In this chapter we price floating strike lookback options, whose payoff with exercise date $T$ is given by the functional

$$C = S_T - \min_{0 \leq t \leq T} S_t$$

of the underlying asset price $(S_t)_{t \in [0,T]}$ in the case of call options, and by

$$C = \left( \max_{0 \leq t \leq T} S_t \right) - S_T,$$

in the case of put options.

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9.1 Average Brownian Extrema

Let

$$m^t_s = \min_{u \in [s,t]} S_u \quad \text{and} \quad M^t_s = \max_{u \in [s,t]} S_u,$$

$0 \leq s \leq t \leq T$, and let $M^T_s$ be either $m^t_s$ or $M^t_s$. In the lookback option case the payoff $\phi(S_T, M^T_0)$ depends not only on the price of the underlying asset at maturity but it also depends on all price values of the underlying asset over the period which starts from the initial time and ends at maturity.
The payoff of such an option is of the form $\phi(S_T, M^T_0)$ with $\phi(x, y) = x - y$ in the case of lookback call options, and $\phi(x, y) = y - x$ in the case of lookback put options. We let

$$e^{-(T-t)r} \mathbb{E}^*[\phi(S_T, M^T_0)|\mathcal{F}_t]$$

denote the price at time $t \in [0, T]$ of such an option.

**Maximum selling price over $[0, T]$**

In the next proposition we start by computing the average of the maximum selling price $M^T_0$ over the time interval $[0, T]$. As in (8.18), we denote

$$\delta^+ \pm (s) := \frac{1}{\sigma \sqrt{T}} \left( \log s + \left( r \pm \frac{1}{2} \sigma^2 \right) \tau \right), \quad s > 0.$$  

**Proposition 9.1.** The average maximum value of $(S_t)_{t \in [0, T]}$ over $[0, T]$ is given by

$$\mathbb{E}^*[M^T_0 | \mathcal{F}_t] = M^T_0 \Phi \left( -\delta^+ (S_t M^T_0) \right) + S_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta^+ (S_t M^T_0) \right) - S_t \frac{\sigma^2}{2r} \left( \frac{M^T_0}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta^+ \left( \frac{M^T_0}{S_t} \right) \right).$$  

When $t = 0$ we have $S_0 = M^0_0$, and given that

$$\delta^+ (1) = \frac{r \pm \sigma^2/2}{\sigma} \sqrt{T},$$  

the formula (9.1) simplifies to

$$\mathbb{E}^*[M^T_0] = S_0 \left( 1 - \frac{\sigma^2}{2r} \right) \Phi \left( \frac{\sigma^2/2 - r}{\sigma} \sqrt{T} \right) + S_0 e^{rT} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \frac{\sigma^2/2 + r}{\sigma} \sqrt{T} \right),$$

with

$$\mathbb{E}^*[M^T_0] = 2S_0 \left( 1 + \frac{\sigma^2T}{4} \Phi \left( \frac{\sigma \sqrt{T}}{2} \right) \right) + \sigma S_0 \sqrt{\frac{T}{2\pi}} e^{-\sigma^2T/8}$$

when $r = 0$, cf. Exercise 9.2.
In general, when $T$ tends to infinity we find that

$$\lim_{T \to \infty} \frac{\mathbb{E}^* [M_T^T | \mathcal{F}_t]}{\mathbb{E}^* [S_T | \mathcal{F}_t]} = \begin{cases} 1 + \frac{\sigma^2}{2r} & \text{if } r > 0, \\ \infty & \text{if } r = 0, \end{cases}$$

see Exercise 8.1-(d) in the case $r = \sigma^2/2$.

**Proof of Proposition 9.1.** We have

$$\mathbb{E}^* [M_T^T | \mathcal{F}_t] = \mathbb{E}^* \left[ \max \left( M_0^t, M_t^T \right) | \mathcal{F}_t \right] = \mathbb{E}^* \left[ M_0^t \mathbb{1}_{\{M_0^t > M_t^T\}} | \mathcal{F}_t \right] + \mathbb{E}^* \left[ M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} | \mathcal{F}_t \right] = M_0^t \mathbb{P}(M_0^t > M_t^T | \mathcal{F}_t) + \mathbb{E}^* \left[ M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} | \mathcal{F}_t \right].$$

Next, we have

$$\mathbb{P}(M_0^t > M_t^T | \mathcal{F}_t) = \mathbb{P} \left( \frac{M_0^t}{S_t} > \frac{M_t^T}{S_t} \bigg| \mathcal{F}_t \right) = \mathbb{P} \left( x > \frac{M_t^T}{S_t} \bigg| \mathcal{F}_t \right)_{x=M_0^t/S_t} = \mathbb{P} \left( \frac{M_0^{T-t}}{S_0} < x \right)_{x=M_0^t/S_t}.$$

