Chapter 7  
Estimation of Volatility

The values of the parameters $r$, $t$, $S_t$, $T$, and $K$ used to price a call option via the Black-Scholes formula can be easily obtained from market data. Estimating the volatility coefficient $\sigma$ can be a more difficult task, and several estimation methods are considered in this section with some examples of how the Black-Scholes formula can be fitted to market data. We cover the historical, implied, and local volatility models, and refer to [37] for stochastic volatility models.

7.1 Historical Volatility

We consider the problem of estimating the parameters $\mu$ and $\sigma$ from market data in the stock price model

$$
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t.
$$

A natural estimator for the trend parameter $\mu$ can be written as

$$
\hat{\mu}_N := \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}}, 
$$

(7.1)

where $(S_{t_{k+1}} - S_{t_k})/S_{t_k}, k = 0, \ldots, N - 1$ is a family of returns observed at different times $t_0, \ldots, t_N$ on the market.

Observe that by replacing (7.1) by actual log-returns with $t_{k+1} - t_k = T/N$, $k = 0, 1, \ldots, N - 1$, one can get the simplified estimate

$$
\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} (\log S_{t_{k+1}} - \log S_{t_k}) = \frac{1}{T} \log \frac{S_T}{S_0}.
$$
Similarly the parameter $\sigma$ can be estimated as by the estimator $\hat{\sigma}_N$ built as

$$
\hat{\sigma}^2_N := \frac{1}{N-1} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} - \hat{\mu}_N(t_{k+1} - t_k) \right)^2.
$$

Parameter estimation based on historical data requires a lot of samples and it can only be valid on a given time interval, or as a moving average.

Fig. 7.1: [110] “The fugazi: it’s a wazy, it’s a woozie. It’s fairy dust.”

7.2 Implied Volatility

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data.

Recall that when $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE is given by

$$
f(t, x, K, \sigma, r, T) = x\Phi(d_+) - Ke^{-(T-t)r}\Phi(d_-),
$$

where

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},
$$

and

* Click on the figure to play the video (works in Acrobat reader on the entire pdf file).
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\[ d_+ = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_- = \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}. \]

Equating

\[ f(t,S_t,K,\sigma,r,T) = M \]

to the observed value \( M \) of a given market price, when \( t, S_t, r, T \) are known, allows one to infer a value for \( \sigma \), as in e.g. Figure 5.10.

This value is called the implied volatility and denoted here by \( \sigma^{\text{imp}}(K,T) \). The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, cf. Figure S.1.

Given two European call options with strikes \( K_1 \), resp. \( K_2 \) and maturities \( T_1 \), resp. \( T_2 \), on the same stock \( S \), this procedure should yield two estimates \( \sigma^{\text{imp}}(K_1,T_1) \) and \( \sigma^{\text{imp}}(K_2,T_2) \) of implied volatilities.

Clearly, there is no reason a priori for the implied volatilities \( \sigma^{\text{imp}}(K_1,T_1) \) and \( \sigma^{\text{imp}}(K_2,T_2) \) to coincide. However, in the standard Black-Scholes model the value of the parameter \( \sigma \) should be unique for a given stock \( S \). This contradiction between a model and market data is a reason for the development of more sophisticated stochastic volatility models.

Plotting the different values of the implied volatility \( \sigma \) as a function of \( K \) and \( T \) will yield a planar curve called the volatility surface.

Figure 7.2 presents an estimation of implied volatility for Asian options whose underlying asset is the price of light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange.
Fig. 7.2: Implied volatility of Asian options on light sweet crude oil futures.

As observed in Figure 7.2, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike values.

### 7.3 The Black-Scholes Formula vs Market Data

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price $S$ of Cheung Kong Holdings (0001.HK) with Strike $K=\$109.99$, Maturity $T =$ December 13, 2010, and entitlement ratio 100.

Fig. 7.3: Graph of the (market) stock price of Cheung Kong Holdings.

* This graph is courtesy of Tan Yu Jia.
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The market price of the option (17838.HK) on September 28 was $12.30, as obtained from http://www.hkex.com.hk/dwrc/search/listsearch.asp

The next graph in Figure 7.4 shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying stock price.

![Fig. 7.4: Graph of the (market) call option price on Cheung Kong Holdings.](image)

In Figure 7.5 we have fitted the path

\[ t \mapsto g_c(t, S_t) \]

of the Black-Scholes price to the data of Figure 7.4 using the stock price data of Figure 7.3, by varying the values of the volatility \( \sigma \).

![Fig. 7.5: Graph of the Black-Scholes call option price on Cheung Kong Holdings.](image)
Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

![Graph of the stock price of HSBC Holdings.](image1)

**Fig. 7.6:** Graph of the (market) stock price of HSBC Holdings.

Next we consider the graph of the price of a call option issued by Societe Generale on 31 December 2008 with strike \( K = 63.704 \), maturity \( T = \text{October 05, 2009} \), and entitlement ratio 100, cf. page 6.

![Graph of the call option price on HSBC Holdings.](image2)

**Fig. 7.7:** Graph of the (market) call option price on HSBC Holdings.

As above, in Figure 7.8 we have fitted the path \( t \mapsto g_c(t, S_t) \) of the Black-Scholes price to the data of Figure 7.7 using the stock price data of Figure 7.6. In this case we are in the money at maturity, and we also check that the option is worth \( 100 \times 0.2650 = 26.650 \) at that time which, by absence of arbitrage, is very close to the value \( \$90 - \$63.703 = \$26.296 \) of its payoff.
For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 with strike \( K = 77.667 \), maturity \( T = \text{October 05, 2009} \), and entitlement ratio 92.593.

One checks easily that at maturity, the price of the put option is worth $0.01 (a market price cannot be lower), which almost equals the option payoff $0, by absence of arbitrage opportunities. Figure 7.10 is a fit of the Black-Scholes put price graph

\[ t \mapsto g_p(t, S_t) \]

to Figure 7.9 as a function of the stock price data of Figure 7.8. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.

