Chapter 7
Estimation of Volatility

While the market parameters $r, t, S_t, T,$ and $K$ used to price an option via the Black-Scholes formula can be easily obtained from market data, estimating the volatility coefficient $\sigma$ can be a more difficult task. Several estimation methods are considered in this chapter, together with examples on how the Black-Scholes formula can be fitted to market data. In particular we cover historical, implied, and local volatility estimation and the VIX® volatility index, as well as some option pricing methods under stochastic volatility.

7.1 Historical Volatility

7.2 Implied Volatility

7.3 Local Volatility

7.4 The CBOE VIX® Volatility Index

7.5 Stochastic Volatility Models

7.6 Volatility Derivatives

7.7 Option Pricing PDE

7.8 Perturbation Analysis

Exercises

7.1 Historical Volatility

We consider the problem of estimating the parameters $\mu$ and $\sigma$ from market data in the stock price model

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{7.1}
\]

Historical trend estimation

By discretization of (7.1) along a family $t_0, t_1, \ldots, t_N$ of observation times as
\[
\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = \mu(t_{k+1} - t_k) + \sigma(B_{t_{k+1}} - B_{t_k}), \quad k = 0, 1, \ldots, N - 1, \quad (7.2)
\]
a natural estimator for the trend parameter \( \mu \) can be constructed as
\[
\hat{\mu}_N := \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \frac{S_{t_{k+1}}^M - S_{t_k}^M}{S_{t_k}^M} \right), \quad (7.3)
\]
where \( \frac{(S_{t_{k+1}}^M - S_{t_k}^M)}{S_{t_k}^M}, \quad k = 0, 1, \ldots, N - 1 \) denotes market returns observed at discrete times \( t_0, t_1, \ldots, t_N \) on the market.

**Historical log-return estimation**

Alternatively, observe that, replacing (7.3) by the log-returns
\[
\log \left( 1 + \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} \right) = \log S_{t_{k+1}} - \log S_{t_k} \simeq \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}},
\]
with \( t_{k+1} - t_k = T/N, \quad k = 0, 1, \ldots, N - 1 \), one can replace (7.3) with the simpler telescoping estimate
\[
\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} (\log S_{t_{k+1}} - \log S_{t_k}) = \frac{1}{T} \log \frac{S_T}{S_0}.
\]

**Historical volatility estimation**

The volatility parameter \( \sigma \) can be estimated by the (unbiased) estimator \( \hat{\sigma}_N \) built from (7.2) as
\[
\hat{\sigma}_N^2 := \frac{1}{N - 1} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left( \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} - \hat{\mu}_N(t_{k+1} - t_k) \right)^2.
\]

* Note that strictly speaking, the Itô formula reads \( d \log S_t = dS_t/S_t - (dS_t)^2/(2S_t^2) \).
Estimation of Volatility

\[
\text{stock.rtn} = \text{diff(log(Ad('0005.HK')))} \\
\text{n} = \text{sum(is.na(stock.rtn))} \\
\text{returns} \leftarrow \text{as.vector(stock.rtn)} \\
\text{times} = \text{index(stock.rtn)} \\
\text{n} = \text{sum(is.na(returns))} + \text{sum(is.na(returns))} \\
\text{plot(times, returns, pch=19, cex=0.05, col="blue", ylab="returns", xlab="n", main = "")} \\
\text{segments(x0 = times, x1 = times, cex=0.05, y0 = 0, y1 = returns, col="blue")} \\
\text{abline(seq(1,n),0,FALSE)} \\
\text{dt=1.0/365} \\
\text{m=mean(returns, na.rm=TRUE)/dt} \\
\text{s=sd(returns, na.rm=TRUE)/sqrt(dt)} \\
\]

\[
\begin{array}{c}
\text{Feb 15} \\
\text{Feb 27} \\
\text{Mar 06} \\
\text{Mar 13} \\
\text{Mar 20} \\
\text{Mar 27} \\
\text{Mar 31}
\end{array}
\]

\[
\text{63} \\
\text{64} \\
\text{65} \\
\text{66} \\
\text{67}
\]

(a) Underlying.

(b) Log returns.

Fig. 7.1: Graph of underlying vs log returns.

\[
\text{library(PerformanceAnalytics)} \\
\text{library(quantmod)} \\
\text{returns} \leftarrow \text{exp(CalculateReturns(stock, method="compound"))} - 1 \\
\text{returns}[1,1] \leftarrow 0 \\
\text{histvol} \leftarrow \text{rollapply(returns, width = 30, FUN=sd.annualized)} \\
\text{myTheme} \leftarrow \text{chart_theme()} \\
\text{myTheme$col$line} \leftarrow \text{"blue"} \\
\text{chart_Series(stock, name="0005.HK", theme=myTheme)} \\
\text{add_TA(histvol, name="Historical Volatility")}
\]

Fig. 7.2: Historical volatility graph.
Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

Fig. 7.3: Scorsese (2013) “The fugazi: it’s a wazy, it’s a woozie. It’s fairy dust.”*

7.2 Implied Volatility

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data.

Recall that when $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE is given by

* Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).
Estimation of Volatility

\[ \text{Bl}(t, x, K, \sigma, r, T) = x \Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \]

where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}, \]

and

\[
\begin{align*}
    d_+(T - t) &= \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \\
    d_-(T - t) &= \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}.
\end{align*}
\]

Equating the Black-Scholes formula

\[ \text{Bl}(t, S_t, K, \sigma, r, T) = M \tag{7.4} \]

to the observed value \( M \) of a given market price allows one to infer a value of \( \sigma \) when \( t, S_t, r, T \) are known, as in e.g. Figure 5.14. This value of \( \sigma \) is called the implied volatility, and it is denoted here by \( \sigma^{\text{imp}}(K, T) \), cf. e.g. Exercise 5.4. Various algorithms can be implemented to solve (7.4) numerically for \( \sigma^{\text{imp}}(K, T) \), such as the bisection method and the Newton-Raphson method.*

```r
BS <- function(S, K, T, r, sig){
d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
d2 <- d1 - sig*sqrt(T);return(S*pnorm(d1) - K*exp(-r*T)*pnorm(d2))
}

implied.vol <- function(S, K, T, r, market){
sig <- 0.20;sig.up <- 1;sig.down <- 0.001;count <- 0
err <- BS(S, K, T, r, sig) - market
while(abs(err) > 0.00001 && count<1000){
    if(err < 0){sig.down <- sig;sig <- (sig.up + sig)/2}
    else{sig.up <- sig;sig <- (sig.down + sig)/2}
    err <- BS(S, K, T, r, sig) - market
    count <- count + 1}
if(count==1000){return(NA)}else{return(sig)}
}

market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02
implied.vol(S, K, T, r, market)
```

The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, cf. Figure S.10.

* Download the corresponding R code or the IPython notebook that can be run here.
Option chain data in R

```r
install.packages("quantmod")
library(quantmod)
getSymbols("^GSPC", src = "yahoo", from = as.Date("2018-01-01"), to = as.Date("2018-03-013"))
head(GSPC)
# Only the front-month expiry
GSPC.OPT <- getOptionChain("^GSPC")
# All expiries
GSPC.OPTS <- getOptionChain("^GSPC", NULL)
# All 2018 to 2020 expiries
GSPC.OPTS <- getOptionChain("^GSPC", "2018/2020")
# Only the front-month expiry
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2018 to 2020 expiries
AAPL.OPTS <- getOptionChain("AAPL", "2018/2020")
```

Exporting option price data

```r
write.table(goog_data, file = "goog")
write.csv(goog_data, file = "goog.csv")
install.packages("xlsx")
library(xlsx)
write.xlsx(goog_data, file = "goog.xlsx")
```

Given two European call options with strike prices $K_1$, resp. $K_2$ and maturities $T_1$, resp. $T_2$, on the same stock $S$, this procedure should yield two estimates $\sigma^\text{imp}(K_1, T_1)$ and $\sigma^\text{imp}(K_2, T_2)$ of implied volatilities. Clearly, there is no reason a priori for the implied volatilities $\sigma^\text{imp}(K_1, T_1)$ and $\sigma^\text{imp}(K_2, T_2)$ to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter $\sigma$ should be unique for a given stock $S$. This contradiction between a model and market data is a reason for the development of more sophisticated stochastic volatility models.
Estimation of Volatility

```r
install.packages("lubridate")
library(lubridate)
library(quantmod)

CHAIN <- getOptionChain("AAPL", "2020-01-17")
estimationDate <- as.Date(Sys.Date(), format="%Y-%m-%d")
optionExpiryDate <- as.Date("2020-01-17", format="%Y-%m-%d")
T <- as.numeric((optionExpiryDate - estimationDate)/365)
r = 0.02; ImpVol <- 1:1

getSymbols("AAPL", from=Sys.Date(), to=Sys.Date(), src="yahoo")
S = as.numeric(Ad(AAPL))

for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i] <- implied.vol(S, CHAIN$calls$Strike[i], T, r, CHAIN$calls$Last[i])}

plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", lwd = 3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)] ~ poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4, data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=2)

CHAIN <- getOptionChain("^GSPC", "2019-12-20")
estimationDate <- as.Date(Sys.Date(), format="%Y-%m-%d")
optionExpiryDate <- as.Date("2019-12-20", format="%Y-%m-%d")
T <- as.numeric((optionExpiryDate - estimationDate)/365); r = 0.02; ImpVol<-1:1

getSymbols("^GSPC", from=Sys.Date(), to=Sys.Date(), src="yahoo")
S = as.numeric(Ad(GSPC))

for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i] <- implied.vol(S, CHAIN$calls$Strike[i], T, r, CHAIN$calls$Last[i])}

plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", lwd = 3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)] ~ poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4, data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=2)
```

Fig. 7.5: S&P500 option prices plotted against strike prices.

Plotting the different values of the implied volatility $\sigma$ as a function of $K$ and $T$ will yield a three-dimensional plot called the volatility surface. Figure 7.6 presents an estimation of implied volatility for Asian options whose underlying asset is the price of light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange.

This version: March 18, 2019
http://www.ntu.edu.sg/home/nprivault/index.html
As observed in Figure 7.6, the volatility surface can exhibit a smile phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

**Black-Scholes Formula vs Market Data**

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price $S$ of Cheung Kong Holdings (0001.HK) with strike price $K=109.99$, Maturity $T = \text{December 13, 2010}$, and entitlement ratio $100$.

The market price of the option (17838.HK) on September 28 was $12.30$, as obtained from [http://www.hkex.com.hk/eng/dwrc/search/listsearch.asp](http://www.hkex.com.hk/eng/dwrc/search/listsearch.asp).

The next graph in Figure 7.8 shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying stock price.

In Figure 7.9 we have fitted the path

* © Tan Yu Jia.

This version: March 18, 2019

Estimation of Volatility

Fig. 7.8: Graph of the (market) call option price on Cheung Kong Holdings.

\[ t \mapsto g_c(t, S_t) \]

of the Black-Scholes price to the data of Figure 7.8 using the stock price data of Figure 7.7, by varying the values of the volatility \( \sigma \).

Fig. 7.9: Graph of the Black-Scholes call option price on Cheung Kong Holdings.

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:
Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price $K = 63.704$, maturity $T =$ October 05, 2009, and entitlement ratio 100, cf. page 7.

As above, in Figure 7.12 we have fitted the path $t \mapsto g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 7.11 using the stock price data of Figure 7.10. In this case we are in the money at maturity, and we also check that the option is worth $100 \times 0.2650 = 26.650$ at that time which, by absence of arbitrage, is very close to the value $90 - 63.703 = 26.296$ of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying HSBC, with strike price $K = 77.667$, maturity $T =$ October 05, 2009, and entitlement ratio 92.593. One checks easily that at maturity, the price of the put option is worth $0.01$ (a market price cannot be lower), which almost equals the option payoff $0$, by absence of arbitrage opportunities. Figure 7.14 is a fit of the Black-Scholes put price graph.
Estimation of Volatility

Fig. 7.12: Graph of the Black-Scholes call option price on HSBC Holdings.