On the other hand, letting $\mu := r/\sigma - \sigma/2$, from (8.8) or (8.12) in Corollary 8.2 we have

$$\mathbb{P} \left( \frac{M_0^T}{S_0} < x \right) = \mathbb{P} \left( \tilde{X}_T < \frac{1}{\sigma} \log x \right) = \Phi \left( \frac{-\mu T + \sigma^{-1} \log x}{\sqrt{T}} \right) - e^{2\mu \sigma^{-1} \log x} \Phi \left( \frac{-\mu T - \sigma^{-1} \log x}{\sqrt{T}} \right) = \Phi \left( \delta_T^{-1}(1/x) \right) - x^{-1+2r/\sigma^2} \Phi \left( \delta_T^{-1}(x) \right).$$

Hence

$$\mathbb{P}(M_0^t > M_t^T | \mathcal{F}_t) = \mathbb{P} \left( \frac{M_0^{T-t}}{S_0} < x \right)_{x=M_0^t/S_t}.$$
By standard square completion arguments, we find

\[ IE = \Phi \left( -\delta_{\geq T-t} \left( \frac{S_t}{M_0^\dagger} \right) \right) - \left( \frac{M_0^\dagger}{S_t} \right)^{-1+2r/\sigma^2} \Phi \left( -\delta_{\geq T-t} \left( \frac{M_0^\dagger}{S_t} \right) \right). \]

Next, we have

\[
\mathbb{E}^* \left[ M_t^T \mathbbm{1}_{\{M_t^T > M_0^\dagger \}} \mid \mathcal{F}_t \right] = S_t \mathbb{E}^* \left[ \frac{M_t^T}{S_t} \mathbbm{1}_{\{M_t^T > M_0^\dagger / S_t \}} \mid \mathcal{F}_t \right]
\]

\[
= S_t \mathbb{E}^* \left[ \max \frac{S_r}{S_t} \mathbbm{1}_{\{\max_{r \in [t,T]} S_r / S_t > x \}} \mid \mathcal{F}_t \right] \bigg|_{x = M_t^T / S_t}
\]

and by Proposition 8.1 we have

\[
\mathbb{E}^* \left[ \max \frac{S_r}{S_0} \mathbbm{1}_{\{\max_{r \in [0,T]} S_r / S_0 > x \}} \right] = \mathbb{E}^* \left[ e^{\sigma \bar{X}_T} \mathbbm{1}_{\{\bar{X}_T > \sigma^{-1} \log x \}} \right]
\]

\[
= \int_{\sigma^{-1} \log x}^{\infty} e^{\sigma z} f_{\bar{X}_T}(z) dz
\]

\[
= \int_{\sigma^{-1} \log x}^{\infty} e^{\sigma z} \left( \sqrt{\frac{2}{\pi T}} e^{-(z-\mu T)^2/(2T)} - 2\mu e^{2\mu z} \Phi \left( \frac{-z - \mu T}{\sqrt{T}} \right) \right) dz
\]

By standard square completion arguments, we find

\[
\frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log x}^{\infty} e^{\sigma z - (z-\mu T)^2/(2T)} dz
\]

\[
= \frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1} \log x}^{\infty} e^{-(z^2 + 2\mu^2 T^2 - 2(\mu + \sigma)Tz)/(2T)} dz
\]

\[
= \frac{1}{\sqrt{2\pi T}} e^{\sigma^2 T / 2 + \mu \sigma T} \int_{\sigma^{-1} \log x}^{\infty} e^{-(z - (\mu + \sigma) T)^2/(2T)} dz
\]

\[
= \frac{1}{\sqrt{2\pi T}} e^{r T} \int_{-(\mu + \sigma) T + \sigma^{-1} \log x}^{\infty} e^{-z^2/(2T)} dz
\]

\[
= e^{r T} \Phi \left( \frac{\delta_{\geq T} \left( \frac{1}{x} \right) }{x} \right),
\]
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since $\mu \sigma + \sigma^2 / 2 = r$. The second integral

$$\int_{\sigma^{-1} \log x}^{\infty} e^{z(\sigma + 2\mu)} \Phi \left( \frac{-z - \mu T}{\sqrt{T}} \right) dz$$

can be computed by integration by parts using the identity

$$\int_a^\infty v'(z) u(z) dz = u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z) u'(z) dz,$$

with $a = \sigma^{-1} \log x$. We let

$$u(z) = \Phi \left( \frac{-z - \mu T}{\sqrt{T}} \right) \quad \text{and} \quad v'(z) = e^{z(\sigma + 2\mu)}$$

which satisfy

$$u'(z) = -\frac{1}{\sqrt{2\pi T}} e^{-(z+\mu T)^2/(2T)} \quad \text{and} \quad v(z) = \frac{1}{\sigma + 2\mu} e^{z(\sigma + 2\mu)},$$

and using (8.19) we find

$$\int_a^\infty e^{z(\sigma + 2\mu)} \Phi \left( \frac{-z - \mu T}{\sqrt{T}} \right) dz = \int_a^\infty v'(z) u(z) dz$$