The normalized market data graph in Figure 7.11 shows how the option price can track the values of the underlying. Note that the range of values [26.55, 26.90] for the underlying corresponds to [0.675, 0.715] for the option.
Fig. 7.10: Graph of the Black-Scholes put option price on HSBC Holdings.

price, meaning 1.36% vs 5.9% in percentage. This is a European call option on the ALSTOM underlying with strike price \( K = €20 \), maturity March 20, 2015, and entitlement ratio 10.

Fig. 7.11: Call option price vs ALSTOM underlying.

### 7.4 Local Volatility

Since the constant volatility assumption in the Black-Scholes model appears to be not satisfying due to the existence of volatility smiles, it makes sense to consider models of the form

\[
\frac{dS_t}{S_t} = r dt + \sigma_t dB_t
\]
where $\sigma_t$ is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$ \frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dB_t $$ (7.2)

where $\sigma(t, x)$ is a deterministic function of time and the stock price. Such models are called local volatility models. The corresponding Black-Scholes PDE can be written as

$$ \begin{cases} rg(t, x, K) = \frac{\partial g}{\partial t}(t, x, K) + rx\frac{\partial g}{\partial x}(t, x, K) + \frac{1}{2}x^2\sigma^2(t, x)\frac{\partial^2 g}{\partial x^2}(t, x, K), \\ g(T, x, K) = (x - K)^+, \end{cases} $$ (7.3)

with terminal condition $g(T, x, K) = (x - K)^+$, i.e. we consider European call options.

Note that the Black-Scholes PDE would allow one to recover the value of $\sigma(t, x)$ as a function of the option price $g(t, x, K)$, as

$$ \sigma(t, x) = \sqrt{\frac{2rg(t, x, K) - 2\frac{\partial g}{\partial t}(t, x, K) - 2rx\frac{\partial g}{\partial x}(t, x, K)}{x^2\frac{\partial^2 g}{\partial x^2}(t, x, K)}}, \quad x, t > 0, $$

however this formula requires the knowledge of the option price for different values of the underlying $x$, in addition to the knowledge of the strike price $K$.

The Dupire formula brings a solution to the local volatility calibration problem by providing an estimator of $\sigma(t, x)$ as a function of $\sigma(t, K)$ based on the values of the strike price $K$.

**Proposition 7.1.** Assume that a family $(C(T, K))_{T, K > 0}$ of market call option prices with maturities $T$ and strikes $K$ is given at time $t$ with $S_t = x$, while the values of $r$ and $x$ are fixed.

The Dupire formula states that, defining the volatility function $\sigma(t, y)$ by

$$ \sigma(t, y) := \sqrt{\frac{2\frac{\partial C}{\partial t}(t, y) + 2ry\frac{\partial C}{\partial y}(t, y)}{y^2\frac{\partial^2 C}{\partial y^2}(t, y)}}, $$ (7.4)
the prices $g(t, x, K)$ computed from the Black-Scholes PDE (7.3) will match the option prices $C(T, K)$ in the sense that

$$g(t, x, K) = C(T, K), \quad T, K > 0.$$  \hfill (7.5)

Proof. We use the probabilistic approach that allows us to write $g(t, x, K)$ as

$$g(t, x, K) = e^{-r(T-t)} \mathbb{E}[(S_T - K)^+ | S_t = x],$$  \hfill (7.6)

where $(S_t)_{t \in \mathbb{R}_+}$ is defined by (7.2), and use stochastic calculus. Hence the condition (7.5) can be written at $t = 0$ as

$$C(T, K) = e^{-rT} \int_{-\infty}^{\infty} (y - K)^+ \varphi_T(y) dy = e^{-rT} \int_{K}^{\infty} (y - K) \varphi_T(y) dy = e^{-rT} \int_{K}^{\infty} y \varphi_T(y) dy - Ke^{-rT} \int_{K}^{\infty} \varphi_T(y) dy,$$  \hfill (7.7)

where $\varphi_T(y)$ is the probability density of $S_T$. By differentiation of (7.7) with respect to $K$, one gets

$$\frac{\partial C}{\partial K}(T, K) = -e^{-rT} K \varphi_T(K) - e^{-rT} \int_{K}^{\infty} \varphi_T(y) dy + e^{-rT} K \varphi_T(K) = -e^{-rT} \int_{K}^{\infty} \varphi_T(y) dy,$$

hence twice differentiation of $C(T, K)$ with respect to $K$ shows that

$$\frac{\partial^2 C}{\partial K^2}(T, K) = e^{-rT} \varphi_T(K),$$  \hfill (7.8)

cf. Relation (1) in [9]. On the other hand, for any sufficiently smooth function $f$, using the Itô formula we have

$$\int_{-\infty}^{\infty} \varphi_T(y) f(y) dy = \mathbb{E}[f(S_T)]$$

$$= \mathbb{E} \left[ f(S_0) + \int_{0}^{T} f'(S_t) dS_t + \frac{1}{2} \int_{0}^{T} f''(S_t) \sigma^2(t, S_t) dt \right]$$

$$= \mathbb{E} \left[ f(S_0) + r \int_{0}^{T} f'(S_t) S_t dt + \sigma \int_{0}^{T} f'(S_t) S_t dB_t + \frac{1}{2} \int_{0}^{T} f''(S_t) \sigma^2(t, S_t) dt \right]$$

$$= f(S_0) + \mathbb{E} \left[ r \int_{0}^{T} f'(S_t) S_t dt + \frac{1}{2} \int_{0}^{T} f''(S_t) \sigma^2(t, S_t) dt \right]$$

$$= f(S_0) + \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{T} y f'(y) \varphi_T(y) dt dy + \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{T} y^2 f''(y) \sigma^2(t, y) \varphi_T(y) dt dy,$$

hence after differentiating both sides of the equality with respect to $T$. 