Fig. 7.13: Graph of the (market) put option price on HSBC Holdings.

\[ t \mapsto g_p(t, S_t) \]

to Figure 7.13 as a function of the stock price data of Figure 7.12. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.

Fig. 7.14: Graph of the Black-Scholes put option price on HSBC Holdings.
The normalized market data graph in Figure 7.15 shows how the option price can track the values of the underlying. Note that the range of values [26.55, 26.90] for the underlying corresponds to [0.675, 0.715] for the option price, meaning 1.36% vs 5.9% in percentage. This is a European call option on the ALSTOM underlying with strike price $K = \e20$, maturity March 20, 2015, and entitlement ratio 10.

![Fig. 7.15: Call option price vs underlying price.](image)

### 7.3 Local Volatility

As the constant volatility assumption in the Black-Scholes model does not appear to be satisfactory due to the existence of volatility smiles, it can make more sense to consider models of the form

$$\frac{dS_t}{S_t} = rd_t + \sigma_t dB_t$$

where $\sigma_t$ is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$\frac{dS_t}{S_t} = rd_t + \sigma(t, S_t) dB_t \tag{7.5}$$

where $\sigma(t, x)$ is a deterministic function of time $t$ and of the underlying stock price $x$. Such models are called local volatility models. The corresponding Black-Scholes PDE can be written as
Estimation of Volatility

\[
\begin{align*}
\left\{
\begin{array}{l}
rg(t, x, K) = \frac{\partial g}{\partial t}(t, x, K) + rx \frac{\partial g}{\partial x}(t, x, K) + \frac{1}{2} x^2 \sigma^2(t, x) \frac{\partial^2 g}{\partial x^2}(t, x, K),
\end{array}
\right.
\end{align*}
\]

\(g(T, x, K) = (x - K)^+\), \quad (7.6)

with terminal condition \(g(T, x, K) = (x - K)^+\), i.e. we consider European call options.

In order to implement a stochastic volatility model such as (7.5), it is important to first calibrate the local volatility function \(\sigma(t, x)\) to market data.

In principle, the Black-Scholes PDE could allow one to recover the value of \(\sigma(t, x)\) as a function of the option price \(g(t, x, K)\), as

\[
\sigma(t, x) = \sqrt{\frac{2rg(t, x, K) - 2 \frac{\partial g}{\partial t}(t, x, K) - 2rx \frac{\partial g}{\partial x}(t, x, K)}{x^2 \frac{\partial^2 g}{\partial x^2}(t, x, K)}}, \quad x, t > 0,
\]

however, this formula requires the knowledge of the option price for different values of the underlying asset price \(x\), in addition to the knowledge of the strike price \(K\).

Partial derivatives in time can be approximated using forward finite difference approximations as

\[
\frac{\partial g}{\partial t}(t_i, x) \simeq \frac{g(t_{i+1}, x_j) - g(t_i, x_j)}{\Delta t}, \quad (7.7)
\]

or, using backward finite difference approximations, as

\[
\frac{\partial g}{\partial t}(t_i, x) \simeq \frac{g(t_i, x_j) - g(t_{i-1}, x_j)}{\Delta t}. \quad (7.8)
\]

First order spatial derivatives can be approximated as

\[
\frac{\partial g}{\partial x}(t, x_j) \simeq \frac{g(t, x_{j+1}) - g(t, x_{j-1})}{\Delta x}, \quad \frac{\partial g}{\partial x}(t, x_{j+1}) \simeq \frac{g(t, x_{j+1}) - g(t, x_j)}{\Delta x}. \quad (7.9)
\]

Reusing (7.9), second order spatial derivatives can be similarly approximated as

\[
\frac{\partial^2 g}{\partial x^2}(t, x_j) \simeq \frac{1}{\Delta x} \left( \frac{\partial g}{\partial x}(t, x_{j+1}) - \frac{\partial g}{\partial x}(t, x_j) \right)
\]

\[
\simeq \frac{g(t_i, x_{j+1}) + g(t_i, x_{j-1}) - 2g(t_i, x_j)}{(\Delta x)^2}. \quad (7.10)
\]
The Dupire (1994) formula brings a solution to the local volatility calibration problem by providing an estimator of \( \sigma(t, x) \) as a function \( \sigma(t, K) \) based on the values of the strike price \( K \).

**Proposition 7.1.** *(Dupire (1994), Derman and Kani (1994))* Consider a family \((C^M(T, K))_{T,K>0}\) of market call option prices with maturities \( T \) and strike prices \( K \) is given at time 0 with \( S_0 = x \), and define the volatility function \( \sigma(t, y) \) by

\[
\sigma(t, y) := \sqrt{\frac{2 \frac{\partial C^M}{\partial t}(t, y) + 2ry \frac{\partial C^M}{\partial y}(t, y)}{y^2 \frac{\partial^2 C^M}{\partial y^2}(t, y)}} = \frac{\sqrt{\frac{\partial C^M}{\partial t}(t, y) + ry \frac{\partial C^M}{\partial y}(t, y)}}{ye^{-rT/2} \sqrt{\varphi_t(y)/2}},
\]

(7.11)

where \( \varphi_t(y) \) denotes the probability density function of \( S_t, t \in [0, T] \). Then the prices \( g(T, K) \) generated from the Black-Scholes PDE (7.6) will be compatible with the market option prices \( C^M(T, K) \) in the sense that

\[
g(T, K) = C^M(T, K), \quad T, K > 0. \quad (7.12)
\]

**Proof.** We apply the probabilistic approach to option pricing, that allows us to write \( g(T, K) \) as

\[
g(T, K) = e^{-rT} \mathbb{E}[(S_T - K)^+],
\]

(7.13)

where \((S_t)_{t \in \mathbb{R}_+}\) is defined by (7.5), and use stochastic calculus. Letting \( \varphi_T(y) \) denote the probability density function of \( S_T \), Condition (7.12) can be written at \( t = 0 \) as

\[
C^M(T, K) = e^{-rT} \mathbb{E}[(S_T - K)^+] = e^{-rT} \int_0^\infty (y - K)^+ \varphi_T(y) dy \]

\[
= e^{-rT} \int_K^\infty (y - K) \varphi_T(y) dy \]

\[
= e^{-rT} \int_K^\infty y \varphi_T(y) dy - K e^{-rT} \int_K^\infty \varphi_T(y) dy \]

\[
= e^{-rT} \int_K^\infty y \varphi_T(y) dy - K e^{-rT} \mathbb{P}(S_T \geq K). \quad (7.14)
\]

By differentiation of (7.14) with respect to \( K \), one gets

\[
\frac{\partial C^M}{\partial K}(T, K) = -e^{-rT} K \varphi_T(K) - e^{-rT} \int_K^\infty \varphi_T(y) dy + e^{-rT} K \varphi_T(K)
\]
Integrating by parts in the above relation yields

\[ \varphi_T(y) = e^{-rT} \int_{K}^{\infty} \varphi_T(y) dy, \]

hence, by twice differentiation of \( C^M(T, K) \) with respect to \( K \) we find

\[ \varphi_T(K) = e^{rT} \frac{\partial^2 C^M}{\partial K^2}(T, K), \quad (7.15) \]

cf. Relation (1) in Breeden and Litzenberger (1978). On the other hand, for any sufficiently smooth function \( f \in C_0^\infty(\mathbb{R}) \), using the Itô formula we have

\[
\int_{-\infty}^{\infty} \varphi_T(y) f(y) dy = \mathbb{E}[f(S_T)]
\]

\[
= \mathbb{E} \left[ f(S_0) + \int_0^T f'(s) dS_s + \frac{1}{2} \int_0^T f''(s) S_s^2 \sigma^2(t, S_t) dt \right]
\]

\[
= \mathbb{E} \left[ f(S_0) + r \int_0^T f'(s) S_s dt + \int_0^T f'(s) S_s \sigma(t, S_t) dB_t + \frac{1}{2} \int_0^T f''(s) S_s^2 \sigma^2(t, S_t) dt \right]
\]

\[
f(S_0) + \mathbb{E} \left[ r \int_0^T f'(s) S_s dt + \frac{1}{2} \int_0^T f''(s) S_s^2 \sigma^2(t, S_t) dt \right]
\]

\[
= f(S_0) + r \int_{-\infty}^{\infty} y f'(y) \int_0^T \varphi_T(y) dt dy + \frac{1}{2} \int_{-\infty}^{\infty} y^2 f''(y) \int_0^T \sigma^2(t, y) \varphi_T(y) dt dy,
\]

hence, after differentiating both sides of the equality with respect to \( T \),

\[
\int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y) f(y) dy = r \int_{-\infty}^{\infty} y f'(y) \varphi_T(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} y^2 f''(y) \sigma^2(T, y) \varphi_T(y) dy.
\]

Integrating by parts in the above relation yields

\[
\int_{-\infty}^{\infty} \frac{\partial \varphi_T}{\partial T}(y) f(y) dy
\]

\[
= -r \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial y} (y \varphi_T(y)) dy + \frac{1}{2} \int_{-\infty}^{\infty} f(y) \frac{\partial^2}{\partial y^2} (y^2 \sigma^2(T, y) \varphi_T(y)) dy,
\]

for all smooth functions \( f(y) \) with compact support, hence

\[
\frac{\partial \varphi_T}{\partial T}(y) = -r \frac{\partial}{\partial y} (y \varphi_T(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (y^2 \sigma^2(T, y) \varphi_T(y)), \quad y \in \mathbb{R}.
\]

Making use of (7.15) we get

\[
- r \frac{\partial^2 C^M}{\partial y^2}(T, y) - \frac{\partial}{\partial T} \frac{\partial^2 C^M}{\partial y^2}(T, y)
\]

\[
= r \frac{\partial}{\partial y} \left( y \frac{\partial^2 C^M}{\partial y^2}(T, y) \right) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y) \right), \quad y \in \mathbb{R}.
\]
After a first integration with respect to $y$ under the limiting condition \( \lim_{K \to +\infty} C^M(T, K) = 0 \), we obtain

\[
-r \frac{\partial C^M}{\partial y}(T, y) - \frac{\partial}{\partial T} \frac{\partial C^M}{\partial y}(T, y) = ry \frac{\partial^2 C^M}{\partial y^2}(T, y) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y) \right),
\]

i.e.

\[
-r \frac{\partial C^M}{\partial y}(T, y) - \frac{\partial}{\partial T} \frac{\partial C^M}{\partial y}(T, y) = r \frac{\partial}{\partial y} \left( y \frac{\partial C^M}{\partial y}(T, y) \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y) \right),
\]

or

\[
- \frac{\partial}{\partial y} \frac{\partial C^M}{\partial T}(T, y) = r \frac{\partial}{\partial y} \left( y \frac{\partial C^M}{\partial y}(T, y) \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y) \right).
\]

Integrating one more time with respect to $y$ yields

\[
- \frac{\partial C^M}{\partial T}(T, y) = ry \frac{\partial C^M}{\partial y}(T, y) - \frac{1}{2} y^2 \sigma^2(T, y) \frac{\partial^2 C^M}{\partial y^2}(T, y), \quad y \in \mathbb{R},
\]

which conducts to (7.11) and is called the Dupire Dupire (1994) PDE. □

Figure 7.16* presents an estimation of local volatility by the finite differences (7.7)-(7.10), based on Boeing NYSE:BA option price data.

![Fig. 7.16: Local volatility estimated from Boeing Co. option price data.](image)

See Achdou and Pironneau (2005) and in particular Figure 8.1 therein for numerical methods applied to local volatility estimation using spline functions instead of the discretization (7.7)-(7.10).