$$= u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z) u'(z) dz$$

$$= -\frac{1}{\sigma + 2\mu} e^{a(\sigma + 2\mu)} \Phi \left( \frac{-a - \mu T}{\sqrt{T}} \right)$$

$$+ \frac{1}{(\sigma + 2\mu)\sqrt{2\pi T}} \int_a^\infty e^{z(\sigma + 2\mu)} e^{-(a + \mu T)^2/(2T)} dz$$

$$= -\frac{1}{\sigma + 2\mu} e^{a(\sigma + 2\mu)} \Phi \left( \frac{-a - \mu T}{\sqrt{T}} \right)$$

$$+ \frac{1}{(\sigma + 2\mu)\sqrt{2\pi}} e^{(T(\sigma + \mu)^2 - \mu^2 T)/2} \int_a^\infty e^{-(z+T(\sigma + \mu))^2/(2T)} dz$$

$$= -\frac{1}{\sigma + 2\mu} e^{a(\sigma + 2\mu)} \Phi \left( \frac{-a - \mu T}{\sqrt{T}} \right)$$

$$+ \frac{1}{(\sigma + 2\mu)\sqrt{2\pi}} e^{(T(\sigma + \mu)^2 - \mu^2 T)/2} \int_{(a-T(\sigma + \mu))/\sqrt{T}}^\infty e^{-z^2/2} dz$$

$$= -\frac{1}{\sigma + 2\mu} e^{a(\sigma + 2\mu)} \Phi \left( \frac{-a - \mu T}{\sqrt{T}} \right)$$

$$+ \frac{1}{\sigma + 2\mu} e^{(T(\sigma + \mu)^2 - \mu^2 T)/2} \Phi \left( \frac{-a + T(\sigma + \mu)}{\sqrt{T}} \right)$$

$$= -\frac{2r}{\sigma} (x)^{2r/\sigma^2} \Phi \left( \frac{-r/\sigma - r/2}{T} - \sigma^{-1} \log x \right)$$
Hence we have
\[ \frac{2r}{\sigma} e^{\sigma T (\sigma + 2\mu)/2} \Phi \left( \frac{T (r - \sigma/2) - \sigma^{-1} \log x}{\sqrt{T}} \right) \]
\[ = \frac{\sigma}{2r} e^{rT} \Phi \left( \frac{\delta^T}{x} \right) - \frac{\sigma}{2r} x^{2r/\sigma^2} \Phi \left( -\delta^T_x \right), \]
cf. pages 317-319 of Shreve (2004) for a different derivation using double integrals. Hence we have
\[
\mathbb{E}^* \left[ M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \mid \mathcal{F}_t \right] = S_t \mathbb{E}^* \left[ \max_{r \in [0,T-t]} \frac{S_r}{S_0} \mathbb{1}_{\{\max_{r \in [0,T-t]} S_r/S_0 > x\}} \right]_{x=M_0^t/S_t}
\]
\[ = 2S_t e^{(T-t)r} \Phi \left( \delta^{T-t}_+ \left( \frac{S_t}{M_0^t} \right) \right) - S_t \frac{\mu \sigma}{r} e^{(T-t)r} \Phi \left( \delta^{T-t}_+ \left( \frac{S_t}{M_0^t} \right) \right) \]
\[ + S_t \frac{\mu \sigma}{r} \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta^{T-t}_- \left( \frac{M_0^t}{S_t} \right) \right), \]
and consequently this yields, since \( \mu \sigma / r = 1 - \sigma^2 / (2r) \),
\[
\mathbb{E}^* \left[ M_0^T \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ M_0^T \mid M_0^t \right]
\]
\[ = M_0^t \mathbb{P} (M_0^t > M_t^T \mid M_0^t) + \mathbb{E}^* \left[ M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \mid M_0^t \right] \]
\[ = M_0^t \Phi \left( -\delta^{T-t}_- \left( \frac{S_t}{M_0^t} \right) \right) - S_t \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta^{T-t}_- \left( \frac{M_0^t}{S_t} \right) \right) \]
\[ + 2S_t e^{(T-t)r} \Phi \left( \delta^{T-t}_+ \left( \frac{S_t}{M_0^t} \right) \right) - S_t \left( 1 - \frac{\sigma^2}{2r} \right) e^{(T-t)r} \Phi \left( \delta^{T-t}_+ \left( \frac{S_t}{M_0^t} \right) \right) \]
\[ + S_t \left( 1 - \frac{\sigma^2}{2r} \right) \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta^{T-t}_- \left( \frac{M_0^t}{S_t} \right) \right) \]
\[ = M_0^t \Phi \left( -\delta^{T-t}_- \left( \frac{S_t}{M_0^t} \right) \right) + S_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta^{T-t}_+ \left( \frac{S_t}{M_0^t} \right) \right) \]
\[ - S_t \frac{\sigma^2}{2r} \left( \frac{M_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta^{T-t}_- \left( \frac{M_0^t}{S_t} \right) \right). \]
This concludes the proof of Proposition 9.1. \( \square \)

9.2 The Lookback Put Option

The standard lookback put option gives its holder the right to sell the underlying asset at its historically highest price. In this case the strike price is \( M_0^T \) and the payoff is given by the terminal value
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\[ C = M_0^T - S_T \]

of the drawdown process \((M_0^t - S_t)_{t \in [0,T]}\).

