Chapter 7
Reduced-Form Approach to Credit Risk

This chapter deals with the reduced-form approach to credit risk, which relies on a failure rate process and an exogeneous random variable, resulting into an enlarged filtration that incorporates the knowledge of the default event.

7.1 Survival Probabilities

Given \( t > 0 \), let \( \mathbb{P}(\tau > t) \) denote the probability that a random system with lifetime \( \tau \) survives at least \( t \) years. Assuming that survival probabilities \( \mathbb{P}(\tau > t) \) are strictly positive for all \( t > 0 \), we can compute the conditional probability for that system to survive up to time \( T \), given that it was still functioning at time \( t \in [0, T] \), as

\[
\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \quad 0 \leq t \leq T,
\]

with

\[
\mathbb{P}(\tau \leq T \mid \tau > t) = 1 - \mathbb{P}(\tau > T \mid \tau > t)
= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}
= \frac{\mathbb{P}(\tau \leq T) - \mathbb{P}(\tau \leq t)}{\mathbb{P}(\tau > t)}
= \frac{\mathbb{P}(t < \tau \leq T)}{\mathbb{P}(\tau > t)}, \quad 0 \leq t \leq T,
\]
and the conditional survival probability distribution
\[
P(\tau \in dx \mid \tau > t) = P(x < \tau \leq x + dx \mid \tau > t) = P(\tau \leq x + dx \mid \tau > t) - P(\tau < x \mid \tau > t)
\]
\[
= P(\tau \leq x + dx) - P(\tau < x) - \frac{P(\tau > t)}{P(\tau > t)}
\]
\[
= \frac{1}{P(\tau > t)} dP(\tau \leq x)
\]
\[
= -\frac{1}{P(\tau > t)} dP(\tau > x), \quad x > t.
\]
Such survival probabilities are typically found in life or mortality tables:

<table>
<thead>
<tr>
<th>Age t</th>
<th>P(\tau \leq t + 1 \mid \tau &gt; t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0894%</td>
</tr>
<tr>
<td>30</td>
<td>0.1008%</td>
</tr>
<tr>
<td>40</td>
<td>0.2038%</td>
</tr>
<tr>
<td>50</td>
<td>0.4458%</td>
</tr>
<tr>
<td>60</td>
<td>0.9827%</td>
</tr>
</tbody>
</table>

Table 7.1: Mortality table.

From this we can deduce the failure rate function
\[
\lambda(t) := \frac{P(\tau \leq t + dt \mid \tau > t)}{dt
\]
\[
= \frac{1}{P(\tau > t)} \frac{P(t < \tau \leq t + dt)}{dt
\]
\[
= \frac{1}{P(\tau > t)} \frac{P(\tau > t) - P(\tau > t + dt)}{dt
\]
\[
= -\frac{d}{dt} \log P(\tau > t)
\]
\[
= -\frac{1}{P(\tau > t)} \frac{d}{dt} P(\tau > t), \quad t > 0,
\]
and the differential equation
\[
\frac{d}{dt} P(\tau > t) = -\lambda(t) P(\tau > t),
\]
which can be solved as
\[
P(\tau > t) = \exp \left( -\int_0^t \lambda(u) du \right), \quad t \in \mathbb{R}_+,
\]
(7.1)
under the initial condition $\mathbb{P}(\tau > 0) = 1$. This allows us to rewrite the (conditional) survival probability as

$$
\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp \left( - \int_t^T \lambda(u) du \right), \quad 0 \leq t \leq T,
$$

with

$$
\mathbb{P}(\tau > t + h \mid \tau > t) = e^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \quad [h \searrow 0],
$$

and

$$
\mathbb{P}(\tau \leq t + h \mid \tau > t) = 1 - e^{-\lambda(t)h} \simeq \lambda(t)h, \quad [h \searrow 0],
$$

as $h$ tends to 0. When the failure rate $\lambda(t) = \lambda > 0$ is a constant function of time, Relation (7.1) shows that

$$
\mathbb{P}(\tau > T) = e^{-\lambda T}, \quad T \geq 0,
$$

i.e. $\tau$ has the exponential distribution with parameter $\lambda$. Note that given $(\tau_n)_{n \geq 1}$ a sequence of i.i.d. exponentially distributed random variables, letting

$$
T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \quad n \geq 1,
$$

defines the sequence of jump times of a standard Poisson process with intensity $\lambda > 0$.

