} \sqrt{T-t} + \Phi^{-1}(\mathbb{P}^*(S_T < K | \mathcal{F}_t)). \]

The probability of default of the firm at a time \( \tau \) before \( T \) can be defined as the probability that the level of its assets falls below the level \( K \) at time \( T \). In this case the conditional distribution of \( \tau \) is given by

\[ \mathbb{P}(\tau < T | \mathcal{F}_t) := \mathbb{P}(S_T < K | \mathcal{F}_t) \]
\[ = \Phi \left( -\frac{(-\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}} \right), \quad T \geq t, \]
assuming that $S_t > K$, with the probability density function

$$
    d\mathbb{P}(\tau \leq T \mid \mathcal{F}_t) = \frac{dT}{2\sigma \sqrt{2\pi(T-t)}} \left( \frac{\sigma^2}{2} - \mu + \frac{\log(S_t/K)}{T-t} \right) \exp \left( - \frac{\left( (\mu - \frac{\sigma^2}{2}) (T-t) + \log(S_t/K) \right)^2}{2\sigma^2(T-t)} \right),
$$

provided that $\mu < \sigma^2/2$. We have

$$
    \mathbb{P}(\tau < T \mid \mathcal{F}_t) = \mathbb{P}(S_T < K \mid \mathcal{F}_t)
    = \Phi \left( \Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma} \sqrt{T-t} \right)
    = \Phi \left( \Phi^{-1}(\mathbb{P}^*(\tau < T \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma} \sqrt{T-t} \right)
$$

and

$$
    \mathbb{P}^*(\tau < T \mid \mathcal{F}_t) = \mathbb{P}^*(S_T < K \mid \mathcal{F}_t)
    = \Phi \left( - \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma \sqrt{T-t}} \right)
    = \Phi \left( \Phi^{-1}(\mathbb{P}(S_T < K \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma} \sqrt{T-t} \right)
    = \Phi \left( \Phi^{-1}(\mathbb{P}(\tau < T \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma} \sqrt{T-t} \right), \quad (6.1)
$$

Note that when $\mu < r$ we have

$$
    \mathbb{P}(\tau < T \mid \mathcal{F}_t) > \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),
$$

whereas when $\mu > r$ we get

$$
    \mathbb{P}(\tau < T \mid \mathcal{F}_t) < \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),
$$

as in the next figure.
Fig. 6.1: Function $x \mapsto \Phi \left( \Phi^{-1}(x) - (\mu - r)\sqrt{T}/\sigma \right)$ for $\mu > r$, $\mu = r$, and $\mu < r$. The discounted expected cash flow $e^{-(T-t)r} \mathbb{E}^{*}\left[ (S_T - K)^+ | \mathcal{F}_t \right]$ received by the equity holder can be estimated at time $t \in [0, T]$ as the price of a European call option from the Black-Scholes formula

$$e^{-(T-t)r} \mathbb{E}^{*}\left[ (S_T - K)^+ | \mathcal{F}_t \right] = S_t \Phi \left( \frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \right) - Ke^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \right), \quad 0 \leq t \leq T.$$

The amount received by the bond holder at maturity is $\min\{S_T, K\}$, including the recovery value $S_T$, and it can be priced at time $t \in [0, T]$ from the value of a put option with strike price $K$ on $S_T$, as

$$e^{-(T-t)r} \mathbb{E}^{*}\left[ \min\{S_T, K\} | \mathcal{F}_t \right] = e^{-(T-t)r} K - e^{-(T-t)r} \mathbb{E}^{*}\left[ (K - S_T)^+ | \mathcal{F}_t \right] = e^{-(T-t)r} K - (S_t \Phi(-d_+) - Ke^{-(T-t)r} \Phi(-d_-)) = Ke^{-(T-t)r} \Phi(d_-) - S_t \Phi(-d_+),$$

and it can be interpreted at the value $P(t, T)$ at time $t \in [0, T]$ of a default bond with face value $1$, maturity $T$ and recovery value $\min(S_T/K, 1)$. Writing

$$P(t, T) = e^{-(T-t)y_{t, T}},$$

$$y_{t, T} = -\frac{1}{T-t} \log(P(t, T)) = -\frac{1}{T-t} \log \left( e^{-(T-t)r} \mathbb{E}^{*}\left[ \min\left(1, \frac{S_T}{K}\right) | \mathcal{F}_t \right] \right)$$