* © Yu Zhi Yu.
Estimation of Volatility

From (7.11), the local volatility \( \sigma(t, y) \) can also be estimated by computing \( \sigma_{\text{M}}(T, y) \) from the Black-Scholes formula, based on a value of the implied volatility \( \sigma \).

**Local volatility from put option prices**

Note that by the call-put parity relation

\[
C_{\text{M}}(T, y) = P_{\text{M}}(T, y) + x - ye^{-rT}, \quad y, T > 0,
\]

where \( S_0 = \), cf. (5.26), we have

\[
\begin{align*}
\frac{\partial C_{\text{M}}}{\partial T}(T, y) &= ry e^{-rT} + \frac{\partial P_{\text{M}}}{\partial T}(T, y), \\
\frac{\partial P_{\text{M}}}{\partial y}(t, y) &= e^{-rT} + \frac{\partial C_{\text{M}}}{\partial y}(t, y), \\
\varphi_T(K) &= e^{rT} \frac{\partial^2 C_{\text{M}}}{\partial y^2}(T, y) = e^{rT} \frac{\partial^2 P_{\text{M}}}{\partial y^2}(T, y),
\end{align*}
\]

and

\[
\frac{\partial C_{\text{M}}}{\partial T}(T, y) + ry \frac{\partial C_{\text{M}}}{\partial y}(T, y) = \frac{\partial P_{\text{M}}}{\partial T}(T, y) + ry \frac{\partial P_{\text{M}}}{\partial y}(T, y).
\]

Consequently, the local volatility in Proposition 7.1 can be rewritten in terms of market put prices as

\[
\sigma(t, y) := \sqrt{\frac{2 \frac{\partial P_{\text{M}}}{\partial t}(t, y) + 2ry \frac{\partial P_{\text{M}}}{\partial y}(t, y)}{y^2 \frac{\partial^2 P_{\text{M}}}{\partial y^2}(t, y)}} = \sqrt{\frac{\frac{\partial P_{\text{M}}}{\partial t}(t, y) + ry \frac{\partial P_{\text{M}}}{\partial y}(t, y)}{ye^{-rT/2} \sqrt{\varphi_t(y)/2}}},
\]

which is formally identical to (7.11) after replacing market call option prices \( C_{\text{M}}(T, K) \) with market put option prices \( P_{\text{M}}(T, K) \).

**7.4 The CBOE VIX® Volatility Index**

Other ways to estimate market volatility include the CBOE Volatility Index® (VIX®) for the S&P 500 stock index, cf. e.g. § 3.1.1 of Papanicolaou and Sircar (2014). Let the asset price process \( (S_t)_{t \in \mathbb{R}^+} \) be given as

\[
dS_t = rS_t dt + \sigma_t S_t dB_t
\]
where $(\sigma_t)_{t \in \mathbb{R}^+}$ is a stochastic volatility process.

**Lemma 7.2.** Assume that the stochastic volatility process $(\sigma_t)_{t \in \mathbb{R}^+}$ is independent of the Brownian motion $(B_t)_{t \in \mathbb{R}^+}$. Then for every $\lambda > 0$ there exists $p_\lambda$ such that

$$
\mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 \, dt \right) \right] = e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \right].
$$

**Proof.** We have

$$
e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \bigg| \mathcal{F}_T^\sigma \right] = \mathbb{E} \left[ \exp \left( p_\lambda \int_0^T \sigma_t \, dB_t - \frac{p_\lambda}{2} \int_0^T \sigma_t^2 \, dt \right) \bigg| \mathcal{F}_T^\sigma \right]
$$

$$
= \exp \left( -\frac{p_\lambda}{2} \int_0^T \sigma_t^2 \, dt \right) \mathbb{E} \left[ \exp \left( p_\lambda \int_0^T \sigma_t \, dB_t \right) \bigg| \mathcal{F}_T^\sigma \right]
$$

$$
= \exp \left( -\frac{p_\lambda}{2} \int_0^T \sigma_t^2 \, dt \right) \exp \left( \frac{p_\lambda^2}{2} \int_0^T \sigma_t^2 \, dt \right)
$$

$$
= \exp \left( p_\lambda (p_\lambda - 1)/2 \int_0^T \sigma_t^2 \, dt \right)
$$

$$
= \exp \left( \lambda \int_0^T \sigma_t^2 \, dt \right),
$$

provided that $\lambda = p_\lambda (p_\lambda - 1)/2$, and in this case we have

$$
e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \right] = e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda} \bigg| \mathcal{F}_T^\sigma \right] = \mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 \, dt \right) \right].
$$

The equation $\lambda = p_\lambda (p_\lambda - 1)/2$, i.e. $p_\lambda^2 - p_\lambda - 2\lambda = 0$, has for solutions

$$
p_\lambda^\pm = \frac{1}{2} \pm \sqrt{1/4 + 2\lambda},
$$

with $p_\lambda^- < 0 < p_\lambda^+$ when $\lambda > 0$. By differentiating the relation

$$
\mathbb{E} \left[ \exp \left( \lambda \int_0^T \sigma_t^2 \, dt \right) \right] = e^{-rp_\lambda T} \mathbb{E} \left[ \left( \frac{S_T}{S_0} \right)^{p_\lambda^+} \right] = \mathbb{E} \left[ \exp \left( -rp_\lambda^+ T + p_\lambda^+ \log \frac{S_T}{S_0} \right) \right]
$$

with respect to $\lambda$ we get

$$
\mathbb{E} \left[ \int_0^T \sigma_t^2 \, dt \exp \left( \lambda \int_0^T \sigma_t^2 \, dt \right) \right] = -rp_\lambda^+ T \mathbb{E} \left[ \exp \left( -rp_\lambda^+ T + p_\lambda^+ \log \frac{S_T}{S_0} \right) \right]
$$

$$
+ \mathbb{E} \left[ \exp \left( -rp_\lambda^+ T + p_\lambda^+ \log \frac{S_T}{S_0} \right) \log \frac{S_T}{S_0} \right]
$$

254
Estimation of Volatility

\[ \pm \frac{rT}{\sqrt{1/4 + 2\lambda}} \mathbb{E} \left[ \exp \left( -rp_{\lambda}^{T} T \right) \left( \frac{S_{T}}{S_{0}} \right) ^{p_{\lambda}^{T}} \right] \]

\[ \pm \frac{1}{\sqrt{1/4 + 2\lambda}} \mathbb{E} \left[ \exp \left( -rp_{\lambda}^{T} T + p_{\lambda}^{T} \log \frac{S_{T}}{S_{0}} \right) \log \frac{S_{T}}{S_{0}} \right], \]

which, when $\lambda = 0$, yields

\[ \mathbb{E} \left[ \int_{0}^{T} \sigma_{t}^{2} dt \right] = 2rT - 2 \mathbb{E} \left[ \log \frac{S_{T}}{S_{0}} \right] = -2 \mathbb{E} \left[ \int_{0}^{T} \sigma_{t} dB_{t} - \frac{1}{2} \int_{0}^{T} \sigma_{t}^{2} dt \right] \]

for $p_{0}^{T} = 0$, and

\[ \mathbb{E} \left[ \int_{0}^{T} \sigma_{t}^{2} dt \right] = 2e^{-rT} \mathbb{E} \left[ \frac{S_{T}}{S_{0}} \log e^{-rT} \frac{S_{T}}{S_{0}} \right] \]

for $p_{0}^{T} = 1$. \hfill \square

The next Proposition 7.3, cf. Friz and Gatheral (2005), shows that the VIX® Volatility Index defined as

\[ \text{VIX}_t := \sqrt{\frac{2e^{r\tau}}{\tau} \left( \int_{0}^{F_{t,t+\tau}} \frac{P(t, t + \tau, K)}{K^2} dK + \int_{F_{t,t+\tau}}^{\infty} \frac{C(t, t + \tau, K)}{K^2} dK \right) \} } \]

(7.16)

at time $t$ can be interpreted as an average of future volatility values. Here, $\tau = 30$ days, $F_{t,t+\tau} := \mathbb{E}^{*}[S_{t+\tau} \mid \mathcal{F}_t]$ is a future price, and $P(t, t + \tau, K)$ and $C(t, t + \tau, K)$ are out of the money put option prices, and out of the money call option prices with strike price $K$ and maturity $t + \tau$, cf. § 3.1.1 of Papanicolaou and Sircar (2014).

**Proposition 7.3.** The value of the VIX® Volatility Index at $t = 0$ is given from the averaged realized variance option as

\[ \text{VIX}_0 := \sqrt{\frac{1}{T} \mathbb{E}^{*} \left[ \int_{0}^{T} \sigma_{t}^{2} dt \right] + 2 \frac{e^{rT} - rT - 1}{T} } \]

**Proof.** We start by showing that for every $\lambda > 0$ we have

\[ \frac{1}{\lambda} \mathbb{E} \left[ \exp \left( rp_{\lambda} T + \lambda \int_{0}^{T} \sigma_{t}^{2} dt \right) - 1 \right] \]

\[ = 2 \frac{e^{rT}}{S_{0}^{p_{\lambda}}} \left( \int_{0}^{S_{0}} P(T, K) \frac{dK}{K^{2-p_{\lambda}}} + \int_{S_{0}}^{\infty} C(T, K) \frac{dK}{K^{2-p_{\lambda}}} \right) + \frac{p_{\lambda}}{\lambda} \left( e^{rT} - 1 \right). \]

Indeed, by integration by parts over the intervals $[0, S_{0}]$ and $[S_{0}, \infty)$ and using the boundary conditions
Finally, taking $p : \lambda \to p_\lambda$ and letting $\lambda$ tend to zero, we find

$$-2rT + \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] = \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbb{E}\left[\exp\left(rp_\lambda T + \lambda \int_0^T \sigma_t^2 dt\right) - 1\right] + 2(1 - e^{-rT})$$

$$= \lim_{\lambda \to 0} \frac{2e^{rT}}{S_0^{p_\lambda}} \left(\int_0^S P(T, K) \frac{dK}{K^{2-p_\lambda}} + \int_S^\infty C(T, K) \frac{dK}{K^{2-p_\lambda}}\right) + 2(1 - e^{-rT}).$$
Estimation of Volatility

\[ = 2 e^{rT} \left( \int_0^{S_0} P(T, K) \frac{dK}{K^2} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2} \right) + 2(1 - e^{rT}), \]

hence

\[ \mathbb{E} \left[ \int_0^T \sigma_t^2 dt \right] = 2 e^{rT} \left( \int_0^{S_0} P(T, K) \frac{dK}{K^2} + \int_{S_0}^{\infty} C(T, K) \frac{dK}{K^2} \right) - 2(e^{rT} - rT - 1). \]

The following code allows us to estimate the VIX® based on the discretization of (7.16) and market option prices on the S&P 500.

```r
library(quantmod)
date = "2019-02-15"
getSymbols("^GSPC", src = "yahoo", from = date)
S0 = as.vector(Ad(GSPC)[1])
GSPC.OPTS <-getOptionChain("^GSPC", "2019-03-15")
Call <- as.data.frame(GSPC.OPTS$calls)
Put <- as.data.frame(GSPC.OPTS$puts)
K0 = max(Put[Put$Strike<0,"Strike",Call[Call$Strike<0,"Strike")
Call_OTM <- Call[Call$Strike>=K0,"Call_OTM$diff = c(S0-K0, diff(Call_OTM$Strike))
Put_OTM <- Put[Put$Strike<=K0,"Put_OTM$diff = c(diff(Put_OTM$Strike),S0-K0)
T = 30/365;r=0.02
VIX_imp = 100*sqrt((2*exp(r*T)/T)*(sum(Call_OTM$Last/(Call_OTM$Strike^2)*Call_OTM$diff)+sum(Put_OTM$Last/(Put_OTM$Strike^2)*Put_OTM$diff]))
getSymbols("^VIX", src = "yahoo", from = date)
VIX_market = as.vector(Ad(VIX)[1])
c("Estimated VIX"= VIX_imp, "market VIX"=VIX_market)
VIX.OPTS <-getOptionChain("^VIX")
```

The next code retrieves VIX® data using the quantmod R package.

```r
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo")
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
c("Estimated VIX"= VIX_imp, "market VIX"=VIX_market)
add_TA(Ad("VIX"), name="VIX")
```

The impact of various events such as the “Brexit” referendum on June 23, 2016 can be observed on the VIX® index in Figure 7.17. Note that the variations of the stock index are negatively correlated to the variations of the volatility index.
library(PerformanceAnalytics)
library(quantmod)
getSymbols("^GSPC", from="2000-01-01", to=Sys.Date(), src="yahoo")
getSymbols("^VIX", from="2000-01-01", to=Sys.Date(), src="yahoo")
SP500=Ad(/grave.ts1 GSPC/)
SP500.rtn <- exp(CalculateReturns(SP500, method="compound")) - 1
SP500.rtn[1,] <- 0
histvol <- rollapply(SP500.rtn, width = 30, FUN=sd.annualized)
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
chart_Series(SP500, name="SP500", theme=myTheme)
add_TA(histvol, name="Historical Volatility")
add_TA(Ad("VIX"), name="VIX")

Figure 7.18 compares the VIX® estimate to the historical volatility of Section 7.1.
7.5 Stochastic Volatility Models

Time-Dependent Stochastic Volatility

The next Figure 7.19 refers to the EURO/SGD exchange rate, and shows some spikes that cannot be generated by Gaussian returns with constant variance.