### 7.2 Stochastic Default

When the random time $\tau$ is a stopping time with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ we have

$$
\{\tau > t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}^+,
$$

i.e. the knowledge of whether default already occurred at time $t$ is contained in $\mathcal{F}_t$, $t \in \mathbb{R}^+$, cf. e.g. Section 9.3 of [51]. As a consequence, we can write

$$
\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E} [\mathbbm{1}_{\{\tau > t\}} \mid \mathcal{F}_t] = \mathbbm{1}_{\{\tau > t\}}, \quad t \in \mathbb{R}^+.
$$

In the sequel we will not assume that $\tau$ is an $\mathcal{F}_t$-stopping time, and by analogy with (7.1) we will write $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ as

$$
\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp \left( - \int_0^t \lambda_u du \right), \quad t > 0, \quad (7.2)
$$

where the failure rate function $(\lambda_t)_{t \in \mathbb{R}^+}$ is modeled as a random process adapted to a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$.

The process $(\lambda_t)_{t \in \mathbb{R}^+}$ can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In [38], the process
$(\lambda_t)_{t \in \mathbb{R}_+}$ is constructed as $\lambda_t := h(X_t)$, $t \in \mathbb{R}_+$, where $h$ is a nonnegative function and $(X_t)_{t \in \mathbb{R}_+}$ is a stochastic process generating the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. The default time $\tau$ is then defined as

$$\tau := \inf \left\{ t \in \mathbb{R}_+ : \int_0^t h(X_u)du \geq L \right\},$$

where $L$ is an exponentially distributed random variable independent of $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. In this case, we have

$$P(\tau > t | \mathcal{F}_t) = \mathbb{P}\left( \int_0^t h(X_u)du \geq L | \mathcal{F}_t \right) = \exp\left( -\int_0^t h(X_u)du \right) = \exp\left( -\int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_+. $$

Let now $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$ be the filtration defined by $\mathcal{G}_\infty := \mathcal{F}_\infty \lor \sigma(\tau)$ and

$$\mathcal{G}_t := \{ B \in \mathcal{G}_\infty : \exists A \in \mathcal{F}_t \text{ such that } A \cap \{ \tau > t \} = B \cap \{ \tau > t \} \}, \quad (7.3)$$

with $\mathcal{F}_t \subset \mathcal{G}_t$, $t \in \mathbb{R}_+$. In other words, $\mathcal{G}_t$ contains the additional information on whether default at time $\tau$ has occurred or not before time $t$.

Taking $F = 1$ in the next key Lemma 7.1 allows one to write the survival probability up to time $T$, given information known up to $t$, as

$$P(\tau > T | \mathcal{G}_t) = \mathbb{E}\left[ \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[ \exp\left( -\int_t^T \lambda_u du \right) | \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (7.4)$$

Note that in general, $\tau$ is not $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-stopping time, and Lemma 7.1 applied for $t = T$ and $F = 1$ shows that

$$\mathbb{E}\left[ \mathbb{1}_{\{\tau > t\}} | \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}},$$

hence $\{\tau > t\} \in \mathcal{G}_t$ for all $t > 0$, and $\tau$ is a $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$-stopping time.