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\[ r - \frac{1}{T-t} \log \left( \mathbb{E}^* \left[ \min \left( 1, \frac{S_T}{K} \right) \mid \mathcal{F}_t \right] \right) \]

\[ = r - \frac{1}{T-t} \log \left( \frac{1}{K} \mathbb{E}^* \left[ \min (K, S_T) \mid \mathcal{F}_t \right] \right) \]

\[ = r - \frac{1}{T-t} \log \left( \Phi (d_-) - \frac{S_t}{K} e^{(T-t)r} \Phi (-d_+) \right) \]

\[ > r. \]

6.2 Black-Cox Model

In the Black-Cox model [2] the firm has to maintain an account balance above the level \( K \) throughout time, therefore default occurs at the first time the process \( S_t \) hits the level \( K \), cf. § 4.2 of [23]. The default time \( \tau_K \) is therefore the first hitting time

\[ \tau_K := \inf \left\{ t \geq 0 : S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t} \leq K \right\}, \]

of the level \( K \) by

\[ (S_t)_{t \in \mathbb{R}_+} = (S_0 e^{\sigma B_t + (\mu - \sigma^2/2)t})_{t \in \mathbb{R}_+}, \]

after starting from \( S_0 > K \). Hence the default probability is given from e.g. Corollary 7.2.2 and pages 297-299 of [57], or from Relation (8.7) in [51], as

\[ \mathbb{P}(\tau_K \leq T) = \mathbb{P} \left( \min_{t \in [0,T]} S_t \leq K \right) \]

\[ = \mathbb{P} \left( \min_{t \in [0,T]} e^{\sigma B_t + (\mu - \sigma^2/2)t} \leq \frac{K}{S_0} \right) \]

\[ = \mathbb{P} \left( \min_{t \in [0,T]} \left( B_t + \frac{(\mu - \sigma^2/2)t}{\sigma} \right) \leq \frac{1}{\sigma} \log \left( \frac{K}{S_0} \right) \right) \]

\[ = \Phi \left( \frac{\log(K/S_0) - (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \]

\[ + \left( \frac{S_0}{K} \right)^{1-2\mu/\sigma^2} \Phi \left( \frac{\log(K/S_0) + (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \]

(6.2)

\[ = \mathbb{P}(S_T \leq K) + \left( \frac{S_0}{K} \right)^{1-2\mu/\sigma^2} \Phi \left( \frac{\log(K/S_0) + (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right), \]

with \( S_0 \geq K \). In this case, the cash flow

\[ \text{(6.2)} \]
\[(S_T - K)^+ \mathbb{1}_{\{\tau_K > T\}} = (S_T - K)^+ \mathbb{1}_{\left\{ \min_{t \in [0, T]} S_t > K \right\}}\]

received at maturity \(T\) by the equity holder can be priced at time \(t \in [0, T]\) as a down-and-out barrier call option with strike price \(K\) and barrier level \(K\), cf. e.g. Chapter 8 of [51], as

\[
\mathbb{E}^* \left[ (S_T - K)^+ \mathbb{1}_{\left\{ \min_{0 \leq t \leq T} S_t > K \right\}} \bigg| \mathcal{F}_t \right] = \mathbb{1}_{\left\{ \min_{t \in [0, T]} S_t > B \right\}} g(t, S_t),
\]