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http://www.ntu.edu.sg/home/nprivault/index.html
\[
\int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y)f(y)dy = r \int_{-\infty}^{\infty} y f'(y) \varphi_T(y)dy + \frac{1}{2} \int_{-\infty}^{\infty} y^2 f''(y) \sigma^2(T, y) \varphi_T(y)dy.
\]

Integrating by parts in the above relation yields
\[
\int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y)f(y)dy = -r \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial y}(y \varphi_T(y))dy + \frac{1}{2} \int_{-\infty}^{\infty} f(y) \frac{\partial^2}{\partial y^2}(y^2 \sigma^2(T, y) \varphi_T(y))dy,
\]
for all smooth functions \( f(y) \) with compact support, hence
\[
\frac{\partial \varphi_T}{\partial T}(y) = -r \frac{\partial}{\partial y}(y \varphi_T(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2}(y^2 \sigma^2(T, y) \varphi_T(y)), \quad y \in \mathbb{R}.
\]

Making use of (7.8) we get
\[
-r \frac{\partial^2 C}{\partial y^2}(T, y) - \frac{\partial}{\partial T} \frac{\partial^2 C}{\partial y^2}(T, y)
= r \frac{\partial}{\partial y} \left( y \frac{\partial^2 C}{\partial y^2}(T, y) \right) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C}{\partial y^2}(T, y) \right), \quad y \in \mathbb{R}.
\]

After a first integration with respect to \( y \) under the limiting condition \( \lim_{K \to +\infty} C(T, K) = 0 \), we obtain
\[
-r \frac{\partial C}{\partial y}(T, y) - \frac{\partial}{\partial T} \frac{\partial C}{\partial y}(T, y) = ry \frac{\partial^2 C}{\partial y^2}(T, y) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C}{\partial y^2}(T, y) \right),
\]
i.e.
\[
-r \frac{\partial C}{\partial y}(T, y) - \frac{\partial}{\partial T} \frac{\partial C}{\partial y}(T, y)
= r \frac{\partial}{\partial y} \left( y \frac{\partial C}{\partial y}(T, y) \right) - r \frac{\partial C}{\partial y}(T, y) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C}{\partial y^2}(T, y) \right),
\]
or
\[
- \frac{\partial}{\partial y} \frac{\partial C}{\partial T}(T, y) = r \frac{\partial}{\partial y} \left( y \frac{\partial C}{\partial y}(T, y) \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C}{\partial y^2}(T, y) \right).
\]

Integrating one more time with respect to \( y \) yields
\[
- \frac{\partial C}{\partial T}(T, y) = ry \frac{\partial C}{\partial y}(T, y) - \frac{1}{2} y^2 \sigma^2(T, y) \frac{\partial^2 C}{\partial y^2}(T, y), \quad y \in \mathbb{R},
\]
which conducts to (7.4) and is called the Dupire [30] PDE. \( \square \)

From (7.4) the local volatility \( \sigma(t, y) \) can be estimated by computing \( C(T, y) \) by the Black-Scholes formula, based on a value of the implied volatil-
ity \( \sigma \). See [1] and in particular Figure 8.1 therein for numerical methods applied to volatility estimation in this framework.

## 7.5 Stochastic Volatility

### Time-Dependent Stochastic Volatility

The next Figure 7.12 refers to the EURO/SGD exchange rate, and shows some spikes that cannot be generated by Gaussian returns with constant variance.

This type data shows that, in addition to jump models that are commonly used to take into account the slow decrease of probability tails observed in market data, other tools should be implemented in order to model a possibly random and time varying volatility.

We consider an asset price driven by the stochastic differential equation

\[
    dS_t = rS_t dt + S_t \sqrt{v_t} dB_t
\]

(7.9)

under the risk-neutral measure \( \mathbb{P}^* \), where \((v_t)_{t \in \mathbb{R}^+}\) is a (possibly random) squared volatility process adapted to the filtration \( \mathcal{F}^{(1)}_t \) generated by \((B_t)_{t \in \mathbb{R}^+}\).

### Time-dependent deterministic volatility

When \((v(t))_{t \in \mathbb{R}^+}\) is a deterministic function of time, the solution
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\[ S_T = S_t \exp \left( r(T - t) + \int_t^T \sqrt{v(s)} dB_s - \frac{1}{2} \int_t^T v(s) ds \right) \]

of (7.9) is a lognormal random variable at time \( T \) with conditional log-variance

\[ \int_t^T v(s) ds \]
given \( \mathcal{F}_t \). In particular, a European call option on \( S_T \) can be priced by the Black-Scholes formula

\[ e^{-r(T-t)} \mathbb{E}^*[ (S_T - K)^+ | \mathcal{F}_t] = \text{BS} \left( S_t, r, K, T - t, \frac{\int_t^T v(s) ds}{T - t} \right), \]

with integrated squared volatility parameter

\[ \hat{v}(t) := \frac{\int_t^T v(s) ds}{T - t}, \quad t \in [0, T). \]

**Independent volatility**

When \( (v_t)_{t \in \mathbb{R}_+} \) is a random process generating a filtration \( \mathcal{F}_t^{(2)} \) independent of the driving Brownian motion \( (B_t^{(1)})_{t \in \mathbb{R}_+} \) under \( \mathbb{P}^* \), the equation (7.9) can still be solved as

\[ S_T = S_t \exp \left( r(T - t) + \int_t^T \sqrt{v_s} dB_s - \frac{1}{2} \int_t^T v_s ds \right), \]

and \( S_T \) is a lognormal random variable with random variance

\[ \int_t^T v_s ds \]
given \( \mathcal{F}_T^{(2)} \). In this case we can still price options with payoff \( \phi(S_T) \) on the underlying \( S_T \) using the tower property

\[ \mathbb{E}^*[\phi(S_T) | \mathcal{F}_t] = \mathbb{E}^* \left[ \mathbb{E}^* \left[ \phi(S_T) \mid \mathcal{F}_t^{(1)} \vee \mathcal{F}_T^{(2)} \right] \mid \mathcal{F}_t^{(1)} \vee \mathcal{F}_t^{(2)} \right]. \]