**Lemma 7.1.** ([24]) For any $\mathcal{F}_T$-measurable integrable random variable $F$ we have

$$\mathbb{E}\left[ F \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[ F \exp\left( -\int_t^T \lambda_u du \right) | \mathcal{F}_t \right].$$

**Proof.** By (7.2) we have
\[
\frac{\mathbb{P}(\tau > T \mid \mathcal{F}_T)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} = \frac{e^{\int_0^T \lambda_u du}}{e^{\int_0^t \lambda_u du}} = \exp \left( - \int_t^T \lambda_u du \right),
\]
hence, since \( F \) is \( \mathcal{F}_T \)-measurable,

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \exp \left( - \int_t^T \lambda_u du \right) \mid \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{P}(\tau > T \mid \mathcal{F}_T) \mid \mathcal{F}_t \right] = \frac{\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_T \right]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} = \mathbb{E} \left[ F \mathbb{1}_{\{\tau > t\}} \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.
\]
In the last step of the above argument we used the key relation

\[
\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] = \frac{\mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)},
\]

cf. Relation (75.2) in § XX-75 of [12], Theorem VI-3-14 of [54], and Lemma 3.1 of [17], under the probability measure \( \mathbb{P}_{|\mathcal{F}_t}, 0 \leq t \leq T \). Indeed, according to (7.3), for any \( B \in \mathcal{G}_t \) we have, for some event \( A \in \mathcal{F}_t, \)

\[
\mathbb{E} \left[ \mathbb{1}_B \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_{B \cap \{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_{A \cap \{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{1}_{\{\tau > T\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} F \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{1}_{\{\tau > T\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] \right] = \mathbb{E} \left[ \mathbb{1}_A \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{1}_{\{\tau > T\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E} \left[ \mathbb{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right] \right].
\]
hence by a standard characterization of conditional expectations, see e.g. § 1.7 of [50], we have

\[ \mathbb{E} \left[ \mathbb{1}_{\{\tau > t\}} F \mathbb{1}_{\{\tau > T\}} \mid G_t \right] = \frac{\mathbb{1}_{\{\tau > t\}}}{\mathbb{P}(\tau > t \mid F_t)} \mathbb{E} \left[ F \mathbb{1}_{\{\tau > T\}} \mid F_t \right] \]

\[ \square \]

The computation of \( \mathbb{P}(\tau > T \mid G_t) \) according to (7.4) is then similar to that of a bond price, by considering the failure rate \( \lambda(t) \) as a virtual short term interest rate. In particular the failure rate \( \lambda(t, T) \) can be modeled in the HJM framework, cf. e.g. Chapter 11.4 of [51], and

\[ \mathbb{P}(\tau > T \mid G_t) = \mathbb{E} \left[ \exp \left( -\int_t^T \lambda(t, u) du \right) \mid F_t \right] \]

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given \( G_t \) as in Lemma 7.1 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration \( G_t \) while the ordinary trader has only access to \( F_t \), therefore generating two different prices \( \mathbb{E}^*[F \mid F_t] \) and \( \mathbb{E}^*[F \mid G_t] \) for the same claim \( F \) under the same risk-neutral probability measure \( \mathbb{P}^* \). This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a \( F_t \)-martingale vs a \( G_t \)-martingale instead of using different forward measures as in e.g. § 8 of [51]. This can be obtained by the technique of enlargement of filtration, cf. [33], [16], [30], [61].

### 7.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition \( P(T, T) = 1 \) according to which the bond payoff at maturity is always equal to 1, and default does not occurs. In this chapter we allow for the possibility of default at a random time \( \tau \), in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price \( P_d(t, T) \) at time \( t \) of a default bond with maturity \( T \), (random) default time \( \tau \) and (possibly random) recovery rate \( \xi \in [0, 1] \) is given by

\[ P_d(t, T) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( -\int_t^T r_u du \right) \mid G_t \right] \]
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\[ + \mathbb{E}^* \left[ \xi 1_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T. \]

**Proposition 7.2.** The default bond with maturity \( T \) and default time \( \tau \) can be priced at time \( t \in [0, T] \) as

\[ P_d(t, T) = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right] \]

\[ + \mathbb{E}^* \left[ \xi 1_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T. \]

**Proof.** We take \( F = \exp \left( - \int_t^T r_u du \right) \) in Lemma 7.1, which shows that

\[ \mathbb{E}^* \left[ 1_{\{\tau > t\}} \exp \left( - \int_t^T r_u du \right) \mid \mathcal{G}_t \right] = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right], \]

cf. e.g. [38], [24], [15]. \( \square \)