The following pricing formula for lookback put options is a direct consequence of Proposition 9.1.

**Proposition 9.2.** The price at time \( t \in [0, T] \) of the lookback put option with payoff \( M_0^T - S_T \) is given by

\[
e^{-r(T-t)} E^* \left[ M_0^T - S_T \mid F_t \right] = M_0^t e^{-r(T-t)} \Phi \left( -\delta_-^{T-t} \left( \frac{S_t}{M_0^t} \right) \right) + S_t \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{M_0^t} \right) \right) - S_t.
\]

Figure 9.1 represents the lookback put price as a function of \( S_t \) and \( M_0^t \), for different values of the time to maturity \( T - t \).

In Figures 9.1 and 9.2 we note that an increase in the underlying asset price \( S_t \) results into a higher put option price when \( S_t \) is close to \( M_0^t \) because in this case the variation of \( S_t \) can increase the value of \( M_0^t \). When \( S_t \) is far from \( M_0^t \) prices behave similarly although an increase in the underlying asset price \( S_t \) is less likely to affect the value of \( M_0^t \), and the put option price becomes lower.

* The animation works in Acrobat Reader on the entire pdf file.
Proof of Proposition 9.2. We have
\[ E^* \left[ M_0^T - S_T \mid F_t \right] = E^* \left[ M_0^T \mid F_t \right] - E^* \left[ S_T \mid F_t \right] \]
\[ = E^* \left[ M_0^T \mid F_t \right] - e^{(T-t)r} S_t, \]
hence Proposition 9.1 shows that
\[ e^{-(T-t)r} E^* \left[ M_0^T - S_T \mid F_t \right] = e^{-(T-t)r} \Phi \left( -\delta_T^+ \left( \frac{S_t}{M_0^T} \right) \right) - S_t \Phi \left( -\delta_T^+ \left( \frac{S_t}{M_0^T} \right) \right) \]
\[ + S_t \frac{\sigma^2}{2r} \Phi \left( \delta_T^+ \left( \frac{S_t}{M_0^T} \right) \right) - S_t \frac{\sigma^2}{2r} e^{-(T-t)r} \left( \frac{M_0^T}{S_t} \right)^{2r/\sigma^2} \Phi \left( -\delta_T^- \left( \frac{S_t}{M_0^T} \right) \right). \]

9.3 PDE Method

Since the couple \((S_t, M_0^T)\) is a Markov process, the price can be written as a function
\[ f(t, S_t, M_0^T) = e^{-(T-t)r} E^* \left[ \phi(S_T, M_0^T) \mid F_t \right] \]
\[ = e^{-(T-t)r} E^* \left[ \phi(S_T, M_0^T) \mid S_t, M_0^T \right], \quad 0 \leq t \leq T, \]
and in this case the function \( f(t, x, y) \) can solve a PDE.
Black-Scholes PDE for lookback put option prices

In the next proposition we derive the partial differential equation (PDE) for the pricing function $f(t, x, y)$ of a self-financing portfolio hedging a lookback option.

**Proposition 9.3.** Let $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ be a portfolio strategy such that

(i) $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t := \eta_t A_t + \xi_t S_t$, $t \in \mathbb{R}_+$, takes the form

$$V_t = f(t, S_t, M^0_t), \quad t \in \mathbb{R}_+,$$

for some function $f \in C^2((0, \infty) \times (0, \infty)^2)$.

Then the function $f(t, x, y)$ satisfies the Black-Scholes PDE

$$rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + rx \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x, y), \quad t, x, y > 0, \quad (9.4)$$

under the boundary conditions

\[
\begin{align*}
&f(t, 0, y) = e^{-(T-t)r} y, \quad 0 \leq t \leq T, \quad y \in \mathbb{R}_+, \quad (9.5a) \\
&\frac{\partial f}{\partial y}(t, x, y)|_{x=y} = 0, \quad 0 \leq t \leq T, \quad y > 0, \quad (9.5b) \\
&f(T, x, y) = y - x, \quad 0 \leq x \leq y. \quad (9.5c)
\end{align*}
\]

The replicating portfolio of the lookback put option is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, M^0_t), \quad t \in [0, T]. \quad (9.6)$$