![Euro / SGD exchange rate](image)

Fig. 7.19: Euro / SGD exchange rate.

This type of data shows that, in addition to jump models that are commonly used to take into account the slow decrease of probability tails observed in market data, other tools should be implemented in order to model a possibly random and time-varying volatility.

We consider an asset price driven by the stochastic differential equation

$$
    dS_t = rS_t dt + S_t \sqrt{v_t} dB_t
$$

under the risk-neutral probability measure $\mathbb{P}^\ast$, with solution

$$
    S_T = S_t \exp \left( r(T - t) + \int_t^T \sqrt{v(s)} dB_s - \frac{1}{2} \int_t^T v(s) ds \right)
$$

where $(v_t)_{t \in \mathbb{R}_+}$ is a (possibly random) squared volatility process adapted to the filtration $\mathcal{F}_t^{(1)}$ generated by $(B_t)_{t \in \mathbb{R}_+}$.

Time-dependent deterministic volatility

When the volatility $(v(t))_{t \in \mathbb{R}_+}$ is a deterministic function of time, the solution (7.18) of (7.17) is a lognormal random variable at time $T$ with conditional log-variance

$$
    \int_t^T v(s) ds
$$
given $\mathcal{F}_t$. In particular, a European call option on $S_T$ can be priced by the Black-Scholes formula as
\[
e^{-(T-t)r} \mathbb{E}^*[\mathbb{P}((S_T - K)^+ \mid \mathcal{F}_t)] = \text{Bl}(S_t, K, r, T-t, \sqrt{\nu(t)}),
\]
with integrated squared volatility parameter
\[
\nu(t) := \frac{\int_t^T v(s)ds}{T-t}, \quad t \in [0,T).
\]

**Independent (stochastic) volatility**

When the volatility $(v_t)_{t \in \mathbb{R}_+}$ is a random process generating a filtration $\mathcal{F}_t^{(2)}$ independent of the filtration $\mathcal{F}_t^{(1)}$ generated by the driving Brownian motion $(B_t^{(1)})_{t \in \mathbb{R}_+}$ under $\mathbb{P}^*$, the equation (7.17) can still be solved as
\[
S_T = S_t \exp \left( r(T-t) + \int_t^T \sqrt{v_s}dB_s^{(1)} - \frac{1}{2} \int_t^T v_sds \right),
\]
and, given $\mathcal{F}_T^{(2)}$, the asset price $S_T$ is a lognormal random variable with random variance
\[
\int_t^T v_sds.
\]
In this case, taking
\[
\mathcal{F}_t := \mathcal{F}_t^{(1)} \lor \mathcal{F}_t^{(2)}, \quad 0 \leq t \leq T,
\]
where $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is the filtration generated by $(B_t^{(1)})_{t \in \mathbb{R}_+}$, we can still price options with payoff $\phi(S_T)$ on the underlying asset price $S_T$ using the tower property
\[
\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_l] = \mathbb{E}^* \left[ \mathbb{E}^* \left[ \phi(S_T) \mid \mathcal{F}_t^{(1)} \lor \mathcal{F}_t^{(2)} \right] \mid \mathcal{F}_t^{(1)} \lor \mathcal{F}_t^{(2)} \right].
\]
As an example, a European call option on $S_T$ can be priced by averaging the Black-Scholes formula as follows:
\[
e^{-(T-t)r} \mathbb{E}^*[\mathbb{P}((S_T - K)^+ \mid \mathcal{F}_t)] = \mathbb{E}^* \left[ \mathbb{E}^* \left[ \mathbb{P}((S_T - K)^+ \mid \mathcal{F}_t^{(1)} \lor \mathcal{F}_t^{(2)}) \mid \mathcal{F}_t^{(1)} \lor \mathcal{F}_t^{(2)} \right] \mid \mathcal{F}_t \right].
\]

Estimation of Volatility

\[
e^{-\left(T-t\right)r} \mathbb{E}^\pi \left[ \mathbb{B}_\pi \left(x, K, r, T-t, \sqrt{\nu(t,T)} \right) \left| \mathcal{F}_t^{(2)} \right. \right] \bigg|_{x=S_t},
\]

which represents an averaged version of Black-Scholes prices, with the random integrated volatility

\[
\hat{\nu}(t,T) := \frac{1}{T-t} \int_t^T \nu_s ds, \quad 0 \leq t \leq T.
\]

On the other hand, when \((\nu_t)_{t \in \mathbb{R}^+}\) is a geometric Brownian motion, the probability distribution of the time integral \(\int_t^T \nu_s ds\) given \(\mathcal{F}_t^{(2)}\) can be computed using integral expressions, cf. Yor (1992) and Proposition 10.1.

Two-factor Stochastic Volatility Models

Evidence based on financial market data, see Figure 7.18, Figure 1 of Papanicolaou and Sircar (2014) or § 2.3.1 of Fouque et al. (2011), shows that the variations in volatility tend to be negatively correlated with the variations of underlying asset prices. For this reason we need to consider an asset price process \((S_t)_{t \in \mathbb{R}^+}\) and a stochastic volatility process \((\nu_t)_{t \in \mathbb{R}^+}\) driven by

\[
\begin{cases}
\quad dS_t = rS_t + \sqrt{\nu_t S_t} dB^{(1)}_t \\
\quad d\nu_t = \mu(t, \nu_t) dt + \beta(t, \nu_t) dB^{(2)}_t,
\end{cases}
\]

Here, \((B^{(1)}_t)_{t \in \mathbb{R}^+}\) and \((B^{(2)}_t)_{t \in \mathbb{R}^+}\) are two standard Brownian motions such that \(\text{Cov}(B^{(1)}_t, B^{(2)}_t) = \rho t\), i.e.

\[
dB^{(1)}_t \cdot dB^{(2)}_t = \rho dt,
\]

where the correlation parameter \(\rho\) satisfies \(-1 \leq \rho \leq 1\), and the coefficients \(\mu(t, x)\) and \(\beta(t, x)\) can be chosen e.g. from mean-reverting models (CIR) or geometric Brownian models, as follows. Note that the observed correlation coefficient \(\rho\) is usually negative, cf. e.g. § 2.1 of Papanicolaou and Sircar (2014).

The Heston model

In the Heston (1993) model, the stochastic volatility \((\nu_t)_{t \in \mathbb{R}^+}\) is chosen to be a CIR Cox et al. (1985) process, i.e. we have
\[
\left\{ \begin{array}{l}
    dS_t = rS_t dt + S_t \sqrt{v_t} dB_t^{(1)} \\
    dv_t = -\lambda(v_t - m) dt + \eta \sqrt{v_t} dB_t^{(2)},
\end{array} \right.
\]

and \( \mu(t, v) = -\lambda(v_t - v) \) and \( \beta(t, v) = \eta \sqrt{v} \), where \( \lambda, m, \eta > 0 \).

Option pricing formulas can be derived in the Heston model using Fourier inversion and complex integrals, cf. (7.32) below.

**The SABR model**

In the SABR model Hagan et al. (2002), which is based on the three parameters \((\alpha, \beta, \rho)\), we have
\[
\left\{ \begin{array}{l}
    dS_t = \sigma_t S_t^{\beta} dB_t^{(1)} \\
    d\sigma_t = \alpha \sigma_t dB_t^{(2)},
\end{array} \right.
\]

where \( \alpha > 0 \) and \( \beta \in (0, 1] \), and \((B_t^{(1)})_{t \in \mathbb{R}_+}\) and \((B_t^{(2)})_{t \in \mathbb{R}_+}\) are two standard Brownian motions with the correlation
\[
dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt.
\]

This model is typically used for the modeling of LIBOR rates and is *not* mean-reverting, hence it is preferably used with a short time horizon. It allows in particular for short time asymptotics of Black implied volatilities that can be used for pricing by inputting them into the Black pricing formula, cf. § 3.3 of Rebonato (2009).

**7.6 Volatility Derivatives**

**Another look at historical volatility**

When \( t_k = kT/N, k = 0, 1, \ldots, N \), a natural estimator for the trend parameter \( \mu \) can be written as
\[
\hat{\mu}_N := \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}
\]
\[
\simeq \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \log \frac{S_{t_k}}{S_{t_{k-1}}}
\]
\[
= \frac{1}{T} \sum_{k=1}^{N} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right)
\]
Estimation of Volatility

\[
= \frac{1}{T} \log \frac{S_T}{S_0}.
\]

Similarly we can use the squared volatility estimator

\[
\hat{\sigma}_N^2 := \frac{1}{N-1} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_k}} \right)^2 - \left( \hat{\mu}_N \right)^2
\]

\[
\simeq \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \left( \log(S_{t_k}) - \log(S_{t_{k-1}}) \right)^2 - \left( \frac{1}{N} \sum_{k=1}^{N} \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_k}} \right)^2
\]

\[
= \frac{1}{T} \sum_{k=1}^{N} \left( \log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \log \frac{S_T}{S_0} \right)^2.
\]

(7.19)

Realized variance swaps are forward contracts that allow for the exchange of the estimated volatility (7.19) against a fixed value \( \kappa_\sigma \). They are priced using the expected value

\[
\mathbb{E} \left[ \hat{\sigma}_N^2 \right] = \frac{1}{T} \mathbb{E} \left[ \sum_{k=1}^{N} \left( \log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \log \frac{S_T}{S_0} \right)^2 \right] - \kappa_\sigma
\]

of their payoff

\[
\frac{1}{T} \sum_{k=1}^{N} \left( \log \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2 - \frac{1}{T^2} \left( \log \frac{S_T}{S_0} \right)^2 - \kappa_\sigma.
\]

Note that the above payoff has to be multiplied by the \emph{vega notional}, which is part of the contract, in order to convert it into currency units.