\(t \in [0, T]\), where

\[
g(t, S_t) = S_t \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{K} \right) \right) - e^{-(T-t)r} K \Phi \left( \delta_-^{T-t} \left( \frac{S_t}{K} \right) \right) - K \left( \frac{S_t}{K} \right)^{2r/\sigma^2} \Phi \left( \delta_+^{T-t} \left( \frac{K}{S_t} \right) \right) + e^{-(T-t)r} S_t \left( \frac{S_t}{K} \right)^{-2r/\sigma^2} \Phi \left( \delta_-^{T-t} \left( \frac{K}{S_t} \right) \right) = BS_c(S_t, r, T-t, \sigma, K) - S_t \left( \frac{K}{S_t} \right)^{2r/\sigma^2} BS_c(K/S_t, r, T-t, \sigma, 1),
\]

\(0 \leq t \leq T\), cf. Relation (8.12) and Exercise 8.2 in [51].

For \(t \geq 0\), taking now

\[
\tau_K := \inf \left\{ u \geq t : S_0 e^{\sigma B_u + (\mu - \sigma^2/2)u} \leq K \right\},
\]

the recovery value received by the bond holder at time \(\min (\tau_K, T)\) is \(K\), and it can be priced after discounting from time \(\min (\tau_K, T)\) to time \(t \in [0, T]\) as

\[
\mathbb{E}^* \left[ K e^{-\min(\tau_K, T-t)r} \big| \mathcal{F}_t \right] = \mathbb{E}^* \left[ K e^{-\tau_K r} \mathbb{1}_{\{t \leq \tau_K \leq T\}} + K e^{-(T-t)r} \mathbb{1}_{\{\tau_K > T\}} \big| \mathcal{F}_t \right] = K \mathbb{E}^* \left[ e^{-\tau_K r} \mathbb{1}_{\{t \leq \tau_K \leq T\}} \big| \mathcal{F}_t \right] + K e^{-(T-t)r} \mathbb{P}^* (\tau_K > T \big| \mathcal{F}_t) = K \mathbb{1}_{\{\tau_K > t\}} \int_t^T e^{-(u-t)r} d\mathbb{P} (\tau_K < u \big| \mathcal{F}_t) + K e^{-(T-t)r} \mathbb{P}^* (\tau_K > T \big| \mathcal{F}_t),
\]
where $P^*(\tau_K \leq u \mid \mathcal{F}_t)$ and $P^*(\tau_K > T \mid \mathcal{F}_t) = 1 - P^*(\tau_K \leq T \mid \mathcal{F}_t)$ can be computed from (6.2) as

$$P^*(\tau_K \leq u \mid \mathcal{F}_t) = \Phi \left( \frac{\log(K/S_t) - (r - \sigma^2/2)(u - t)}{\sigma \sqrt{u - t}} \right) + \left( \frac{S_t}{K} \right)^{1 - 2r/\sigma^2} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(u - t)}{\sigma \sqrt{u - t}} \right),$$

with $S_t \geq K$ and $u > t$, from which the probability density function of the hitting time $\tau_K$ can be estimated by derivation with respect to $u > t$.

Note also that we have

$$P^*(\tau_K < \infty \mid \mathcal{F}_t) = \lim_{u \to \infty} P^*(\tau_K \leq u \mid \mathcal{F}_t) = \begin{cases} \left( \frac{K}{S_t} \right)^{-1 + 2r/\sigma^2} & \text{if } r > \sigma^2/2 \\ 1 & \text{if } r \leq \sigma^2/2. \end{cases}$$

### 6.3 Correlated Default Times

In order to model correlated default and possible “domino effects”, one can regard two given default times $\tau_1$ and $\tau_2$ are correlated random variables.

Namely, given $\tau_1$ and $\tau_2$ two default times we can consider the correlation

$$\rho = \frac{\text{Cov}(\tau_1, \tau_2)}{\sqrt{\text{Var}[\tau_1] \text{Var}[\tau_2]}} \in [-1, 1].$$

When trying to build a dependence structure for the default times $\tau_1$ and $\tau_2$, the idea of [40] is to use the normalized Gaussian copula $C_\Sigma(x, y)$, with

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with correlation parameter $\rho \in [-1, 1]$, and to model the joint default probability $P(\tau_1 \leq T \text{ and } \tau_2 \leq T)$ as

$$P(\tau_1 \leq T \text{ and } \tau_2 \leq T) := C_\Sigma(P(\tau_1 \leq T), P(\tau_2 \leq T)).$$
where $C_\Sigma$ is given by (5.4).