In the case of complete default (zero-recovery) we have \( \xi = 0 \) and

\[ P_d(t, T) = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (7.5) \]

From the above expression (7.5) we note that the effect of the presence of a default time \( \tau \) is to decrease the bond price, which can be viewed as an increase of the short rate by the amount \( \lambda_u \). In a simple setting where the interest rate \( r > 0 \) and failure rate \( \lambda > 0 \) are constant, the default bond price becomes

\[ P_d(t, T) = 1_{\{\tau > t\}} e^{-(r+\lambda)(T-t)}, \quad 0 \leq t \leq T. \]

Finally, from e.g. Proposition 12.1 of [51] the bond price (7.5) can also be expressed under the forward measure \( \hat{\mathbb{P}} \) with maturity \( T \), as

\[ P_d(t, T) = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right] \]

\[ = 1_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right] \hat{\mathbb{E}} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \mid \mathcal{F}_t \right] \]

\[ = 1_{\{\tau > t\}} \hat{N}_t Q(t, T), \]

where \((N_t)_{t \in \mathbb{R}_+}\) is the numéraire process

\[ N_t := P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \]

and by (7.4),

\[ \hat{\mathbb{E}} \]
\[
Q(t, T) := \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[ \exp \left( - \int_t^T \lambda_s ds \right) \right] = \hat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t)
\]
denotes the survival probability under the forward measure \(\hat{\mathbb{P}}\) defined as
\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} := \frac{N_T}{N_0} e^{-\int_0^T r_t dt},
\]
cf. [8], [7],

**Estimating the default rates**

Recall that the price of a default bond with maturity \(T\), (random) default time \(\tau\) and (possibly random) recovery rate \(\xi \in [0, 1]\) is given by
\[
P_d(t, T) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \right] \mathcal{F}_t
\]
\[
+ \mathbb{E}^* \left[ \xi \mathbb{1}_{\{\tau \leq T\}} \exp \left( - \int_t^T r_u du \right) \right] \mathcal{G}_t, \quad 0 \leq t \leq T,
\]
where \(\xi\) denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure
\[
\{t = T_0 < T_1 < \cdots < T_n = T\},
\]
where
\[
r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l, T_{l+1})}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l, T_{l+1})}(t), \quad t \in \mathbb{R}_+. \quad (7.6)
\]

i) Estimating the default rates from default bond prices.

We have
\[
P_d(t, T_k) = \mathbb{1}_{\{\tau > t\}} \exp \left( - \int_t^{T_k} (r(u) + \lambda(u)) du \right)
\]
\[
= \mathbb{1}_{\{\tau > t\}} \exp \left( - \sum_{l=0}^{k-1} (r_l + \lambda_l)(T_{l+1} - T_l) \right),
\]
k = 1, 2, \ldots, n, from which we can infer
\[
\lambda_k = -r_k + \frac{1}{T_{k+1} - T_k} \log P_d(T_{k+1}, T) P_d(T_k, T), \quad k = 0, 1, \ldots, n - 1.
\]
ii) Estimating (implied) default probabilities from default rates.

Based on the expression

\[
P^*(\tau > T \mid G_t) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \mid G_t \right]
= \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \mid F_t \right], \quad 0 \leq t \leq T,
\]

of the survival probability up to time \( T \), and given information known up to \( t \), in terms of the hazard rate process \( (\lambda_u)_{u \in \mathbb{R}_+} \) adapted to a filtration \( (F_t)_{t \in \mathbb{R}_+} \), we find

\[
P(\tau > T \mid G_{T_k}) = \mathbb{1}_{\{\tau > T_k\}} \exp \left( - \int_{T_k}^T \lambda_u du \right)
= \mathbb{1}_{\{\tau > t\}} \exp \left( - \sum_{l=k}^{n-1} \lambda_l(T_{l+1} - T_l) \right), \quad k = 0, 1, \ldots, n - 1,
\]

where

\[G_t = F_t \vee \sigma(\{\tau \leq u\} : 0 \leq u \leq t), \quad t \in \mathbb{R}_+,
\]

i.e. \( G_t \) contains the additional information on whether default at time \( \tau \) has occurred or not before time \( t \).