**Proof.** The existence of $f(t, x, y)$ follows from the Markov property, more precisely, the function $f(t, x, y)$ satisfies

$$f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi(S_T, M^T_0) \mid S_t = x, \ M^0_t = y \right]$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( \frac{S_T}{S_t}, \max \left( y, M^T_t \right) \right) \right]$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( \frac{S_{T-t}}{S_0}, \max \left( y, M^{T-t}_0 \right) \right) \right], \quad t \in [0, T],$$

from the time homogeneity of the asset price process $(S_t)_{t \in \mathbb{R}_+}$. Applying the change of variable formula to the discounted portfolio value
\[ \bar{f}(t, x, y) := e^{-rt} f(t, x, y) = e^{-rT} \mathbb{E}^* \left[ \phi(S_T, M_T^t) \mid S_t = x, \ M_0^t = y \right] \]

which is a martingale for \( t \in [0, T] \), we have

\[
\begin{align*}
&d\bar{f}(t, S_t, M_0^t) = -r e^{-rt} f(t, S_t, M_0^t) dt + e^{-rt} df(t, S_t, M_0^t) \\
&= -r e^{-rt} f(t, S_t, M_0^t) dt + e^{-rt} \frac{\partial f}{\partial t}(t, S_t, M_0^t) dt + r e^{-rt} S_t \frac{\partial f}{\partial x}(t, S_t, M_0^t) dt \\
&+ \frac{1}{2} e^{-rt} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_0^t) dt \\
&+ e^{-rt} \frac{\partial f}{\partial y}(t, S_t, M_0^t) dM_0^t \\
&+ e^{-rt} \sigma S_t \frac{\partial f}{\partial x}(t, S_t, M_0^t) dB_t.
\end{align*}
\]

(9.7)

Since \((\bar{f}(t, S_t, M_0^t))_{t \in [0,T]} = (e^{-rT} \mathbb{E}^*[\phi(S_T, M_T^t) \mid \mathcal{F}_t])_{t \in [0,T]}\) is a martingale under \( \mathbb{P} \) and \((M_0^t)_{t \in [0,T]}\) has finite variation (it is in fact a nondecreasing process), (9.7) yields:

\[
d\bar{f}(t, S_t, M_0^t) = \sigma S_t \frac{\partial f}{\partial x}(t, S_t, M_0^t) dB_t, \quad t \in [0, T],
\]

(9.8)

and the function \( f(t, x, y) \) satisfies the equation

\[ \frac{\partial f}{\partial t}(t, S_t, M_0^t) dt + r S_t \frac{\partial f}{\partial x}(t, S_t, M_0^t) dt \\
+ \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_0^t) dt + \frac{\partial f}{\partial y}(t, S_t, M_0^t) dM_0^t = r f(t, S_t, M_0^t) dt, \]

which implies

\[ \frac{\partial f}{\partial t}(t, S_t, M_0^t) + r S_t \frac{\partial f}{\partial x}(t, S_t, M_0^t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_0^t) = r f(t, S_t, M_0^t), \]

which is (9.4), and

\[ \frac{\partial f}{\partial y}(t, S_t, M_0^t) dM_0^t = 0, \]

because \( M_0^t \) increases only on a set of zero measure (which has no isolated points), see the Lebesgue decomposition theorem and also the Cantor function. This implies

\[ \frac{\partial f}{\partial y}(t, S_t, M_0^t) = 0, \]

when \( dM_0^t > 0 \), hence since

\[ \{ S_t = M_0^t \} \iff dM_0^t > 0 \]

and

\[ \{ S_t < M_0^t \} \iff dM_0^t = 0, \]

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we have
\[ \frac{\partial f}{\partial y}(t, S_t, S_t) = \frac{\partial f}{\partial y}(t, x, y)_{x=S_t, y=S_t} = 0, \]
since \( M_0^t \) hits \( S_t \), \text{i.e.} \( M_0^t = S_t \), only when \( M_0^t \) increases at time \( t \), and this shows the boundary condition (9.5b). On the other hand, (9.8) shows that
\[ \phi(S_T, M_0^T) = \mathbb{E}^* \left[ \phi(S_T, M_0^T) \right] + \sigma \int_0^T S_t \frac{\partial f}{\partial x}(t, x, M_0^t)_{x=S_t} dB_t, \]
\( 0 \leq t \leq T \), which implies (9.6) as in the proof of Proposition 5.12 or 8.4.