Given two default events $A = \{\tau_1 \leq T\}$ and $B = \{\tau_2 \leq T\}$ with probabilities

\[
\mathbb{P}(\tau_1 \leq T) = 1 - \exp \left( - \int_0^T \lambda_1(s)ds \right) \quad \text{and} \quad \mathbb{P}(\tau_2 \leq T) = 1 - \exp \left( - \int_0^T \lambda_2(s)ds \right)
\]

we can also define the default correlation

\[
\rho^D = \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\sqrt{\mathbb{P}(A)(1 - \mathbb{P}(A))}\sqrt{\mathbb{P}(B)(1 - \mathbb{P}(B))}} \in [-1, 1]. \quad (6.3)
\]

In this case, the default correlation $\rho^D$ in (6.3) can be written as

\[
\rho^D = \frac{C_\Sigma (\mathbb{P}(\tau_1 \leq T), \mathbb{P}(\tau_2 \leq T)) - \mathbb{P}(\tau_1 \leq T)\mathbb{P}(\tau_2 \leq T)}{\sqrt{\mathbb{P}(\tau_1 \leq T)(1 - \mathbb{P}(\tau_1 \leq T))}\sqrt{\mathbb{P}(\tau_2 \leq T)(1 - \mathbb{P}(\tau_2 \leq T))}}.
\]

When the default probabilities are specified in the Merton model of credit risk as

\[
\mathbb{P}(\tau_i \leq T) = \mathbb{P}(S_T < K)
\]

\[
= \mathbb{P} \left( e^{\sigma_i B_T + (\mu_i - \sigma_i^2 / 2)T} < \frac{K}{S_0} \right)
\]

\[
= \mathbb{P} \left( B_T \leq -\frac{(\mu_i - \sigma_i^2 / 2)T}{\sigma_i} + \frac{1}{\sigma_i} \log \frac{K}{S_0} \right)
\]

\[
= \Phi \left( \frac{\log (K/S_0) - (\mu_i - \sigma_i^2 / 2)T}{\sigma_i \sqrt{T}} \right), \quad i = 1, 2,
\]

where

\[
(A_i^t)_{t \in \mathbb{R}_+} := (S_0 e^{\sigma_i B_t + (\mu_i - \sigma_i^2 / 2)t})_{t \in \mathbb{R}_+}, \quad i = 1, 2,
\]

the default correlation $\rho^D$ becomes

\[
\rho^D = \frac{\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T) - \mathbb{P}(\tau_1 \leq T)\mathbb{P}(\tau_2 \leq T)}{\sqrt{\mathbb{P}(\tau_1 \leq T)(1 - \mathbb{P}(\tau_1 \leq T))}\sqrt{\mathbb{P}(\tau_2 \leq T)(1 - \mathbb{P}(\tau_2 \leq T))}}
\]

\[
= \Phi_\Sigma \left( \frac{\log (S_0/K) + (\mu_1 - \sigma_1^2 / 2)T}{\sigma_1 \sqrt{T}}, \frac{\log (S_0/K) + (\mu_2 - \sigma_2^2 / 2)T}{\sigma_2 \sqrt{T}} \right) - \mathbb{P}(\tau_1 \leq T)\mathbb{P}(\tau_2 \leq T)
\]

\[
= \frac{\Phi_\Sigma \left( \log (S_0/K) + (\mu_1 - \sigma_1^2 / 2)T, \frac{\log (S_0/K) + (\mu_2 - \sigma_2^2 / 2)T}{\sigma_2 \sqrt{T}} \right) - \mathbb{P}(\tau_1 \leq T)\mathbb{P}(\tau_2 \leq T)}{\sqrt{\mathbb{P}(\tau_1 \leq T)(1 - \mathbb{P}(\tau_1 \leq T))}\sqrt{\mathbb{P}(\tau_2 \leq T)(1 - \mathbb{P}(\tau_2 \leq T))}}.
\]