**Realized variance call options**

Consider the realized variance call option with payoff

\[
\left( \int_0^T \sigma_t^2 dt - \kappa_\sigma^2 \right)^+.
\]

In case \( \int_0^t \sigma_u^2 du \geq \kappa_\sigma^2 \), the price of the realized variance call option is

\[
e^{-\left( T-t \right)r} \mathbb{E}^* \left[ \left( \int_0^T \sigma_u^2 du - \kappa_\sigma^2 \right)^+ \left| \mathcal{F}_t \right. \right]
\]

\[= e^{-\left( T-t \right)r} \mathbb{E}^* \left[ \left( x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2 \right)^+ \left| \mathcal{F}_t \right. \right]_{x=\int_0^t \sigma_u^2 du}
\]
\[
= e^{-(T-t)r} \mathbb{E}^{*} \left[ x + \int_{t}^{T} \sigma_{u}^{2} du - \kappa_{\sigma}^{2} \left| \mathcal{F}_{t} \right. \right]_{x=\int_{0}^{t} \sigma_{u}^{2} du} \\
= e^{-(T-t)r} \int_{0}^{t} \sigma_{u}^{2} du - e^{-(T-t)r} \kappa_{\sigma}^{2} + e^{-(T-t)r} \mathbb{E}^{*} \left[ \int_{t}^{T} \sigma_{u}^{2} du \left| \mathcal{F}_{t} \right. \right].
\]

**Lognormal approximation**

In order to estimate the price

\[
e^{-(T-t)r} \mathbb{E}^{*} \left[ \left( x + \int_{t}^{T} \sigma_{u}^{2} du - \kappa_{\sigma}^{2} \right)^{+} \left| \mathcal{F}_{t} \right. \right]_{x=\int_{0}^{t} \sigma_{u}^{2} du},
\]

of the realized variance call option we can approximate \( \int_{t}^{T} \sigma_{u}^{2} du \) by a lognormal random variable

\[
\int_{t}^{T} \sigma_{u}^{2} du \simeq e^{\mu_{t,T}+\eta_{t,T}X}
\]

with mean \( \mu_{t,T} \) and variance \( \eta_{t,T}^{2} \), where \( X \simeq \mathcal{N}(0,1) \) is a standard normal random variable. The parameters \( \mu_{t,T} \) and \( \eta_{t,T}^{2} \) can be estimated from the variance swap price

\[
e^{-(T-t)r} \mathbb{E}^{*} \left[ \int_{t}^{T} \sigma_{u}^{2} du \left| \mathcal{F}_{t} \right. \right].
\]

and from the variance power option price

\[
e^{-(T-t)r} \mathbb{E}^{*} \left[ \left( \int_{t}^{T} \sigma_{u}^{2} du \right)^{2} \left| \mathcal{F}_{t} \right. \right].
\]

Letting \( R_{i,T}^{2} := \int_{t}^{T} \sigma_{u}^{2} du \), under the (unconditional) lognormal approximation we have

\[
f_{R_{i,T}^{2}}(x) \approx \frac{1}{x \sigma_{t,T} \sqrt{2\pi(T-t)}} \exp \left( \frac{-(\mu_{t,T}-\log x)^{2}}{2(T-t)\sigma_{t,T}^{2}} \right), \quad x > 0, \quad (7.20)
\]

where

\[
\sigma_{t,T}^{2}(T-t) := \log(\mathbb{E}[R_{i,T}^{4}]/(\mathbb{E}[R_{i,T}^{2}])^{2}) \quad \text{and} \quad \mu_{t,T} := -\frac{\sigma_{t,T}^{2}(T-t)}{2} + \log \mathbb{E}[R_{i,T}^{2}].
\]

Consequently, in case \( R_{0,t}^{2} = \int_{0}^{t} \sigma_{u}^{2} dt < \kappa_{\sigma}^{2} \) we can estimate the price
Estimation of Volatility

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2 \right)^+ \right]_{x=\int_0^t \sigma_u^2 du} \]

of the realized variance call option by approximating \( R_{t,T}^2 = \int_t^T \sigma_u^2 du \) by a lognormal random variable.

Recall that under the lognormal approximation we have

\[ f_{R_{t,T}^2}(x) \approx \frac{1}{x \sigma_{t,T} \sqrt{2\pi(T-t)}} e^{-(\mu_{t,T} + \log x) / (2(T-t)\sigma_{t,T}^2)}, \quad x > 0, \tag{7.21} \]

where the parameters \( \mu_{t,T} \) and \( \sigma_{t,T} \) are estimated by matching the first and second conditional moments

\[ \mathbb{E}[R_{t,T}^2] = e^{\mu_{t,T} + \sigma_{t,T}^2(T-t)/2} \quad \text{and} \quad \mathbb{E}[R_{t,T}^4] = e^{2(\mu_{t,T} + \sigma_{t,T}^2(T-t))} \]

to those of the lognormal distribution with mean \( \mu_{t,T} \) and variance \( \sigma_{t,T}^2(T-t) \), i.e.

\[ \mu_{t,T} = -\frac{\sigma_{t,T}^2(T-t)}{2} + \log \mathbb{E}[R_{t,T}^2] \quad \text{and} \quad \sigma_{t,T}^2(T-t) = \log \left( 1 + \frac{\text{Var}[R_{t,T}^2]}{(\mathbb{E}[R_{t,T}^2])^2} \right). \tag{7.22} \]

Hence, under the (unconditional) lognormal approximation we have

\[ \text{VC}_{t,T}(\kappa_\sigma) = e^{-(T-t)r} \mathbb{E} \left[ (x + R_{t,T}^2 - \kappa_\sigma)^+ \right]_{x=R_{0,t}^2} \]
\[ = e^{-(T-t)r} \int_{\kappa_\sigma}^{\infty} (y - (\kappa_\sigma - R_{0,t}^2))^+ f_{R_{t,T}^2}(y) dy \]
\[ \approx e^{-(T-t)r} \left( e^{\mu_{t,T} + \sigma_{t,T}^2(T-t)/2} \Phi(d_1) - (\kappa_\sigma - R_{0,t}^2) \Phi(d_2) \right) \]
\[ \approx e^{-(T-t)r} \mathbb{E}[R_{t,T}^2] \Phi(d_1) - e^{-(T-t)r} (\kappa_\sigma - R_{0,t}^2) \Phi(d_2), \]

see Carr and Lee (May 2007), Friz and Gatheral (2005), where \( \Phi \) denotes the standard Gaussian cumulative distribution function and

\[ d_1 := \frac{\log \left( \mathbb{E}[R_{t,T}^2] / (\kappa_\sigma - R_{0,t}^2) \right)}{\sigma_{t,T} \sqrt{T-t}} + \frac{\sqrt{T-t}}{2} \]
\[ = -\frac{\log(\kappa_\sigma - R_{0,t}^2)}{\sigma_{t,T} \sqrt{T-t}} + \frac{\mu_{t,T} + \sigma_{t,T}^2(T-t)}{\sigma_{t,T} \sqrt{T-t}}, \]

and

\[ d_2 := d_1 - \sigma_{t,T} \sqrt{T-t} = -\frac{\log(\kappa_\sigma - R_{0,t}^2)}{\sigma_{t,T} \sqrt{T-t}} + \frac{\mu_{t,T}}{\sigma_{t,T} \sqrt{T-t}}. \]
Gamma approximation

In case $R_{0,t}^2 = \int_0^t \sigma_u^2 dt < \kappa^2$, we can derive an estimate for the price

$$e^{-(T-t)\theta} \mathbb{E}^*[\left(x + \int_t^T \sigma_u^2 du - \kappa^2\right)^+ | \mathcal{F}_t]_{x = \int_t^0 \sigma_u^2 du}$$

of the realized variance call option by approximating $R_{t,T}^2 = \int_t^T \sigma_u^2 du$ by a gamma variable.

Recall that under the gamma approximation we have

$$f_{R_{t,T}^2}(x) \approx \frac{e^{-x/\theta_{t,T}} (x/\theta_{t,T})^{1+\nu_{t,T}}}{\Gamma(\nu_{t,T})}, \quad x > 0, \quad (7.24)$$

where $\theta_{t,T}, \nu_{t,T}$ are parameters estimated by matching the first and second conditional moments of $R_{t,T}^2$ to those of the gamma distribution, as

$$\theta_{t,T} = \frac{\mathbb{E}[R_{t,T}^2]}{\mathbb{E}[R_{t,T}^2]} \quad \text{and} \quad \nu_{t,T} = \frac{\mathbb{E}[R_{t,T}^2]}{\theta_{t,T}} = \frac{\left(\mathbb{E}[R_{t,T}^2]\right)^2}{\mathbb{V}[R_{t,T}^2]}. \quad (7.25)$$

Hence, under the (unconditional) gamma approximation we have

$$f_{R_{t,T}^2}(x) \approx \frac{1}{\Gamma(\nu_{t,T})} \times \frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}}, \quad (7.26)$$

where

$$\theta_{t,T} = \frac{\mathbb{E}[R_{t,T}^2]}{\mathbb{E}[R_{t,T}^2]} \quad \text{and} \quad \nu_{t,T} = \frac{\mathbb{E}[R_{t,T}^2]}{\theta_{t,T}} = \frac{\left(\mathbb{E}[R_{t,T}^2]\right)^2}{\mathbb{V}[R_{t,T}^2]}. \quad (7.27)$$

Therefore the Asian option price can be approximated as

$$\mathbb{E}[(R_{t,T}^2 - \kappa\sigma)^+] = \int_{\kappa\sigma}^{\infty} (x - \kappa\sigma)^+ f_{R_{t,T}^2}(x) dx$$

$$\approx \frac{1}{\Gamma(\nu_{t,T})} \int_{\kappa\sigma}^{\infty} (x - \kappa\sigma)^{\nu_{t,T} - 1} e^{-x/\theta_{t,T}} dx$$

$$= \frac{\theta_{t,T}}{\Gamma(\nu_{t,T})} \int_{\kappa\sigma}^{\infty} x^{\nu_{t,T}} e^{-x/\theta_{t,T}} dx - \frac{\kappa\sigma}{\Gamma(\nu_{t,T})} \int_{\kappa\sigma}^{\infty} x^{\nu_{t,T} - 1} e^{-x/\theta_{t,T}} dx$$

$$= \theta_{t,T} \nu_{t,T} Q\left(1 + \nu_{t,T}, \frac{\kappa\sigma}{\theta_{t,T}}\right) - \kappa\sigma Q\left(\nu_{t,T}, \frac{\kappa\sigma}{\theta_{t,T}}\right),$$

266
where
\[ Q(\lambda, z) := \frac{1}{\Gamma(\lambda)} \int_z^{\infty} t^{\lambda-1} e^{-t} dt, \quad z > 0, \]
is the (normalized) upper incomplete gamma function, which yields
\[
\begin{align*}
\mathcal{E}(\kappa \sigma, T) &= e^{-(T-t)r} \mathbb{E} \left[ (x + R_{t,T}^2 - \kappa \sigma)^+ \right] \\
&\approx e^{-(T-t)r} \left( \nu_{t,T} \theta_{t,T} Q \left( 1 + \nu_{t,T}, \frac{\kappa \sigma}{\theta_{t,T}} \right) - \kappa \sigma Q \left( \nu_{t,T}, \frac{\kappa \sigma}{\theta_{t,T}} \right) \right) \\
&= e^{-(T-t)r} \left( \mathbb{E}[R_{t,T}^2] Q \left( 1 + \nu_{t,T}, \frac{\kappa \sigma}{\theta_{t,T}} \right) - \kappa \sigma Q \left( \nu_{t,T}, \frac{\kappa \sigma}{\theta_{t,T}} \right) \right).
\end{align*}
\]

Realized variance in the Heston model

Consider now the Heston model driven by the stochastic differential equation
\[ dv_t = (a - bv_t)dt + \sigma \sqrt{v_t} dW_t, \]
we have
\[ \mathbb{E}[v_T] = v_0 e^{-bT} + \frac{a}{b} \left( 1 - e^{-bT} \right), \]
from which it follows
\[ \mathbb{E}[R_{0,T}^2] = v_0 \frac{1 - e^{-bT}}{b} + a \frac{e^{-bT} + bT - 1}{b^2}. \]
We also have
\[ \text{Var}[v_T] = v_0 \frac{\sigma^2}{b} \left( e^{-bT} - e^{-2bT} \right) + \frac{a \sigma^2}{2b} \left( 1 - e^{-bT} \right)^2, \]
and
\[ \text{Var}[R_{0,T}^2] = \sigma^2 v_0 \frac{1 - 2bT e^{-bT} - e^{-2bT}}{b^3} + a \sigma^2 \frac{e^{-2bT} + 2bT + 4(bT + 1) e^{-bT} - 5}{2b^4}. \]
Using the parameters \( \sigma = 1, b = 1.15, a = 0.04 \times b, v_0 = 0.04, T = 1, r = 0, \)
we plot the graphs of the gamma and lognormal variance call option prices
for \( \kappa \sigma \in [0, 0.2]. \)

The following graph is obtained using an R code, see also § 11 of Gatheral (2006).
The gamma variance approximation seems better than the lognormal approximations for large values of $\kappa \sigma$, which can be consistent with the fact that the long run distribution of the CIR-Heston process has the gamma density

$$f(x) = \frac{1}{\Gamma(2a/\sigma^2)} \left( \frac{2b}{\sigma^2} \right)^{2a/\sigma^2} x^{-1+2a/\sigma^2} e^{-2bx/\sigma^2}, \quad x > 0.$$ 

with shape parameter $2a/\sigma^2$ and scale parameter $\sigma^2/(2b)$, which is also the invariant distribution of $v_t$.