For example, a European call option on \( S_T \) can be priced by averaging the Black-Scholes formula as follows:

\[ e^{-r(T-t)} \mathbb{E}^*[ (S_T - K)^+ | \mathcal{F}_t] = e^{-r(T-t)} \mathbb{E}^* \left[ \mathbb{E}^* \left[ (S_T - K)^+ \mid \mathcal{F}_t^{(1)} \vee \mathcal{F}_T^{(2)} \right] \right]. \]

\[ = e^{-r(T-t)} \mathbb{E}^* \left[ \text{BS} \left( S_t, r, K, T - t, \frac{\int_t^T v_s ds}{T - t} \right) \mid \mathcal{F}_t \right] \]

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\[ e^{-r(T-t)} \mathbb{E}^* \left[ BS \left( S_t, r, K, T - t, \frac{\int_t^T v_s ds}{T - t} \right) \big| \mathcal{F}_t^{(2)} \right], \]

with the random integrated volatility

\[ \hat{v}_t := \frac{1}{T - t} \int_t^T v_s ds. \]

On the other hand, when \((v_t)_{t \in \mathbb{R}}^+\) is a geometric Brownian motion, the probability distribution of the time integral \(\int_t^T v_s ds\) given \(\mathcal{F}_t^{(2)}\) can be computed using integral expressions.

**Two-factor Stochastic Volatility Models**

Evidence based on financial market data shows that the variations in volatility tend to be negatively correlated with the variations of underlying asset prices, Figure 1 of [86] and cf. § 2.3.1 of [38]. For this reason we need to consider an asset price process \((S_t)_{t \in \mathbb{R}}^+\) and a stochastic volatility process \((v_t)_{t \in \mathbb{R}}^+\) driven by

\[
\begin{aligned}
    dS_t &= rS_t + \sqrt{v_t} S_t dB_t^{(1)} \\
    dv_t &= \mu(t, v_t) dt + \beta(t, v_t) dB_t^{(2)},
\end{aligned}
\]

Here, \((B_t^{(1)})_{t \in \mathbb{R}}^+\) and \((B_t^{(2)})_{t \in \mathbb{R}}^+\) are two Brownian motions such that

\[ dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt, \]

where the correlation parameter \(\rho\) satisfies \(-1 \leq \rho \leq 1\), and the coefficients \(\mu(t, x)\) and \(\beta(t, x)\) can be chosen e.g. from mean-reverting models (CIR) or geometric Brownian models, as follows.

**Heston model**

In the Heston model [52], the stochastic volatility \((v_t)_{t \in \mathbb{R}}^+\) is chosen to be a CIR process, i.e. we have

\[
\begin{aligned}
    dS_t &= rS_t dt + S_t \sqrt{v_t} dB_t^{(1)} \\
    dv_t &= -\lambda(v_t - m) dt + \eta \sqrt{v_t} dB_t^{(2)},
\end{aligned}
\]
and $\mu(t, v) = -\lambda(v_t - v)$ and $\beta(t, v) = \eta \sqrt{v}$, where $\lambda, \eta > 0$.

Option pricing formulas can be derived in the Heston model using complex integrals.

**SABR model**

In the SABR model ([48], here with $\beta = 1$), we have

$$\begin{align*}
\frac{dS_t}{S_t} &= \sigma_t S_t^\beta dB_t^{(1)} \\
\frac{d\sigma_t}{\sigma_t} &= \alpha \sigma_t dB_t^{(2)},
\end{align*}$$

with $\alpha > 0$ and $\beta \in (0, 1]$, which is not mean-reverting, i.e. it is preferably used in short time. This model is typically used for the modeling of LIBOR rates and it allows for short-time asymptotics of Black implied volatilities that can be used for pricing by inputting them into the Black pricing formula, cf. § 3.3 of [100].

**Pricing PDE with Stochastic Volatility**

Consider a portfolio priced as

$$V_t = f(t, v_t, S_t) = e^{-r(T-t)} \mathbb{E}^* \left[ h(S_T) \mid \mathcal{F}_t \right],$$

$0 \leq t \leq T$, for an option with payoff $h(S_T)$ on $S_T$.

In the sequel we will assume that $(B_t^{(1)})_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, i.e. the discounted price process $(e^{-rt}S_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. For simplicity of exposition we will make the assumption that $(B_t^{(2)})_{t \in \mathbb{R}_+}$ is also a standard Brownian motion under $\mathbb{P}^*$.

By Itô calculus with respect to the correlated Brownian motions $(B_t^{(1)})_{t \in \mathbb{R}_+}$ and $(B_t^{(2)})_{t \in \mathbb{R}_+}$, the portfolio value $f(t, v_t, S_t)$ can be differentiated as follows:

$$df(t, v_t, S_t) \quad (7.10)$$

$$= \frac{\partial f}{\partial t}(t, v_t, S_t)dt + r S_t \frac{\partial f}{\partial x}(t, v_t, S_t)dt + \sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t)dB_t^{(1)}$$

$$+ \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t, v_t, S_t)dt + \mu(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t)dt$$

$$+ \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t)dB_t^{(2)} + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t)dt$$
\[ + \rho \beta(t,v_t) \sqrt{v_t S_t} \frac{\partial^2 f}{\partial v \partial x}(t,v_t,S_t) dt. \]

Assuming that \((B_t^{(2)})_{t \in \mathbb{R}_+}\) is also a standard Brownian motion under the risk-neutral measure* \(\mathbb{P}^*\) and knowing that the discounted portfolio price process \((e^{-rt} f(t,v_t,S_t))_{t \in \mathbb{R}_+}\) is also a martingale under \(\mathbb{P}^*\), from the relation

\[
d(e^{-rt} f(t,v_t,S_t)) = -re^{-rt} f(t,v_t,S_t) dt + e^{-rt} df(t,v_t,S_t),
\]

we obtain

\[
- r f(t,v_t,S_t) dt + \frac{\partial f}{\partial t}(t,v_t,S_t) dt + r S_t \frac{\partial f}{\partial x}(t,v_t,S_t) dt + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t,v_t,S_t) dt
\]

\[
+ \mu(t,v_t) \frac{\partial f}{\partial v}(t,v_t,S_t) dt + \frac{1}{2} \beta^2(t,v_t) \frac{\partial^2 f}{\partial v^2}(t,v_t,S_t) dt
\]

\[
+ \rho \beta(t,v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x}(t,v_t,S_t) dt = 0,
\]

and the pricing PDE

\[
\frac{\partial f}{\partial t}(t,v,x) + r x \frac{\partial f}{\partial x}(t,v,x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t,v,x) + \mu(t,v) \frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2} \beta^2(t,v) \frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho \beta(t,v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t,v,x) = rf(t,v,x),
\]

under the terminal condition \(f(T,v,x) = h(x)\).