In [40] it was suggested to use a single average correlation estimate, see (8.1) page 82 of the Credit Metrics™ Technical Document [25], and also the Appendix F therein.
It is worth noting that the outcomes of this methodology have been discussed in a number of magazine articles in recent years, to name a few:

“Recipe for disaster: the formula that killed Wall Street”, *Wired Magazine*, 23 February 2009, by F. Salmon [56];

“The formula that felled Wall Street”, *Financial Times Magazine*, April 24 2009, by S. Jones [34];

“Formula from hell”, *Forbes.com*, August 8 2009, by S. Lee [39], see also here.

On the other hand, a more proper definition of the default correlation $\rho^D$ should be

$$
\rho^D := \frac{\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T) - \mathbb{P}(\tau_1 \leq T) \mathbb{P}(\tau_2 \leq T)}{\sqrt{\mathbb{P}(\tau_1 \leq T)(1 - \mathbb{P}(\tau_1 \leq T))} \sqrt{\mathbb{P}(\tau_2 \leq T)(1 - \mathbb{P}(\tau_2 \leq T))}},
$$

which requires the actual computation of the joint default probability $\mathbb{P}(\tau_1 \leq T \text{ and } \tau_2 \leq T)$. An exact expression for this joint default probability in the first passage time Black-Cox model, and the associated correlation, have been recently obtained in [41].

**Multiple default times**

Consider now a sequence $(\tau_k)_{k=1,2,\ldots,n}$ of random default times. As in the Merton [43] model, cf. § 6.1, a common practice [60], [21], [28] is to parametrize the default probability associated to each $\tau_k$ by the conditioning

$$
\mathbb{P}(\tau_k \leq T \mid M = m) = \Phi \left( \Phi^{-1} \left( \mathbb{P}(\tau_k \leq T) \right) - a_k m \right) \sqrt{1 - a_k^2}^{-1},
$$

$k = 1, 2, \ldots, n,$

see (6.1), where $a_k \in (-1, 1), k = 1, 2, \ldots, n$, and $M$ is a standardized random variable with probability density function $\phi(m)$ and variance $\text{Var}[M] = 1$. Note that we have

$$
\mathbb{P}(\tau_k \leq T) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leq T \mid M = m) \phi(m) dm
$$

$$
= \int_{-\infty}^{\infty} \Phi \left( \Phi^{-1} \left( \mathbb{P}(\tau_k \leq T) \right) - a_k m \right) \phi(m) dm,
$$

and $\phi(m)$ can be typically chosen as a standard normal Gaussian density function.
The dependence structure presented in the next proposition provides an implementation of the Gaussian copula correlation method [40] in the case of multiple default times.

**Proposition 6.1.** Define the conditional Gaussian samples $X_1, X_2, \ldots, X_n$ by

$$X_k := a_k M + \sqrt{1 - a_k^2} Z_k, \quad k = 1, 2, \ldots, n,$$

(6.6)

where $Z_1, Z_2, \ldots, Z_n$ are normal random variables with same cumulative distribution function $\Phi$, independent of $M$, and let

$$\tau_k := F_{\tau_k}^{-1}(\Phi(X_k)), \quad k = 1, 2, \ldots, n,$$

(6.7)

Then the default times $(\tau_k)_{k=1,2,\ldots,n}$ have the joint distribution

$$\mathbb{P}(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = C(\mathbb{P}(\tau_1 \leq y_1), \ldots, \mathbb{P}(\tau_n \leq y_n)),$$

where

$$C(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm,$$

$x_1, x_2, \ldots, x_n \in [0, 1]$, is a Gaussian copula on $[0, 1]^n$ with covariance matrix

$$\Sigma = \begin{bmatrix}
1 & a_1 a_2 & \cdots & a_1 a_{n-1} & a_1 a_n \\
& a_2 a_1 & \ddots & \vdots & \vdots \\
& & \ddots & 1 & a_n a_{n-1} \\
& & & a_n a_1 & a_n a_{n-1} & 1
\end{bmatrix}$$