In Table 7.2, bond ratings are determined according to hazard (or failure) rate thresholds.

<table>
<thead>
<tr>
<th>Bond rating categories</th>
<th>Moody’s Municipal</th>
<th>Moody’s Corporate</th>
<th>S&amp;P Municipal</th>
<th>S&amp;P Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa/AAAAs</td>
<td>0.00</td>
<td>0.52</td>
<td>0.00</td>
<td>0.60</td>
</tr>
<tr>
<td>Aa/AA</td>
<td>0.06</td>
<td>0.52</td>
<td>0.00</td>
<td>1.50</td>
</tr>
<tr>
<td>A/A</td>
<td>0.03</td>
<td>1.29</td>
<td>0.23</td>
<td>2.91</td>
</tr>
<tr>
<td>Baa/BBB</td>
<td>0.13</td>
<td>4.64</td>
<td>0.32</td>
<td>10.29</td>
</tr>
<tr>
<td>Ba/BB</td>
<td>2.65</td>
<td>19.12</td>
<td>1.74</td>
<td>29.93</td>
</tr>
<tr>
<td>B/B</td>
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<td>43.34</td>
<td>8.48</td>
<td>53.72</td>
</tr>
<tr>
<td>Caa-C/CCC-C</td>
<td>16.58</td>
<td>69.18</td>
<td>44.81</td>
<td>69.19</td>
</tr>
<tr>
<td>Investment Grade</td>
<td>0.07</td>
<td>2.09</td>
<td>0.20</td>
<td>4.14</td>
</tr>
<tr>
<td>Non-Invest Grade</td>
<td>4.29</td>
<td>31.37</td>
<td>7.37</td>
<td>42.35</td>
</tr>
<tr>
<td>All</td>
<td>0.10</td>
<td>9.70</td>
<td>0.29</td>
<td>12.98</td>
</tr>
</tbody>
</table>

Table 7.2: Cumulative historic default rates (in percentage).*

* Source: Moody’s, S&P.
Exercises

Exercise 7.1  Consider a standard zero-coupon bond with constant yield \( r > 0 \) and a defaultable (risky) bond with constant yield \( r_d \) and default probability \( \alpha \in (0, 1) \). Find a relation between \( r, r_d, \alpha \) and the bond maturity \( T \).

Exercise 7.2  A standard zero-coupon bond with constant yield \( r > 0 \) and maturity \( T \) is priced \( P(t, T) = e^{-(T-t)r} \) at time \( t \in [0, T] \). Assume that the company can get bankrupt at a random time \( t + \tau \), and default on its final $1 payment if \( \tau < T - t \).

a) Explain why the defaultable bond price \( P_d(t, T) \) can be expressed as

\[
P_d(t, T) = e^{-(T-t)r} \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T-t\}} \right].
\]  

(7.8)

b) Assuming that the default time \( \tau \) is exponentially distributed with parameter \( \lambda > 0 \), compute the default bond price \( P_d(t, T) \) using (7.8).

c) Find a formula that can estimate the parameter \( \lambda \) from the risk-free rate \( r \) and the market data \( P_M(t, T) \) of the defaultable bond price at time \( t \in [0, T] \).

Exercise 7.3  Consider a (random) default time \( \tau \) with cumulative distribution function

\[
\mathbb{P}(\tau > t \mid F_t) = \exp \left( - \int_0^t \lambda_u du \right), \quad t \in \mathbb{R}_,
\]

where \( \lambda_t \) is a (random) default rate process which is adapted to the filtration \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \). Recall that the probability of survival up to time \( T \), given information known up to time \( t \), is given by

\[
\mathbb{P}(\tau > T \mid G_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T \lambda_u du \right) \bigg| \mathcal{F}_t \right],
\]

where \( G_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} : 0 \leq u \leq t), t \in \mathbb{R}_+, \) is the filtration defined by adding the default time information to the history \( (\mathcal{F}_t)_{t \in \mathbb{R}_+} \). In this framework, the price \( P(t, T) \) of defaultable bond with maturity \( T \), short term interest rate \( r_t \) and (random) default time \( \tau \) is given by

\[
P(t, T) = \mathbb{E}^* \left[ \mathbb{1}_{\{\tau > T\}} \exp \left( - \int_t^T r_u du \right) \bigg| G_t \right],
\]