### 7.7 Option Pricing PDE

In the sequel we will assume that $(B_t^{(1)})_{t \in \mathbb{R}^+}$ is a standard Brownian motion under $\mathbb{P}^*$, i.e. the discounted price process $(e^{-rt}S_t)_{t \in \mathbb{R}^+}$ is a martingale under $\mathbb{P}^*$. For simplicity of exposition we will make the assumption that $(B_t^{(2)})_{t \in \mathbb{R}^+}$ is also a standard Brownian motion under $\mathbb{P}^*$.

**Proposition 7.4.** Consider a vanilla option with payoff $h(S_T)$ priced as

$$V_t = f(t, v_t, S_t) = e^{-(T-t)r} \mathbb{E}^* \left[ h(S_T) \left| \mathcal{F}_t \right. \right], \quad 0 \leq t \leq T.$$ 

The function $f(t, y, x)$ satisfies the PDE
Estimation of Volatility

\[
\frac{\partial f}{\partial t} (t, v, x) + r x \frac{\partial f}{\partial x} (t, v, x) + \frac{1}{2} v x^2 \frac{\partial^2 f}{\partial x^2} (t, v, x) \\
+ \mu(t, v) \frac{\partial f}{\partial v} (t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2} (t, v, x) + \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x} (t, v, x) \\
= rf(t, v, x),
\]

under the terminal condition \(f(T, v, x) = h(x)\).

**Proof.** By Itô calculus with respect to the correlated Brownian motions \((B_t^{(1)})_{t \in \mathbb{R}_+}\) and \((B_t^{(2)})_{t \in \mathbb{R}_+}\), the portfolio value \(f(t, v, S_t)\) can be differentiated as follows:

\[
df(t, v_t, S_t) = \frac{\partial f}{\partial t} (t, v_t, S_t) dt + r S_t \frac{\partial f}{\partial x} (t, v_t, S_t) dt + \sqrt{v_t} S_t \frac{\partial f}{\partial x} (t, v_t, S_t) dB_t^{(1)} \\
+ \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2} (t, v_t, S_t) dt + \mu(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) dt \\
+ \beta(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) dB_t^{(2)} + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2} (t, v_t, S_t) dt \\
+ \beta(t, v_t) \sqrt{v_t} S_t \frac{\partial^2 f}{\partial v \partial x} (t, v_t, S_t) dB_t^{(1)} \cdot dB_t^{(2)} \\
= \frac{\partial f}{\partial t} (t, v_t, S_t) dt + r S_t \frac{\partial f}{\partial x} (t, v_t, S_t) dt + \sqrt{v_t} S_t \frac{\partial f}{\partial x} (t, v_t, S_t) dB_t^{(1)} \\
+ \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2} (t, v_t, S_t) dt + \mu(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) dt \\
+ \beta(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) dB_t^{(2)} + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2} (t, v_t, S_t) dt \\
+ \rho \beta(t, v_t) \sqrt{v_t} S_t \frac{\partial^2 f}{\partial v \partial x} (t, v_t, S_t) dt.
\]

Assuming that \((B_t^{(2)})_{t \in \mathbb{R}_+}\) is also a standard Brownian motion under the risk-neutral probability measure* \(\mathbb{P}^*\) and knowing that the discounted portfolio price process \((e^{-rt} f(t, v_t, S_t))_{t \in \mathbb{R}_+}\) is also a martingale under \(\mathbb{P}^*\), from the relation

\[d(e^{-rt} f(t, v_t, S_t)) = -re^{-rt} f(t, v_t, S_t) dt + e^{-rt} df(t, v_t, S_t),\]

we obtain

* When this condition is not satisfied we need to introduce a drift that yields a market price of volatility.
\[- rf(t, vt, St)dt + \frac{\partial f}{\partial t}(t, vt, St)dt + rSt \frac{\partial f}{\partial x}(t, vt, St)dt + \frac{1}{2} vt S_t^2 \frac{\partial^2 f}{\partial x^2}(t, vt, St)dt + \mu(t, vt) \frac{\partial f}{\partial v}(t, vt, St)dt + \frac{1}{2} \beta^2(t, vt) \frac{\partial^2 f}{\partial v^2}(t, vt, St)dt + \rho \beta(t, vt) S_t \sqrt{vt} \frac{\partial^2 f}{\partial v \partial x}(t, vt, St)dt = 0,\]

and the pricing PDE \((7.29)\). \qed

**Heston model**

In the Heston model with \(\mu(t, v) = -\lambda (v - m)\) and \(\beta(t, v) = \eta \sqrt{v}\), from \((7.29)\) we find the PDE

\[
\frac{\partial f}{\partial t}(t, v, x) + rx \frac{\partial f}{\partial x}(t, v, x) + \frac{1}{2} vx^2 \frac{\partial^2 f}{\partial x^2}(t, v, x) - \lambda (v - m) \frac{\partial f}{\partial v}(t, v, x) + \frac{1}{2} \eta^2 \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \eta v \frac{\partial^2 f}{\partial v \partial x}(t, v, x) = rf(t, v, x).
\]  

\((7.31)\)

The solution of this PDE has been expressed in Heston (1993) as a complex integral by inversion of a characteristic function.

Using the change of variable \(y = \log x\) with \(g(t, v, y) = f(t, v, e^y)\), the PDE \((7.31)\) is transformed into

\[
\frac{\partial g}{\partial t}(t, v, y) + r \frac{\partial g}{\partial y}(t, v, y) + \frac{1}{2} v \frac{\partial^2 g}{\partial y^2}(t, v, y) - \frac{1}{2} \frac{\partial g}{\partial y}(t, v, x) - \lambda (v - m) \frac{\partial g}{\partial v}(t, v, y) + \frac{1}{2} \eta^2 \frac{\partial^2 g}{\partial v^2}(t, v, y) + \rho \eta v \frac{\partial^2 g}{\partial v \partial y}(t, v, y) = rg(t, v, y).
\]

Using the Fourier transform

\[
\hat{g}(t, v, z) := \int_{-\infty}^{\infty} e^{-iyz} g(t, v, y) dy
\]

and the relation

\[
iz \hat{g}(t, v, z) = \int_{-\infty}^{\infty} e^{-iyz} \frac{\partial g}{\partial y}(t, v, y) dy,
\]

we find, using the rule \(i^2 = -1\), that \(\hat{g}(t, v, z)\) satisfies the equation

\[
\frac{\partial \hat{g}}{\partial t}(t, v, z) + irz \hat{g}(t, v, z) - \frac{1}{2} vz^2 \hat{g}(t, v, z) - iz \frac{1}{2} v \hat{g}(t, v, z)
\]
Estimation of Volatility

\[- \lambda (v - m) \frac{\partial \hat{g}}{\partial v} (t, v, z) + v \frac{\eta^2}{2} \frac{\partial^2 \hat{g}}{\partial v^2} (t, v, z) + i \rho \eta v \frac{\partial \hat{g}}{\partial v} (t, v, z) = r \hat{g} (t, v, z), \]

which is an affine PDE with respect to the variable $v$ with $z$ a constant parameter. This equation can be solved in closed form, and the final solution $g(t, v, y)$ can then be obtained by the Fourier inversion

\[ g(t, v, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izy} \hat{g}(t, v, z) dz, \quad (7.32) \]


**Delta hedging in the Heston model**

Consider a portfolio of the form

\[ V_t = \eta_t e^{rt} + \xi_t S_t \]

based on the risk-free asset $A_t = e^{rt}$ and on the risky asset $S_t$. When this portfolio is self-financing we have

\[
\begin{align*}
    dV_t &= df(t, v_t, S_t) \\
    &= r\eta_t e^{rt} dt + \xi_t dS_t \\
    &= r\eta_t e^{rt} dt + \xi_t (rS_t dt + S_t \sqrt{v_t} dB_t^{(1)}) \\
    &= rV_t dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} \\
    &= rf(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)}. \quad (7.33)
\end{align*}
\]

However, trying to match (7.30) to (7.33) yields

\[
\sqrt{v_t} S_t \frac{\partial f}{\partial x} (t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) dB_t^{(2)} = \xi_t S_t \sqrt{v_t} dB_t^{(1)}, \quad (7.34)
\]

which admits no solution unless $\beta(t, v) = 0$, i.e. when volatility is deterministic. A solution to that problem is to consider instead a portfolio

\[ V_t = f(t, v_t, S_t) = \eta_t e^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t) \]

that includes an additional asset with price $P(t, v_t, S_t)$, which can be an option depending on the volatility $v_t$.

**Proposition 7.5.** The self-financing portfolio allocation $(\xi_t, \zeta_t)_{t \in [0, T]}$ in the assets $(e^{rt}, S_t, P(t, v_t, S_t))_{t \in [0, T]}$ with portfolio price

\[ V_t = f(t, v_t, S_t) = \eta_t e^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t) \]

is given by

\[ \bigcirc \]
$$\zeta_t = \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\partial P}{\partial v}(t, v_t, S_t), \quad (7.35)$$

and

$$\xi_t = \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\partial P}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\partial P}{\partial v}(t, v_t, S_t). \quad (7.36)$$

Proof. Here, \((7.33)\) is replaced with

\[
dV_t = df(t, v_t, S_t) = r\eta_t e^{rt} dt + \xi_t dB_t + \zeta_t dP(t, v_t, S_t) dt
\]

\[
= r\eta_t e^{rt} dt + \xi_t (rS_t dt + \sqrt{v_t} dB_t^{(1)}) + r\zeta_t dt \frac{\partial P}{\partial v}(t, v_t, S_t) dt
\]

\[
+ \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t dt \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t dt \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt
\]

\[
+ \frac{1}{2} \zeta_t \beta^2(t, v_t) dt \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} dt \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt
\]

\[
+ \zeta_t S_t \sqrt{v_t} dt \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) dt \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}
\]

\[
= (V_t - \zeta_t P(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t dt \frac{\partial P}{\partial x}(t, v_t, S_t) dt
\]

\[
+ \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t dt \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t dt \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt
\]

\[
+ \frac{1}{2} \zeta_t \beta^2(t, v_t) dt \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} dt \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt
\]

\[
+ \zeta_t S_t \sqrt{v_t} dt \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) dt \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}
\]

\[
= rf(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t dt \frac{\partial P}{\partial x}(t, v_t, S_t) dt
\]

\[
+ \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t dt \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t dt \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt
\]

\[
+ \frac{1}{2} \zeta_t \beta^2(t, v_t) dt \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} dt \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt
\]

\[
+ \zeta_t S_t \sqrt{v_t} dt \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) dt \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}
\]

and by matching \((7.37)\) to \((7.30)\), the equation \((7.34)\) now becomes

\[
\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)}
\]
Estimation of Volatility

\[ = \xi_t S_t \sqrt{v_t} dB^{(1)}_t + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x} (t, v_t, S_t) dB^{(1)}_t + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v} (t, v_t, S_t) dB^{(2)}_t. \]