In other words, the price of the lookback put option takes the form
\[ f(t, S_t, M_0^t) = e^{-(T-t)r} \mathbb{E}^* \left[ M_0^T - S_T \left| \mathcal{F}_t \right. \right], \]
where the function \( f(t, x, y) \) is given by
\[
\begin{align*}
f(t, x, y) &= ye^{-(T-t)r} \Phi \left( -\delta_-(T-t) \frac{x}{y} \right) + x \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( \delta_+(T-t) \frac{x}{y} \right) \\
&\quad - x \frac{\sigma^2}{2r} e^{-(T-t)r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( -\delta_-(T-t) \frac{y}{x} \right) - x.
\end{align*}
\]

Checking the boundary conditions

The boundary condition (9.5a) is explained by the fact that
\[
\begin{align*}
f(t, 0, y) &= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^T - S_T \right| S_t = 0, M_0^t = y] \\
&= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^t - S_T \right| S_t = 0, M_0^t = y] \\
&= e^{-(T-t)r} \mathbb{E}^* \left[ M_0^t \right| M_0^t = y] - e^{-(T-t)r} \mathbb{E}^* \left[ S_T \right| S_t = 0] \\
&= ye^{-(T-t)r},
\end{align*}
\]
since \( \mathbb{E}^* \left[ S_T \right| S_t = 0] = 0 \) as \( S_t = 0 \) implies \( S_T = 0 \) from the relation
\[ S_T = S_t e^{\sigma(B_T-B_t) + (\mu - \sigma^2/2)(T-t)}, \quad 0 \leq t \leq T. \]

On the other hand, (9.5c) follows from the fact that
\[ f(T, x, y) = \mathbb{E}^* \left[ M_0^T - S_T \right| S_T = x, M_0^T = y] = y - x. \]
Remark 9.4. We have
\[ f(t, x, x) = xC(T - t), \]
with
\[ C(\tau) = e^{-r\tau} \Phi(-\delta_-(1)) + \left( 1 + \frac{\sigma^2}{2r} \right) \Phi(\delta_+(1)) - \frac{\sigma^2}{2r} e^{-r\tau} \Phi(-\delta_-(1)) - 1, \]
\( \tau > 0, \) hence
\[ \frac{\partial f}{\partial x}(t, x, x) = C(T - t), \quad t \in [0, T], \]
while we also have
\[ \frac{\partial f}{\partial y}(t, x, y)_{y=x} = 0, \quad 0 \leq x \leq y, \]

Scaling property of lookback put option prices

From (9.9) and the following argument we note the scaling property
\[ f(t, x, y) = e^{-(T-t)r} E^* \left[ M_0^T - S_T \middle| S_t = x, M_0^t = y \right] \]
\[ = e^{-(T-t)r} E^* \left[ (M_0^t, M_t^T - S_T) \middle| S_t = x, M_0^t = y \right] \]
\[ = e^{-(T-t)r} E^* \left[ \max \left( \frac{M_0^t}{S_t}, \frac{M_t^T}{S_t} - \frac{S_T}{S_t} \right) \middle| S_t = x, M_0^t = y \right] \]
\[ = e^{-(T-t)r} E^* \left[ \max \left( \frac{y}{x}, \frac{M_t^T}{x} - \frac{S_T}{x} \right) \middle| S_t = x, M_0^t = y \right] \]
\[ = e^{-(T-t)r} E^* \left[ \max \left( M_0^t, M_t^T - S_T \right) \middle| S_t = 1, M_0^t = \frac{y}{x} \right] \]
\[ = e^{-(T-t)r} E^* \left[ M_0^T - S_T \middle| S_t = 1, M_0^t = \frac{y}{x} \right] \]
\[ = xf(t, 1, y/x) \]
\[ = xg(T - t, x/y), \]
where we let
\[ g(\tau, z) := \]
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\[
\frac{1}{z}e^{-r\tau}\Phi(-\delta^-(z)) + \left(1 + \frac{\sigma^2}{2r}\right)\Phi(\delta^+(z)) - \frac{\sigma^2}{2r}e^{-r\tau}\left(\frac{1}{z}\right)^{2r/\sigma^2}\Phi\left(-\delta^-\left(\frac{1}{z}\right)\right) - 1,
\]

with the boundary condition

\[
\begin{align*}
\frac{\partial g}{\partial z}(\tau, 1) &= 0, \quad \tau > 0, \quad (9.10a) \\
g(0, z) &= \frac{1}{z} - 1, \quad z \in (0, 1). \quad (9.10b)
\end{align*}
\]

The next Figure 9.3 shows a graph of the function \( g(\tau, z) \).

![Graph of the normalized lookback put option price](image)

Fig. 9.3: Graph of the normalized lookback put option price.

Black-Scholes approximation of lookback put option prices

Letting

\[
BS_p(x, K, r, \sigma, \tau) := Ke^{-r\tau}\Phi\left(-\delta^-(\frac{x}{K})\right) - x\Phi\left(-\delta^+(\frac{x}{K})\right)
\]

denote the standard Black-Scholes formula for the price of a European put option, we observe that when \( S_t < M_0 \), the lookback put option price satisfies

\[
e^{-(T-t)r}\mathbb{E}[M_0^T - S_T | \mathcal{F}_t] = BS_p(S_t, M_0^t, r, \sigma, T-t) + S_t\sigma^2\left(\Phi\left(\delta^+ - \left(\frac{S_t}{M_0^t}\right)\right) - e^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta^- - \left(\frac{M_0^t}{S_t}\right)\right)\right),
\]

i.e.