**Hedging**

Consider a portfolio of the form

\[ V_t = \eta_t e^{rt} + \xi_t S_t \]

based on the riskless asset \(A_t = e^{rt}\) and on the risky asset \(S_t\). When this portfolio is self-financing we have

\[
dV_t = df(t,v_t,S_t)
\]

\[
= r \eta_t e^{rt} dt + \xi_t dS_t
\]

\[
= r \eta e^{rt} dt + \xi_t (r S_t dt + S_t \sqrt{v_t} dB^{(1)}_t)
\]

\[
= r V_t dt + \xi_t S_t \sqrt{v_t} dB^{(1)}_t
\]

* When this condition is not satisfied we need to introduce a drift that yields a market price of volatility.

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\[ = r f(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)}. \]  

(7.12)

However, trying to match (7.12) to (7.10) yields

\[ \sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} = \xi_t S_t \sqrt{v_t} dB_t^{(1)}, \]  

(7.13)

which admits no solution unless \( \beta(t, v) = 0 \), i.e. when volatility is deterministic. A solution to that problem is to consider instead a portfolio

\[ V_t = f(t, v_t, S_t) = \eta_t e^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t) \]

that includes an additional asset with price \( P(t, v_t, S_t) \), which can be an option depending on the volatility \( v_t \). In that case, (7.12) is replaced with

\[ dV_t = df(t, v_t, S_t) \]

\[ = r \eta_t e^{rt} dt + \xi_t dS_t + \zeta_t dP(t, v_t, S_t) \]

\[ = r \eta_t e^{rt} dt + \xi_t (r S_t dt + S_t \sqrt{v_t} dB_t^{(1)}) + r \xi_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt + \xi_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt \]

\[ + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \xi_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt + \frac{1}{2} \xi_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt \]

\[ + \rho \xi_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} \]

\[ + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}, \]

\[ = r (V_t - \zeta_t P(t, v_t, S_t)) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r \xi_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \]

\[ + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt \]

\[ + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \xi_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt + \frac{1}{2} \xi_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt \]

\[ + \rho \xi_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} \]

\[ + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}, \]

\[ \frac{(7.14)}{\zeta} \]

and by matching (7.14) to (7.10), the equation (7.13) now becomes
\[ \sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} \]
\[ = \xi_t S_t \sqrt{v_t} dB_t^{(1)} + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}. \]

This leads to the equations
\[
\begin{cases}
\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) = \xi_t S_t \sqrt{v_t} + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t), \\
\beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) = \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t),
\end{cases}
\]

hence
\[ \zeta_t = \frac{\frac{\partial f}{\partial v}(t, v_t, S_t)}{\frac{\partial P}{\partial v}(t, v_t, S_t)}, \quad (7.15) \]

and
\[ \xi_t = \frac{1}{S_t \sqrt{v_t}} \left( \sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) \right), \]
\[ = \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t \frac{\partial P}{\partial x}(t, v_t, S_t) \]
\[ = \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial x}(t, v_t, S_t) \frac{\partial P}{\partial v}(t, v_t, S_t) \frac{\partial P}{\partial v}(t, v_t, S_t). \quad (7.16) \]

In addition, identifying the “dt” terms when equating (7.14) to (7.10) would now lead to the more complicated PDE
\[ r(f(t, v_t, S_t) - \zeta_t P(t, v_t, S_t)) + r \xi_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) \]
\[ + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) + 2 \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) + \frac{1}{2} \zeta_t^2 (t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t) \]
\[ + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial x \partial v}(t, v_t, S_t) \]
\[ = \frac{\partial f}{\partial t}(t, v_t, S_t) + r S_t \frac{\partial f}{\partial x}(t, v_t, S_t) + 2 v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t, v_t, S_t) \]
\[ + \mu(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t) + \rho \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x}(t, v_t, S_t), \]

which can be rewritten using (7.15) as
\[ \frac{\partial f}{\partial v}(t, v, x) \left( -r P(t, v, x) + r x \frac{\partial P}{\partial x}(t, v, x) + \mu(t, v) \frac{\partial P}{\partial v}(t, v, x) + \frac{\partial P}{\partial t}(t, v, x) \right) \]
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\[ + \frac{\partial f}{\partial v}(t, v, x) \left( \frac{1}{2} x^2 \frac{\partial^2 P}{\partial x^2}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 P}{\partial v^2}(t, v, x) + \rho \beta(t, v)x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t, v, x) \right) \]

\[ = \frac{\partial P}{\partial v}(t, v, x) \left( -rf(t, v, x) + \frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) \right) \]

\[ + \frac{\partial P}{\partial v}(t, v, x) \left( \mu(t, v) \frac{\partial f}{\partial v}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \beta(t, v)x \sqrt{v} \frac{\partial^2 f}{\partial x \partial v}(t, v, x) \right), \]

or

\[ \frac{1}{\sigma(t, v, x)} \left( -rf(t, v, x) + \frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) \right) \]

\[ + \frac{1}{\sigma(t, v, x)} \left( \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \beta(t, v)x \sqrt{v} \frac{\partial^2 f}{\partial x \partial v}(t, v, x) \right) \]

\[ = \frac{1}{\sigma(t, v, x)} \left( -rP(t, v, x) + rx \frac{\partial P}{\partial x}(t, v, x) + \frac{\partial P}{\partial t}(t, v, x) \right) \]

\[ + \frac{1}{\sigma(t, v, x)} \left( \frac{1}{2} x^2 \frac{\partial^2 P}{\partial x^2}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 P}{\partial v^2}(t, v, x) + \rho \beta(t, v)x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t, v, x) \right) \]

\[ = \lambda(t, v, x), \]

where \( \lambda(t, v, x) \) is a function that does not depend of \( P \), without requiring \((B^{(2)}_t)_{t \in \mathbb{R}^+}\) to be a standard Brownian motion under \( P^* \). The function \( \lambda(t, v, x) \) is linked to the market price of volatility risk, cf. Chapter 1 of [40] and § 2.4.1 of [38] for details.