(7.9)

\[
= \mathbb{1}_{\{\tau > t\}} \mathbb{E}^* \left[ \exp \left( - \int_t^T (r_u + \lambda_u) du \right) \bigg| \mathcal{F}_t \right].
\]
In the sequel we assume that the processes \((r_t)_{t \in \mathbb{R}^+}\) and \((\lambda_t)_{t \in \mathbb{R}^+}\) are modeled according to the Vasicek processes

\[
\begin{align*}
    dr_t &= -ar_t dt + \sigma dB^{(1)}_t, \\
    d\lambda_t &= -b\lambda_t dt + \eta dB^{(2)}_t,
\end{align*}
\]

where \((B^{(1)}_t)_{t \in \mathbb{R}^+}\) and \((B^{(2)}_t)_{t \in \mathbb{R}^+}\) are two standard \(\mathcal{F}_t\)-Brownian motions with correlation \(\rho \in [-1, 1]\), and \(dB^{(1)}_t \cdot dB^{(2)}_t = \rho dt\).

a) Give a justification for the fact that

\[
\mathbb{E}^* \left[ \exp \left( -\int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right]
\]

can be written as a function \(F(t, r_t, \lambda_t)\) of \(t, r_t\) and \(\lambda_t, t \in [0, T]\).

b) Show that

\[
t \mapsto \exp \left( -\int_0^t (r_s + \lambda_s) ds \right) \mathbb{E}^* \left[ \exp \left( -\int_t^T (r_u + \lambda_u) du \right) \mid \mathcal{F}_t \right]
\]

is an \(\mathcal{F}_t\)-martingale under \(\mathbb{P}\).

c) Use the Itô formula with two variables to derive a PDE on \(\mathbb{R}^2\) for the function \(F(t, x, y)\).

d) Show that we have

\[
\int_t^T r_s ds = C(a, t, T)r_t + \sigma \int_t^T C(a, s, T) dB^{(1)}_s,
\]

and

\[
\int_t^T \lambda_s ds = C(b, t, T)\lambda_t + \eta \int_t^T C(b, s, T) dB^{(2)}_s,
\]

where

\[
C(a, t, T) = -\frac{1}{a} (e^{-a(T-t)} - 1).
\]

e) Show that the random variable

\[
\int_t^T r_s ds + \int_t^T \lambda_s ds
\]

is Gaussian and compute its conditional mean

\[
\mathbb{E}^* \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right]
\]

and variance

\[
\text{Var} \left[ \int_t^T r_s ds + \int_t^T \lambda_s ds \mid \mathcal{F}_t \right],
\]
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conditionally to $\mathcal{F}_t$.

f) Compute $P(t, T)$ from its expression (7.9) as a conditional expectation.

h) Show that the defaultable bond price $P(t, T)$ can also be written as

$$P(t, T) = e^{U(t, T)} \mathbb{P}(\tau > T \mid \mathcal{G}_t) \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \bigg| \mathcal{F}_t \right],$$

where

$$U(t, T) = \rho \sigma \eta a b (T - t - C(a, t, T) - C(b, t, T) + C(a + b, t, T)).$$

i) By partial differentiation of $\log P(t, T)$ with respect to $T$, compute the corresponding instantaneous short rate

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$

j) Show that $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ can be written using an HJM type default rate as

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = 1_{\{\tau > t\}} \exp \left( - \int_t^T f_2(t, u) du \right),$$

where

$$f_2(t, u) = \lambda_t e^{-b(u-t)} - \frac{\eta^2}{2} C^2(b, t, u).$$

k) Show how the result of Question (h) can be simplified when $(B_t^{(1)})_{t \in \mathbb{R}^+}$ and $(B_t^{(2)})_{t \in \mathbb{R}^+}$ are independent.