This leads to the equations

\[
\begin{align*}
\sqrt{v_t} S_t \frac{\partial f}{\partial x} (t, v_t, S_t) &= \xi_t S_t \sqrt{v_t} + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x} (t, v_t, S_t), \\
\beta(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) &= \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v} (t, v_t, S_t),
\end{align*}
\]

which show that

\[ \zeta_t = \frac{\partial f}{\partial v} (t, v_t, S_t) \frac{\partial P}{\partial v} (t, v_t, S_t), \]

and

\[ \xi_t = \frac{1}{S_t \sqrt{v_t}} \left( \sqrt{v_t} S_t \frac{\partial f}{\partial x} (t, v_t, S_t) - \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x} (t, v_t, S_t) \right) = \frac{\partial f}{\partial x} (t, v_t, S_t) - \zeta_t \frac{\partial P}{\partial x} (t, v_t, S_t) \]

\[ = \frac{\partial f}{\partial x} (t, v_t, S_t) - \frac{\partial f}{\partial v} (t, v_t, S_t) \frac{\partial P (t, v_t, S_t)}{\partial v} (t, v_t, S_t). \]

\[ \square \]

We note in addition that identifying the “\(dt\)” terms when equating \((7.37)\) to \((7.30)\) would now lead to the more complicated PDE

\[ (f(t, v_t, S_t) - \zeta_t P(t, v_t, S_t)) r + r \zeta_t S_t \frac{\partial P}{\partial x} (t, v_t, S_t) + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v} (t, v_t, S_t) \]

\[ + \zeta_t \frac{\partial^2 P}{\partial x^2} (t, v_t, S_t) + \frac{1}{2} \zeta_t S_t^2 \frac{\partial^2 P}{\partial x^2} (t, v_t, S_t) + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2} (t, v_t, S_t) \]

\[ + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v} (t, v_t, S_t) \]

\[ = \frac{\partial f}{\partial t} (t, v_t, S_t) + r S_t \frac{\partial f}{\partial x} (t, v_t, S_t) + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2} (t, v_t, S_t) + \mu(t, v_t) \frac{\partial f}{\partial v} (t, v_t, S_t) \]

\[ + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2} (t, v_t, S_t) + \rho \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x} (t, v_t, S_t), \]

which can be rewritten using \((7.35)\) as

\[ \frac{\partial f}{\partial v} (t, v, x) \left( -r P(t, v, x) + r x \frac{\partial P}{\partial x} (t, v, x) + \mu(t, v) \frac{\partial P}{\partial v} (t, v, x) + \frac{\partial P}{\partial t} (t, v, x) \right) \]

\[ \diamond \]
Consider the time-rescaled model

We refer to Chapter 4 of Fouque et al. (2011) for the contents of this section.

7.8 Perturbation Analysis

ing to a vanishing market price of volatility risk.

and (7.29) corresponds to the choice

\[ \frac{\partial P}{\partial v}(t, v, x) = \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t, v, x) \]

Therefore, letting

\[
\lambda(t, v, x) := \frac{1}{\frac{\partial P}{\partial v}(t, v, x)} \left( -r P(t, v, x) + r x \frac{\partial P}{\partial x}(t, v, x) + \frac{\partial P}{\partial v}(t, v, x) \right)
\]

\[
+ \frac{1}{\frac{\partial P}{\partial v}(t, v, x)} \left( \frac{x^2 v}{2} \frac{\partial^2 P}{\partial x^2}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 P}{\partial v^2}(t, v, x) + \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t, v, x) \right)
\]

\[
= \frac{1}{\frac{\partial f}{\partial v}(t, v, x)} \left( -r f(t, v, x) + r x \frac{\partial f}{\partial x}(t, v, x) + \frac{\partial f}{\partial v}(t, v, x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t, v, x) \right) \]

\[
+ \frac{1}{\frac{\partial f}{\partial v}(t, v, x)} \left( \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t, v, x) \right)
\]

defines a function \( \lambda(t, v, x) \) that depends only on the parameters \((t, v, x)\) and not on \(P\), without requiring \((B_t^{(2)})_{t \in \mathbb{R}^+}\) to be a standard Brownian motion under \( \mathbb{P}^* \). The function \( \lambda(t, v, x) \) is linked to the market price of volatility risk, cf. Chapter 1 of Gatheral (2006) § 2.4.1 of Fouque et al. (2011) and Fouque et al. (2000) for details.

Combining (7.38)-(7.40) allows us to rewrite the pricing PDE as

\[
\frac{\partial f}{\partial t}(t, v, x) + r x \frac{\partial f}{\partial x}(t, v, x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x)
\]

\[+ \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t, v, x) = r f(t, v, x) + \lambda(t, v, x) \frac{\partial f}{\partial v}(t, v, x), \]

and (7.29) corresponds to the choice \( \lambda(t, v, x) = -\mu(t, v) \), which corresponding to a vanishing market price of volatility risk.

7.8 Perturbation Analysis

We refer to Chapter 4 of Fouque et al. (2011) for the contents of this section. Consider the time-rescaled model
Estimation of Volatility

\[
\begin{align*}
\left\{ \begin{array}{l}
    dS_t &= rS_t dt + S_t \sqrt{v_t/\varepsilon} dB^{(1)}_t \\
    dv_t &= \mu(v_t) dt + \beta(v_t) dB^{(2)}_t
\end{array} \right. \\
\text{(7.41)}
\end{align*}
\]

We note that \( v^\varepsilon_t := v_{t/\varepsilon} \) satisfies the SDE

\[
\begin{align*}
dv^\varepsilon_t &= dv_{t/\varepsilon} \\
&\simeq v_{(t+dt)/\varepsilon} - v_{t/\varepsilon} \\
&= \frac{1}{\varepsilon} \mu(v_{t/\varepsilon}) dt + \beta(v_{t/\varepsilon}) dB^{(2)}_{t/\varepsilon},
\end{align*}
\]

with

\[
(dB^{(2)}_{t/\varepsilon})^2 \simeq \frac{dt}{\varepsilon} \simeq \frac{1}{\varepsilon} (dB^{(2)}_t)^2 \simeq \left( \frac{1}{\sqrt{\varepsilon}} dB^{(2)}_t \right)^2,
\]

hence the SDE for \( v^\varepsilon_t \) can be rewritten as

\[
dv^\varepsilon_t = \frac{1}{\varepsilon} \mu(v^\varepsilon_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(v^\varepsilon_t) dB^{(2)}_t.
\]

In other words, \( \varepsilon \to 0 \) corresponds to fast mean-reversion and (7.41) can be rewritten as

\[
\begin{align*}
\left\{ \begin{array}{l}
    dS_t &= rS_t dt + \sqrt{v^\varepsilon_t} S_t dB^{(1)}_t \\
    dv^\varepsilon_t &= \frac{1}{\varepsilon} \mu(v^\varepsilon_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(v^\varepsilon_t) dB^{(2)}_t, \\
    &\quad \varepsilon > 0.
\end{array} \right.
\end{align*}
\]

The perturbed PDE

\[
\begin{align*}
\frac{\partial f^\varepsilon}{\partial t}(t, v, x) + r x \frac{\partial f^\varepsilon}{\partial x}(t, v, x) + \frac{v^2 x^2}{2} \frac{\partial^2 f^\varepsilon}{\partial x^2}(t, v, x) + \frac{1}{\varepsilon} \mu(v) \frac{\partial f^\varepsilon}{\partial v}(t, v, x) &+ \frac{1}{2\varepsilon} \beta^2(v) \frac{\partial^2 f^\varepsilon}{\partial v^2}(t, v, x) + \frac{\rho}{\sqrt{\varepsilon}} \beta(v) x \sqrt{v} \frac{\partial^2 f^\varepsilon}{\partial v \partial x}(t, v, x) = r f^\varepsilon(t, v, x)
\end{align*}
\]

with terminal condition \( f^\varepsilon(T, v, x) = (x-K)^+ \) rewrites as

\[
\begin{align*}
\frac{1}{\varepsilon} \mathcal{L}_0 f^\varepsilon(t, v, x) + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 f^\varepsilon(t, v, x) + \mathcal{L}_2 f^\varepsilon(t, v, x) = r f^\varepsilon(t, v, x),
\end{align*}
\]

(7.42)
The solution

\[
\begin{align*}
\mathcal{L}_0 f_\varepsilon(t, v, x) &:= \frac{1}{2} \beta^2(v) \frac{\partial^2 f_\varepsilon}{\partial v^2}(t, v, x) + \mu(v) \frac{\partial f_\varepsilon}{\partial v}(t, v, x), \\
\mathcal{L}_1 f_\varepsilon(t, v, x) &:= \rho x \beta(v) \sqrt{v} \frac{\partial^2 f_\varepsilon}{\partial v \partial x}(t, v, x), \\
\mathcal{L}_2 f_\varepsilon(t, v, x) &:= \frac{\partial f_\varepsilon}{\partial t}(t, v, x) + rx \frac{\partial f_\varepsilon}{\partial x}(t, v, x) + \frac{vx^2}{2} \frac{\partial^2 f_\varepsilon}{\partial x^2}(t, v, x).
\end{align*}
\]

Note that

- $\mathcal{L}_0$ is the infinitesimal generator of the process $(v^1_s)_{s \in \mathbb{R}^+}$, see (7.46) below, and
- $\mathcal{L}_2$ is the Black-Scholes operator, i.e. $\mathcal{L}_2 f = r f$ is the Black-Scholes PDE.

The solution $f_\varepsilon(t, v, x)$ will be expanded as

\[
f_\varepsilon(t, v, x) = f^{(0)}(t, v, x) + \sqrt{\varepsilon} f^{(1)}(t, v, x) + \varepsilon f^{(2)}(t, v, x) + \cdots \quad (7.43)
\]

with $f(T, v, x) = (x - K)^+$, $f^{(1)}(T, v, x) = 0$, and $f^{(2)}(T, v, x) = 0$.