The pricing PDE rewrites as

\[ \frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) \]

\[ + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \beta(t, v)x \sqrt{v} \frac{\partial^2 f}{\partial x \partial v}(t, v, x) = rf(t, v, x) + \lambda(t, v, x) \frac{\partial f}{\partial v}(t, v, x), \]

and (7.11) corresponds to the choice \( \lambda(t, v, x) = -\mu(t, v) \), i.e. a vanishing market price of volatility risk.

**Heston model**

In the Heston model with \( \mu(t, v) = -\lambda(v_t - v) \) and \( \beta(t, v) = \eta \sqrt{v} \), we find the PDE

\[ \frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) \]

\[ -\lambda(v - m) \frac{\partial f}{\partial v}(t, v, x) + \frac{1}{2} \eta^2 v \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \eta x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t, v, x) = rf(t, v, x). \]
The solution of this PDE has been expressed in [52] as a complex integral by inversion of a characteristic function.

Using the change of variable $y = \log x$ with $g(t, v, x) = f(t, v, e^y)$ we find
\[
\frac{\partial g}{\partial t}(t, v, y) + r \frac{\partial g}{\partial y}(t, v, y) + \frac{1}{2} v \frac{\partial^2 g}{\partial y^2}(t, v, y) - \frac{1}{2} v \frac{\partial g}{\partial y}(t, v, x)
- \lambda(v - m) \frac{\partial g}{\partial v}(t, v, y) + \frac{\eta^2}{2} \frac{\partial^2 g}{\partial v^2}(t, v, y) + \rho \eta v \frac{\partial g}{\partial v}(t, v, y)
= rg(t, v, y).
\]

Using the Fourier transform
\[
\tilde{g}(t, v, z) := \int_{-\infty}^{\infty} e^{-iyz} g(t, v, y) dy
\]
and the relation
\[
-iz\tilde{g}(t, v, z) = \int_{-\infty}^{\infty} e^{-iyz} \frac{\partial g}{\partial y}(t, v, y) dy,
\]
we find, using the rule $i^2 = -1$, that $\tilde{g}(t, v, z)$ satisfies the equation
\[

r \tilde{g}(t, v, z) + \frac{\partial \tilde{g}}{\partial t}(t, v, z) - i rz \tilde{g}(t, v, z) - \frac{1}{2} vz^2 \tilde{g}(t, v, z) + izv \frac{1}{2} \partial \tilde{g}(t, v, z)
- \lambda(v - m) \frac{\partial \tilde{g}}{\partial v}(t, v, z) + \frac{\eta^2}{2} \frac{\partial^2 \tilde{g}}{\partial v^2}(t, v, z) - \rho \eta zv \frac{\partial \tilde{g}}{\partial v}(t, v, z)
= 0,
\]
which is an affine PDE with respect to the variable $v$ with $z$ a constant parameter. This equation can be solved in closed form, and the final solution $g(t, v, y)$ can then be obtained by the Fourier inversion
\[
g(t, v, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izy} \tilde{g}(t, v, z) dz,
\]

\textbf{Perturbation Analysis}

We refer to Chapter 4 of [38] for the contents of this section. Consider the time-rescaled model
\[
\begin{align*}
\quad
\begin{cases}
    dS_t = rS_t dt + S_t \sqrt{\nu_t / \varepsilon} dB^{(1)}_t \\
    dv_t = \mu(v_t) dt + \beta(v_t) dB^{(2)}_t.
\end{cases}
\end{align*}
\]
(7.17)

We note that $\nu_{t/\varepsilon}$ satisfies the SDE
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\[ dv_{t/\epsilon} \approx v_{(t+dt)/\epsilon} - v_{t/\epsilon} \]
\[ = v_{t/\epsilon + dt/\epsilon} - v_{t/\epsilon} \]
\[ = \frac{1}{\epsilon} \mu(v_{t/\epsilon}) dt + \beta(v_{t/\epsilon}) dB_{t/\epsilon}^{(2)}, \]

with

\[ (dB_{t/\epsilon}^{(2)})^2 \approx \frac{dt}{\epsilon} \approx \frac{1}{\epsilon} (dB_{t}^{(2)})^2 \approx \left( \frac{1}{\sqrt{\epsilon}} dB_{t}^{(2)} \right)^2, \]

hence the SDE can be rewritten as

\[ dv_{t} = \frac{1}{\epsilon} \mu(v_{t}) dt + \frac{1}{\sqrt{\epsilon}} \beta(v_{t}) dB_{t}^{(2)}. \]

In other words, \( \epsilon \to 0 \) corresponds to fast mean-reversion and (7.17) can be rewritten as

\[
\begin{align*}
\frac{dS_t}{S_t} &= rS_t dt + \sqrt{v_t} dB_t^{(1)} \\
\frac{dv_t}{v_t} &= \frac{1}{\epsilon} \mu(v_t) dt + \frac{1}{\sqrt{\epsilon}} \beta(v_t) dB_t^{(2)}, \quad \epsilon > 0.
\end{align*}
\]