\[
e^{-(T-t)r}\mathbb{E}[M_0^T - S_T | \mathcal{F}_t] = BS_p(S_t, M_0^t, r, \sigma, T-t) + S_t h_p \left( T - t, \frac{S_t}{M_0^t} \right)
\]
where the function
\[
h_p(\tau, z) = \frac{\sigma^2}{2r} \left( \Phi(\delta_+^r (z)) - e^{-r\tau} z^{-2r/\sigma^2} \Phi\left(-\delta_+^r \left(\frac{1}{z}\right)\right) \right), \tag{9.11}
\]
depends only on time \(\tau\) and \(z = S_t/M_t^t\). In other words, due to the relation
\[
BS_p(x, y, r, \sigma, \tau) = ye^{-r\tau} \Phi\left(-\delta_+^r \left(\frac{x}{y}\right)\right) - x\Phi\left(-\delta_+^r \left(\frac{x}{y}\right)\right)
= xBS_p(1, y/x, r, \sigma, \tau)
\]
for the standard Black-Scholes put price formula, we observe that \(f(t, x, y)\) satisfies
\[
f(t, x, y) = xBS_p(1, y/x, r, \sigma, T-t) + xh(T-t, x/y),
\]
i.e.
\[
f(t, x, y) = xg(T-t, x/y),
\]
with
\[
g(\tau, z) = BS_p \left( 1, \frac{1}{z}, r, \sigma, \tau \right) + h_p(\tau, z), \tag{9.12}
\]
where the function \(h_p(\tau, z)\) is a correction term given by (9.11) which is small when \(z = x/y\) or \(\tau\) become small.

Note that \((x, y) \mapsto xh_p(T - t, x/y)\) also satisfies the Black-Scholes PDE (9.4), in particular \((\tau, z) \mapsto BS_p(1, 1/z, r, \sigma, \tau)\) and \(h_p(\tau, z)\) both satisfy the PDE
\[
\frac{\partial h_p}{\partial \tau} (\tau, z) = z (r + \sigma^2) \frac{\partial h_p}{\partial z} (\tau, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 h_p}{\partial z^2} (\tau, z), \tag{9.13}
\]
\(\tau \in \mathbb{R}_+, \ z \in [0, 1]\), under the boundary condition
\[
h_p(0, z) = 0, \quad 0 \leq z \leq 1.
\]
The next Figures 9.4 and 9.5 illustrate the decomposition (9.12) of the normalized lookback put option price \(g(\tau, z)\) in Figure 9.3 into the Black-Scholes put price function \(BS_p(1, 1/z, r, \sigma, \tau)\) and \(h_p(\tau, z)\).
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Fig. 9.4: Black-Scholes put price in the decomposition (9.12).

Fig. 9.5: Correction term $h_p(\tau, z)$ in the decomposition (9.12).

Note that in Figures 9.4-9.5 the condition $h_p(0, z) = 0$ is not fully respected as $z \to 1$ due to numerical error in the approximation of the function $\Phi$.

9.4 The Lookback Call Option

The following result gives the value of the average minimum $\mathbb{E}^*[m_T^0 \mid \mathcal{F}_t]$ of $(S_t)_{t \in [0, T]}$ over the interval $[0, T]$.

**Proposition 9.5.** The average minimum value of $(S_t)_{t \in [0, T]}$ over $[0, T]$ is given by
\[
\mathbb{E}^* \left[ m_0^T \mid \mathcal{F}_t \right] = m_0^t \Phi \left( \delta_{-t} \left( \frac{S_t}{m_0^t} \right) \right) - S_t \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_{T-t} \left( \frac{m_0^t}{S_t} \right) \right) + S_t e^{(T-t)r} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\delta_{T-t} \left( \frac{S_t}{m_0^t} \right) \right). \tag{9.14}
\]

We note a certain symmetry between the expressions of (9.1) and (9.14).

When \( t = 0 \) we have \( S_0 = m_0^0 \), and given (9.2) the formula (9.14) simplifies to

\[
\mathbb{E}^* \left[ m_0^T \right] = S_0 \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T}} \right) - S_0 \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \frac{r - \sigma^2/2}{\sigma \sqrt{T}} \right) + S_0 e^{rT} \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\sigma^2/2 + r/\sqrt{T} \right),
\]

with

\[
\mathbb{E}^* \left[ m_0^T \right] = 2S_0 \left( 1 + \frac{\sigma^2 T}{4} \right) \Phi \left( -\frac{\sigma^2 T/2}{\sigma \sqrt{T}} \right) - \sigma S_0 \frac{T}{2\pi} e^{-\sigma^2 T/8}.
\]

when \( r = 0 \), cf. Exercise 9.1.

In general, when \( T \) tends to infinity we find that

\[
\lim_{T \to \infty} \mathbb{E}^* \left[ m_0^T \mid \mathcal{F}_t \right] = 0, \quad r \geq 0,
\]

see Exercise 8.1-(f) in the case \( r = \sigma^2/2 \).