Since $\mathcal{L}_0$ contains only differentials with respect to $v$, we will choose $f^{(0)}(t, v, x)$ of the form

\[
f^{(0)}(t, v, x) = f^{(0)}(t, x),
\]

cf. § 4.2 of § 4.1.1 of Fouque et al. (2011) for details, with

\[
\mathcal{L}_0 f^{(0)}(t, x) = \mathcal{L}_1 f^{(0)}(t, x) = 0. \quad (7.44)
\]

By identifying the terms of order $1/\sqrt{\varepsilon}$ when plugging (7.43) in (7.42) we also find

\[
\mathcal{L}_0 f^{(1)}(t, v, x) + \mathcal{L}_1 f^{(0)}(t, x) = 0,
\]

hence $\mathcal{L}_0 f^{(1)}(t, v, x) = 0$. Similarly, by identifying the terms that do not depend on $\varepsilon$ in (7.42) and taking $f^{(1)}(t, v, x) = f^{(1)}(t, x)$, we have $\mathcal{L}_1 f^{(1)} = 0$ and

\[
\mathcal{L}_0 f^{(2)}(t, v, x) + \mathcal{L}_2 f^{(0)}(t, x) = 0. \quad (7.45)
\]

Using the Itô formula, we have

\[
\mathbb{E} \left[ f^{(2)}(t, v^1_s, x) \right] = f^{(2)}(t, v^1_0, x) + \mathbb{E} \left[ \int_0^s \frac{\partial f^{(2)}}{\partial x}(t, v^1_{\tau}, x) dB^{(2)}_{\tau} \right] \\
+ \mathbb{E} \left[ \int_0^s \left( \mu(v^1_{\tau}) \frac{\partial f^{(2)}}{\partial v}(t, v^1_{\tau}, x) + \frac{1}{2} \beta^2(v^1_{\tau}) \frac{\partial^2 f^{(2)}}{\partial v^2}(t, v^1_{\tau}, x) \right) d\tau \right] \\
= \mathbb{E} \left[ f^{(2)}(t, v^1_0, x) \right] + \int_0^s \mathbb{E} \left[ \mathcal{L}_0 f^{(2)}(t, v^1_{\tau}, x) \right] d\tau. \quad (7.46)
\]
Estimation of Volatility

When the process \((v^1_t)_{t \in \mathbb{R}_+}\) is started under its stationary (or invariant) probability distribution with density function \(\phi(v)\) we have

\[
\mathbb{E} \left[ f^{(2)} (t, v^1_\tau, x) \right] = \int_0^\infty f^{(2)} (t, v, x) \phi(v) dv, \quad \tau \in \mathbb{R}_+,
\]

hence (7.46) rewrites as

\[
\int_0^\infty f^{(2)} (t, v, x) \phi(v) dv = \int_0^\infty f^{(2)} (t, v, x) \phi(v) dv + \int_0^\infty \int_0^\infty \mathcal{L}_0 f^{(2)} (t, v, x) \phi(v) dv d\tau.
\]

By differentiation with respect to \(s > 0\) this yields

\[
\int_0^\infty \mathcal{L}_0 f^{(2)} (t, v, x) \phi(v) dv = 0,
\]

hence by (7.45) we find

\[
\int_0^\infty \mathcal{L}_2 f^{(0)} (t, x) \phi(v) dv = 0,
\]

cf. § 3.2 of Fouque et al. (2011), i.e.

\[
\frac{\partial f^{(0)}}{\partial t} (t, x) + rx \frac{\partial f^{(0)}}{\partial x} (t, x) + \frac{\eta^2}{2} \int_0^\infty v \phi(v) dv \frac{\partial^2 f^{(0)}}{\partial x^2} (t, x) = r f^{(0)} (t, x),
\]

with the terminal condition \(f^{(0)} (T, x) = (x - K)^+\).

Consequently, the first expansion term \(f^{(0)} (t, x)\) in (7.42) is the Black-Scholes function

\[
f^{(0)} (t, x) = \text{Bl} \left( S_t, K, r, T - t, \sqrt{\int_0^\infty v \phi(v) dv} \right),
\]

with the averaged squared volatility

\[
\int_0^\infty v \phi(v) dv = \mathbb{E} \left[ v^1_\tau \right], \quad \tau \in \mathbb{R}_+,
\]

under the stationary distribution of the process with infinitesimal generator \(\mathcal{L}_0\), i.e. the stationary distribution of the solution to

\[
dv^1_t = \mu(v^1_t) dt + \beta(v^1_t) dB^{(2)}_t.
\]

**Perturbation analysis in the Heston model**

We have
\[
\begin{aligned}
dS_t &= rS_t dt + S_t \sqrt{v^\varepsilon_t} dB^{(1)}_t \\
dv^\varepsilon_t &= \frac{-\lambda}{\varepsilon} (v^\varepsilon_t - m) dt + \eta \sqrt{v^\varepsilon_t / \varepsilon} dB^{(2)}_t,
\end{aligned}
\]

under the modified short mean-reversion time scale, and the SDE can be rewritten as

\[
dv^\varepsilon_t &= \frac{-\lambda}{\varepsilon} (v^\varepsilon_t - m) dt + \eta \sqrt{v^\varepsilon_t / \varepsilon} dB^{(2)}_t.
\]

In other words, \( \varepsilon \to 0 \) corresponds to fast mean reversion, in which \( v^\varepsilon_t \) becomes close to its mean (7.47).

Recall, cf. (13.5), that the CIR process \( (v^1_t)_{t \in \mathbb{R}^+} \) has a gamma invariant (or stationary) distribution with shape parameter \( 2\lambda m / \eta^2 \), scale parameter \( \eta^2 / (2\lambda) \), and probability density function \( \phi \) given by

\[
\phi(v) = \frac{1}{\Gamma(2\lambda m / \eta^2) (\eta^2 / (2\lambda))^{2\lambda m / \eta^2}} v^{-1+2\lambda m / \eta^2} e^{-2\lambda v / \eta^2} \mathbb{1}_{[0, \infty)}(v), \quad v \in \mathbb{R},
\]

and mean

\[
\int_0^\infty v \phi(v) dv = m.
\]

Hence the first expansion term \( f^{(0)}(t, x) \) in (7.42) reads

\[
f^{(0)}(t, x) = B \left( S_t, K, r, T - t, \sqrt{m} \right),
\]

with the averaged squared volatility

\[
\int_0^\infty v \phi(v) dv = m = \mathbb{E} [v^1_\tau], \quad \tau \in \mathbb{R}^+,
\]

under the stationary distribution of the process with infinitesimal generator \( \mathcal{L}_0 \), i.e. the stationary distribution of the solution to

\[
dv^1_t = \mu(v^1_t) dt + \beta(v^1_t) dB^{(2)}_t.
\]

In Figure 7.21, cf. Privault and She (2016), related approximations of put option prices are plotted against the value of \( v \) with correlation \( \rho = -0.5 \) and \( \varepsilon = 0.01 \) in the \( \alpha \)-hypergeometric stochastic volatility model of Fonseca and Martini (2016), based on the series expansion of Han et al. (2013), and compared to a Monte Carlo curve requiring 300,000 samples and 30,000 time steps.
Exercises

Exercise 7.1  ( Gatheral (2006), Chapter 11) Compute the expected total realized variance in the Heston model, with

\[ dv_t = -\lambda(v_t - m)dt + \eta \sqrt{v_t} dB_t. \]

Exercise 7.2  Compute the variance swap rate

\[ VST := \frac{1}{T} \mathbb{E} \left[ \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{S_{kT/n} - S_{(k-1)T/n}}{S_{(k-1)T/n}} \right)^2 \right] = \frac{1}{T} \mathbb{E} \left[ \int_0^T \frac{1}{S_t^2} (dS_t)^2 \right] \]

on the index whose level \( S_t \) is given in the following two models.

a) Heston model. Here, \((S_t)_{t \in \mathbb{R}_+}\) is given by the system of stochastic differential equations

\[
\begin{aligned}
    dS_t &= (r - \alpha v_t)S_t dt + S_t \sqrt{\beta + v_t} dB_t^{(1)} \\
    dv_t &= -\lambda(v_t - m) dt + \gamma \sqrt{v_t} dB_t^{(2)},
\end{aligned}
\]

where \((B_t^{(1)})_{t \in \mathbb{R}_+}\) and \((B_t^{(2)})_{t \in \mathbb{R}_+}\) are standard Brownian motions with correlation \( \rho \in [-1, 1] \) and \( \alpha \geq 0, \beta \geq 0, \lambda > 0, m > 0, r > 0, \gamma > 0. \)

b) SABR model with \( \beta = 1. \) The index level \( S_t \) is given by the system of stochastic differential equations

Fig. 7.21: Option price approximations plotted against \( v \) with \( \rho = -0.5. \)
\[ \begin{cases} dS_t = \sigma_t S_t dB_t^{(1)} \\ d\sigma_t = \alpha \sigma_t dB_t^{(2)}, \end{cases} \]

where \( \alpha > 0 \) and \( (B_t^{(1)})_{t \in \mathbb{R}_+} \) and \( (B_t^{(2)})_{t \in \mathbb{R}_+} \) are standard Brownian motions with correlation \( \rho \in [-1, 1] \).

Exercise 7.3 Let \( \sigma_{imp}(K) \) denote the implied volatility of a call option with strike price \( K \), defined from the relation

\[ MC(K, S, r, \tau) = C(K, S, \sigma_{imp}(K), r, \tau), \]

where \( MC \) is the market price of the call option, \( C(K, S, \sigma_{imp}(K), r, \tau) \) is the Black-Scholes call pricing function, \( S \) is the underlying asset price, \( \tau \) is the time remaining until maturity, and \( r \) is the risk-free interest rate.

a) Compute the partial derivative

\[ \frac{\partial MC}{\partial K} (K, S, r, \tau). \]

using the functions \( C \) and \( \sigma_{imp} \).

b) Knowing that market call option prices \( MC(K, S, r, \tau) \) are decreasing in the strike prices \( K \), find an upper bound for the slope \( \sigma'_{imp}(K) \) of the implied volatility curve.

c) Similarly, knowing that the market put option prices \( MP(K, S, r, \tau) \) are increasing in the strike prices \( K \), find a lower bound for the slope \( \sigma'_{imp}(K) \) of the implied volatility curve.

Exercise 7.4 Hagan et al. (2002) Consider a European option priced as \( e^{-rT} \mathbb{E}^*[(S_T - K)^+] \) in a local volatility model \( dS_t = \sigma_{loc}(S_t) S_t dB_t \). The implied volatility \( \sigma_{imp}(K, S_0) \), computed from the equation

\[ \text{Bl}(S_0, K, T, \sigma_{imp}(K, S_0), r) = e^{-rT} \mathbb{E}^*[(S_T - K)^+], \]

is known to admit the approximation

\[ \sigma_{imp}(K, S_0) \simeq \sigma_{loc} \left( \frac{K + S_0}{2} \right). \]

a) Taking a local volatility of the form \( \sigma_{loc}(x) := \sigma_0 + \beta (x - S_0)^2 \), estimate the implied volatility \( \sigma_{imp}(K, S) \) when the underlying is at the level \( S \).

b) Express the Delta of the Black Scholes call option price given by

\[ \text{Bl}(S, K, T, \sigma_{imp}(K, S), r), \]
Estimation of Volatility

using the standard Black-Scholes Delta and the Black-Scholes Vega.

Exercise 7.5 Carr and Lee (2008) Consider an underlying share price $S_t \in \mathbb{R}_+$ given by $dS_t = rS_t dt + \sigma_t S_t dB_t$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $(\sigma_t)_{t \in \mathbb{R}_+}$ is an (adapted) stochastic volatility process. The risk-free asset is priced $A_t := e^{rt}$, $t \in [0, T]$. We consider a realized variance swap with strike $\kappa_\sigma = 0$ and payoff $\int_0^T \sigma_t^2 dt$.

a) Show that the payoff $\int_0^T \sigma_t^2 dt$ of the realized variance swap satisfies

$$\int_0^T \sigma_t^2 dt = 2 \int_0^T dS_t \frac{S_t}{S_0} - 2 \log \frac{S_T}{S_0}. \quad (7.48)$$

b) Show that the price $V_t := e^{-r(T-t)} \mathbb{E}^* \left[ \int_0^T \sigma_t^2 dt \bigg| \mathcal{F}_t \right]$ of the variance swap at time $t \in [0, T]$ satisfies

$$V_t = L_t + 2(T-t) r e^{-(T-t)r} + 2 e^{-(T-t)r} \int_0^t \frac{dS_u}{S_u}, \quad (7.49)$$

where

$$L_t := -2 e^{-(T-t)r} \mathbb{E}^* \left[ \log \frac{S_T}{S_0} \bigg| \mathcal{F}_t \right]$$

is the price at time $t$ of the log-contract Neuberger (1994), Demeterfi et al. (1999), with payoff $-2 \log(S_T/S_0)$, see also Exercises 5.7 and 6.8.

c) Show that the portfolio made at time $t \in [0, T]$ of:

- one log-contract priced $L_t$,
- $2 e^{-(T-t)r} / S_t$ in shares priced $S_t$,
- $2 e^{-rT} \left( \int_0^t \frac{dS_u}{S_u} + (T-t)r - 1 \right)$ in the risk-free asset $A_t = e^{rt}$,

hedges the realized variance swap.

d) Show that the above portfolio is self-financing.