The perturbed PDE

\[
\begin{align*}
\frac{\partial f_{\epsilon}(t,v,x)}{\partial t} + rx \frac{\partial f_{\epsilon}(t,v,x)}{\partial x} + \frac{1}{2} vx^2 \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial x^2} \\
+ \frac{1}{\epsilon} \mu(v) \frac{\partial f_{\epsilon}(t,v,x)}{\partial v} + \frac{1}{2\epsilon} \beta^2(v) \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial v^2} + \rho \sqrt{\epsilon} \beta(v) x \sqrt{v} \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial v \partial x}
\end{align*}
\]
\[ = rf_{\epsilon}(t,v,x) \]

with terminal condition \( f_{\epsilon}(T,v,x) = (x - K)^+ \), rewrites as

\[ \frac{1}{\epsilon} L_0 f_{\epsilon}(t,v,x) + \frac{1}{\sqrt{\epsilon}} L_1 f_{\epsilon}(t,v,x) + L_2 f_{\epsilon}(t,v,x) = rf_{\epsilon}(t,v,x), \quad (7.18) \]

where

\[
\begin{align*}
L_0 f_{\epsilon}(t,v,x) &= \frac{1}{2} \beta^2(v) \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial v^2} + \mu(v) \frac{\partial f_{\epsilon}(t,v,x)}{\partial v}, \\
L_1 f_{\epsilon}(t,v,x) &= \rho x \beta(v) \sqrt{v} \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial v \partial x}, \\
L_2 f_{\epsilon}(t,v,x) &= \frac{\partial f_{\epsilon}(t,v,x)}{\partial t} + rx \frac{\partial f_{\epsilon}(t,v,x)}{\partial x} + \frac{1}{2} vx^2 \frac{\partial^2 f_{\epsilon}(t,v,x)}{\partial x^2}.
\end{align*}
\]

Note that
\[ L_0 \] is the infinitesimal generator of the process \( (v^1_t)_{t \in \mathbb{R}^+} \), see (7.22) below, and
\[ L_2 \] is the Black-Scholes operator, \textit{i.e.} \( L_2 f = rf \) is the Black-Scholes PDE.

The solution \( f_\varepsilon(t, v, x) \) will be expanded as
\[
f_\varepsilon(t, v, x) = f^{(0)}(t, v, x) + \sqrt{\varepsilon} f^{(1)}(t, v, x) + \varepsilon f^{(2)}(t, v, x) + \cdots	ag{7.19}
\]
with \( f(T, v, x) = (x - K)^+ \), \( f^{(1)}(T, v, x) = 0 \), and \( f^{(2)}(T, v, x) = 0 \).

Since \( L_0 \) contains only differentials with respect to \( v \) we will choose \( f^{(0)}(t, v, x) \) of the form
\[
f^{(0)}(t, v, x) = f^{(0)}(t, x), \tag{7.20}
\]
for details, with
\[
L_0 f^{(0)}(t, x) = L_1 f^{(0)}(t, x) = 0.
\]

By identifying the terms of order \( 1/\sqrt{\varepsilon} \) when plugging (7.19) in (7.18) we also find
\[
L_0 f^{(1)}(t, v, x) + L_1 f^{(0)}(t, x) = 0,
\]
hence \( L_0 f^{(1)}(t, v, x) = 0 \). Similarly, by identifying the terms that do not depend on \( \varepsilon \) in (7.18) and taking \( f^{(1)}(t, v, x) = f^{(1)}(t, x) \), we have \( L_1 f^{(1)} = 0 \) and
\[
L_0 f^{(2)}(t, v, x) + L_2 f^{(0)}(t, x) = 0.	ag{7.21}
\]

Using the Itô formula we have
\[
\mathbb{E} \left[ f^{(2)}(t, v^1_\tau, x) \right] = f^{(2)}(t, v^1_0, x) + \left[ \int_0^t \frac{\partial f^{(2)}}{\partial x} (t, v^1_\tau, x) \, dB^{(2)}_\tau \right] + \mathbb{E} \left[ \int_0^t \left( \mu(v^1_\tau) \frac{\partial f^{(2)}}{\partial v} (t, v^1_\tau, x) + \frac{1}{2} \beta^2(v^1_\tau) \frac{\partial^2 f^{(2)}}{\partial v^2} (t, v^1_\tau, x) \right) \, d\tau \right]
= \mathbb{E} \left[ f^{(2)}(t, v^1_0, x) \right] + \int_0^t \mathbb{E} \left[ L_0 f^{(2)}(t, v^1_\tau, x) \right] \, d\tau.	ag{7.22}
\]

When the process \( (v^1_t)_{t \in \mathbb{R}^+} \) is started under its stationary probability distribution with density function \( \phi(v) \) we have
\[
\mathbb{E}[f^{(2)}(t, v^1_\tau, x)] = \int_0^\infty f^{(2)}(t, v, x) \phi(v) \, dv, \quad \tau \in \mathbb{R}^+,
\]
hence (7.22) rewrites as
\[
\int_0^\infty f^{(2)}(t, v, x) \phi(v) \, dv = \int_0^\infty f^{(2)}(t, v, x) \phi(v) \, dv + \int_0^\infty \int_0^\infty L_0 f^{(2)}(t, v, x) \phi(v) \, dv \, d\tau.
\]
By differentiation with respect to \( s > 0 \) this yields
\[
\int_0^\infty L_0 f^{(2)}(t, v, x) \phi(v) \, dv = 0,
\]
hence by (7.21) we find
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\[ \int_0^\infty \mathcal{L}_2 f^{(0)}(t, x) \phi(v) dv = 0, \]

cf. § 3.2 of [38], i.e.

\[ \frac{\partial f^{(0)}}{\partial t}(t, x) + rx \frac{\partial f^{(0)}}{\partial x}(t, x) + \frac{1}{2} \eta^2 \int_0^\infty v \phi(v) dv \frac{\partial^2 f^{(0)}}{\partial x^2}(t, x) = r f^{(0)}(t, x), \]

with the terminal condition \( f^{(0)}(T, x) = (x - K)^+ \).