(6.8)

**Proof.** We start by recovering the conditional distribution (6.4) as follows:

$$\mathbb{P}(\tau_k \leq T \mid M = m) = \mathbb{P}(F_{\tau_k}^{-1}(\Phi(X_k)) \leq T \mid M = m)$$

$$= \mathbb{P}(\Phi(X_k) \leq F_{\tau_k}(T) \mid M = m)$$

$$= \mathbb{P}(X_k \leq \Phi^{-1}(F_{\tau_k}(T)) \mid M = m)$$

$$= \mathbb{P} \left( a_k m + \sqrt{1 - a_k^2} Z_k \leq \Phi^{-1}(F_{\tau_k}(T)) \right)$$

$$= \mathbb{P} \left( \sqrt{1 - a_k^2} Z_k \leq \Phi^{-1}(F_{\tau_k}(T)) - a_k m \right)$$

$$= \mathbb{P} \left( Z_k \leq \frac{1}{\sqrt{1 - a_k^2}} (\Phi^{-1}(F_{\tau_k}(T)) - a_k m) \right)$$
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\[ P(\tau_k \leq y_k) = P(\tau_1 \leq \infty, \ldots, \tau_k-1 \leq \infty, \tau_k \leq y_k, \tau_{k+1} \leq \infty, \ldots, \tau_n \leq \infty) \]

\[ = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(P(\tau_k \leq y_k)) - a_k m}{\sqrt{1 - a_k^2}} \right) \phi(m) \, dm \]

\[ = \int_{-\infty}^{\infty} P(\tau_k \leq T \mid M = m) \phi(m) \, dm, \quad k = 1, 2, \ldots, n. \]

Note that the above recovers the correct marginal distributions \((6.5)\), i.e. we have

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n \mid M = m) = P(\tau_1 \leq y_1 \mid M = m) \times \cdots \times P(\tau_n \leq y_n \mid M = m), \]

conditionally to \(M = m\). This yields

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = \int_{-\infty}^{\infty} P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n \mid M = m) \phi(m) \, dm \]

\[ = \int_{-\infty}^{\infty} P(\tau_1 \leq y_1 \mid M = m) \cdots P(\tau_n \leq y_n \mid M = m) \phi(m) \, dm \]

\[ = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(P(\tau_1 \leq y_1)) - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(P(\tau_n \leq y_n)) - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) \, dm. \]

In other words, we have

\[ P(\tau_1 \leq y_1, \ldots, \tau_n \leq y_n) = C(P(\tau_1 \leq y_1), \ldots, P(\tau_n \leq y_n)), \]

where the function

\[ C(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) \, dm, \]

\(x_1, x_2, \ldots, x_n \in [0, 1]\), is a Gaussian copula on \([0, 1]^n\), built as

\[ C(x_1, \ldots, x_n) = F(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n)), \]

from the Gaussian cumulative distribution function.
\[ F(x_1, \ldots, x_n) := \int_{-\infty}^{\infty} \Phi \left( \frac{x_1 - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \Phi \left( \frac{x_n - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm \]

\[ = \int_{-\infty}^{\infty} \mathbb{P} \left( Z_1 \leq \frac{x_1 - a_1 m}{\sqrt{1 - a_1^2}} \right) \cdots \mathbb{P} \left( Z_n \leq \frac{x_n - a_n m}{\sqrt{1 - a_n^2}} \right) \phi(m) dm \]

\[ = \int_{-\infty}^{\infty} \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n \mid M = m) \phi(m) dm \]

\[ = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n), \quad 0 \leq x_1, x_2, \ldots, x_n \leq 1, \]

of the vector \((X_1, \ldots, X_n)\), with covariance matrix given by (6.8). \hfill \square