Consequently the first expansion term \( f^{(0)}(t, x) \) in (7.18) is the Black-Scholes function

\[ f^{(0)}(t, x) = BS(S_t, r, K, T - t, \int_0^\infty v \phi(v) dv), \]

with the averaged squared volatility

\[ \int_0^\infty v \phi(v) dv = \mathbb{E}\left[v_\tau^1\right], \quad \tau \in \mathbb{R}_+, \]

under the stationary distribution of the process with infinitesimal generator \( L_0 \), i.e. the stationary distribution of the solution to

\[ dv_t^1 = \mu(v_t^1) dt + \beta(v_t^1) dB_t^{(2)}. \]

**Heston model**

We have

\[
\begin{cases}
    dS_t = rS_t dt + S_t \sqrt{v_t^\varepsilon} dB_t^{(1)} \\
    dv_t^\varepsilon = -\frac{\lambda}{\varepsilon} (v_t^\varepsilon - m) dt + \eta \sqrt{v_t^\varepsilon / \varepsilon} dB_t^{(2)},
\end{cases}
\]

under the modified short mean-reversion time scale, and the SDE can be rewritten as

\[ dv_t^\varepsilon = -\frac{\lambda}{\varepsilon} (v_t^\varepsilon - m) dt + \eta \sqrt{v_t^\varepsilon / \varepsilon} dB_t^{(2)}. \]

In other words, \( \varepsilon \to 0 \) corresponds to fast mean-reversion.

The CIR process \( (v_t^1)_{t \in \mathbb{R}_+} \) has a gamma stationary distribution with shape parameter \( 2\lambda m / \eta^2 \), scale parameter \( \eta^2 / (2\lambda) \), probability density function

\[ \phi(v) = \frac{1}{\Gamma(2\lambda m / \eta^2)(\eta^2 / (2\lambda))^{2\lambda m / \eta^2} v^{-1+2\lambda m / \eta^2}} e^{-2x\lambda / \eta^2} \mathbb{1}_{[0, \infty)}(v), \]

and mean
\[ \int_0^\infty v \phi(v) dv = m. \]

Hence the first expansion term \( f^{(0)}(t, x) \) in (7.18) reads

\[ f^{(0)}(t, x) = BS(S_t, r, K, T - t, m), \]

with the averaged squared volatility

\[ \int_0^\infty v \phi(v) dv = m = \mathbb{E}[v^1_\tau], \quad \tau \in \mathbb{R}_+, \]

under the stationary distribution of the process with infinitesimal generator \( L_0 \), i.e. the stationary distribution of the solution to

\[ dv^1_t = \mu(v^1_t) dt + \beta(v^1_t) dB^{(2)}_t. \]

### 7.6 Volatility Derivatives

**Another look at historical volatility**

When \( t_k = kT/N, \) \( k = 0, 1, \ldots, N \), a natural estimator for the trend parameter \( \mu \) can be written as

\[
\hat{\mu}_N := \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}
\]

\[
\approx \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \log \frac{S_{t_k}}{S_{t_{k-1}}}
\]

\[
= \frac{1}{T} \sum_{k=1}^{N} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right)
\]

\[
= \frac{1}{T} \log \frac{S_T}{S_0}.
\]

Similarly we can use

\[
\hat{\sigma}_N^2 := \frac{1}{N-1} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \right)^2 - (\hat{\mu}_N)^2
\]

\[
\approx \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right)^2 - \left( \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \right)^2
\]

\[
= \frac{1}{T} \sum_{k=1}^{N} \left( \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \log \frac{S_T}{S_0} \right)^2. \tag{7.23}
\]
Volatility swaps are forward contracts that allow for the exchange of the estimated volatility (7.23) against a fixed value \( \kappa \sigma \), with the “payoff”

\[
\frac{1}{T} \sum_{k=1}^{N} \left( \log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \log \frac{S_T}{S_0} \right)^2 - \kappa \sigma.
\]

Note that the above payoff has to be multiplied by the vega notional, which is part of the contract, in order to convert it into currency units.

Exercise. ([40], Ch. 11) Compute the expected total realized variance in the Heston model with

\[
dv_t = -\lambda(v_t - m)dt + \eta \sqrt{v_t} dB_t.
\]

Answer: We need to compute

\[
\frac{1}{T} \mathbb{E} \left[ \int_0^T v_t dt \right] = \frac{1}{T} \int_0^T \mathbb{E} [v_t] dt = \frac{1}{T} \int_0^T u(t) dt,
\]

where \( u(t) := \mathbb{E} [v_t] \) satisfies the ordinary differential equation

\[
u'(t) = \lambda m - \lambda u(t),
\]

i.e.

\[
(e^{\lambda t} u(t))' = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t) = \lambda m e^{\lambda t},
\]

hence

\[
u(t) = e^{-\lambda t} \left( u(0) + \lambda m \int_0^t e^{\lambda s} ds \right) = \mathbb{E} [v_0] e^{-\lambda t} + m (1 - e^{-\lambda t}), \quad t \in \mathbb{R}_+.
\]

Exercises

Exercise 7.1 Consider an index whose level \( S_t \) is given in the Heston stochastic volatility model

\[
\begin{cases}
    dS_t = (r - \alpha v_t) S_t dt + S_t \sqrt{\beta + v_t} dB_t^{(1)} \\
    dv_t = -\lambda (v_t - m) dt + \gamma \sqrt{v_t} dB_t^{(2)},
\end{cases}
\]

where \( (B_t^{(1)})_{t \in \mathbb{R}_+} \) and \( (B_t^{(2)})_{t \in \mathbb{R}_+} \) are standard Brownian motions with correlation \( \rho \in [-1, 1] \) and \( \alpha \geq 0, \beta \geq 0, \lambda > 0, m > 0, r > 0, \gamma > 0 \). Compute the variance swap rate.
\[ V_{ST} := \frac{1}{T} \mathbb{E} \left[ \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{S_{kT/n} - S_{(k-1)T/n}}{S_{(k-1)T/n}} \right)^2 \right] = \frac{1}{T} \mathbb{E} \left[ \int_{0}^{T} \frac{1}{S_{t}^2} (dS_{t})^2 \right]. \]