When \(n = 2\) we find

\[ \Sigma = \begin{bmatrix} 1 & a_1 a_2 \\ a_2 a_1 & 1 \end{bmatrix}, \]

and letting

\[ \alpha^2 := 1 + \frac{a_1^2}{1 - a_1^2} + \frac{a_2^2}{1 - a_2^2} \]

\[ = \frac{(1 - a_1^2)(1 - a_2^2) + a_1^2(1 - a_2^2) + a_2^2(1 - a_1^2)}{(1 - a_1^2)(1 - a_2^2)} \]

\[ = \frac{1 - a_2^2 a_1^2}{(1 - a_1^2)(1 - a_2^2)}, \]

we have

\[ \Sigma^{-1} = \frac{1}{\alpha^2} \begin{bmatrix} \frac{\alpha^2(1 - a_1^2) - a_1^2}{(1 - a_1^2)^2} & -\frac{a_1 a_2}{(1 - a_1^2)(1 - a_2^2)} \\ -\frac{a_2 a_1}{(1 - a_2^2)(1 - a_1^2)} & \frac{(1 - a_1^2)(1 - a_2^2)}{\alpha^2(1 - a_2^2) - a_2^2} \end{bmatrix} \]

\[ = \frac{1}{\alpha^2} \begin{bmatrix} \frac{\alpha^2}{1 - a_1^2} & -\frac{a_1 a_2}{\alpha^2(1 - a_2^2)} \\ -\frac{a_2 a_1}{(1 - a_2^2)(1 - a_1^2)} & \frac{1 - a_2^2 a_1^2}{\alpha^2(1 - a_2^2) - a_2^2} \end{bmatrix} \]

\[ = \frac{1}{\alpha^2} \begin{bmatrix} \frac{\alpha^2}{1 - a_1^2} & -\frac{a_1 a_2}{\alpha^2 a_1^2} \\ -\frac{a_2 a_1}{(1 - a_2^2)(1 - a_1^2)} & \frac{1 - a_2^2 a_1^2}{\alpha^2 a_1^2} \end{bmatrix} \]

\[ = \frac{(1 - a_1^2)(1 - a_2^2)}{1 - a_2^2 a_1^2} \begin{bmatrix} \alpha^2 & -\frac{a_1 a_2}{\alpha^2} \\ -\frac{a_2 a_1}{(1 - a_2^2)(1 - a_1^2)} & \frac{1 - a_2^2 a_1^2}{\alpha^2} \end{bmatrix} \]

\[ = \frac{1}{\alpha^2} \begin{bmatrix} \frac{\alpha^2}{1 - a_1^2} & -\frac{a_1 a_2}{\alpha^2 a_1^2} \\ -\frac{a_2 a_1}{(1 - a_2^2)(1 - a_1^2)} & \frac{1 - a_2^2 a_1^2}{\alpha^2 a_1^2} \end{bmatrix} \]
In particular, the case \( n = 2 \) is able to recover all two-dimensional copulas by setting the correlation coefficient \( \rho = a_1 a_2 \). In the general case, \( \Sigma \) is parametrized by \( n \) numbers, which offers less degrees of freedom compared with the joint Gaussian copula correlation method which relies on \( n(n - 1)/2 \) coefficients, see also Exercise 6.2.

**Exercises**

Exercise 6.1  Credit Default Contract. The assets of a company are modeled using a geometric Brownian motion \( (S_t)_{t \in \mathbb{R}^+} \) with drift \( r > 0 \) under the risk-neutral probability measure \( \mathbb{P}^* \). A Credit Default Contract pays $1 as soon as the asset \( S_t \) hits a level \( K > 0 \). Price this contract at time \( t > 0 \) assuming that \( S_t > K \).

Exercise 6.2

a) Check that the vector \( (X_1, X_2, \ldots, X_n) \) defined in (6.6) has the covariance matrix given by (6.8).

b) Show that the vector \( (X_1, X_2, \ldots, X_n) \), with covariance matrix (6.8) has standard Gaussian marginals.

c) By computing explicitly the probability density function of \( (X_1, \ldots, X_n) \), recover the fact that it is a jointly Gaussian random vector with covariance matrix (6.8).