**Proof of Proposition 9.5.** We have

\[
\mathbb{E}^* \left[ m_0^T \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ \min \left( m_0^t, m_t^T \right) \mid \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ m_0^t \mathbb{1}_{\{m_0^t < m_t^T\}} \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \mid \mathcal{F}_t \right]
\]

\[
= m_0^t \mathbb{E}^* \left[ \mathbb{1}_{\{m_0^t < m_t^T\}} \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \mid \mathcal{F}_t \right]
\]

\[
= m_0^t \mathbb{P}(m_0^t < m_t^T \mid \mathcal{F}_t) + \mathbb{E}^* \left[ m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \mid \mathcal{F}_t \right].
\]

By (8.11) we find the cumulative distribution function

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\[
P \left( \frac{m_{0,t} - m_{t}^T}{S_0} > x \right) = \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{S_t}{m_0} \right)}{\frac{m_0}{S_t}} \right) - \left( \frac{m_0}{S_t} \right)^{-1+2r/\sigma^2} \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{m_0}{S_t} \right)}{\frac{m_0}{S_t}} \right),
\]

of the minimum \( m_{0,t}^{T-t} \) of \((S_t)_{t \in \mathbb{R}^+}\) over the time interval \([0, T - t]\), hence

\[
P(m_0^T < m_t^T | F_t) = P \left( \frac{m_0^T}{S_t} < \frac{m_t^T}{S_t} \right| F_t) = \Phi \left( x \right)_{x=m_0^T/S_t} = \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{S_t}{m_0} \right)}{\frac{m_0}{S_t}} \right) - \left( \frac{m_0}{S_t} \right)^{-1+2r/\sigma^2} \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{m_0}{S_t} \right)}{\frac{m_0}{S_t}} \right).
\]

Next, by integration with respect to the probability density function (8.10) as in (9.3) in the proof of Proposition 9.1, we find

\[
E^* \left[ m_t^T 1_{\{m_0^T > m_t^T\}} | F_t \right] = S_t E^* \left[ \min_{r \in [t,T]} \frac{S_r}{S_t} 1_{\{\text{min}_r \in [t,T] S_r/S_t < x\}} \right]_{x=m_0^T/S_t} = 2S_t e^{(T-t)r} \Phi \left( -\delta_{t}^{T-t} \left( \frac{S_t}{m_0} \right) \right) - S_t \frac{\mu \sigma}{r} e^{(T-t)r} \Phi \left( -\delta_{t}^{T-t} \left( \frac{S_t}{m_0} \right) \right)
\]

Given the relation \( \mu \sigma/r = 1 - \sigma^2/(2r) \), this yields

\[
E^* \left[ m_0^T | F_t \right] = m_0^t P \left( \frac{m_{0,t} - m_{t}^T}{S_0} > x \right)_{x=m_0^T/S_t} + S_t E^* \left[ \min_{r \in [0,T-t]} \frac{S_r}{S_0} 1_{\{\text{min}_r \in [0,T-t] S_r/S_0 < x\}} \right]_{x=m_0^T/S_t} = m_0^t \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{S_t}{m_0} \right)}{\frac{m_0}{S_t}} \right) - m_0^t \left( \frac{m_0}{S_t} \right)^{-1+2r/\sigma^2} \Phi \left( \frac{\delta_{t}^{T-t} \left( \frac{m_0}{S_t} \right)}{\frac{m_0}{S_t}} \right).
\]
Proposition 9.6. The price at time $t \in [0,T]$ of the lookback call option with payoff $S_T - m_0^T$ is given by

$$
e^{-(T-t)r} \mathbb{E}^*[S_T - m_0^T | \mathcal{F}_t] = S_t \Phi \left( \delta_-^{T-t} \left( \frac{S_t}{m_0^t} \right) \right) - m_0^t e^{-(T-t)r} \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{m_0^t} \right) \right)$$

$$+ e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_-^{T-t} \left( \frac{m_0^t}{S_t} \right) \right) - S_t \frac{\sigma^2}{2r} \Phi \left( -\delta_+^{T-t} \left( \frac{S_t}{m_0^t} \right) \right).$$

Proof. We have

$$e^{-(T-t)r} \mathbb{E}^*[S_T - m_0^T | \mathcal{F}_t] = S_t - e^{-(T-t)r} \mathbb{E}^*[m_0^T | \mathcal{F}_t]$$

$$= S_t \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{m_0^t} \right) \right) - e^{-(T-t)r} m_0^t \Phi \left( \delta_+^{T-t} \left( \frac{S_t}{m_0^t} \right) \right)$$

$$+ e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta_-^{T-t} \left( \frac{m_0^t}{S_t} \right) \right) - e^{(T-t)r} \Phi \left( -\delta_+^{T-t} \left( \frac{S_t}{m_0^t} \right) \right).$$

Figure 9.6 represents the price of the lookback call option as a function of $m_0^t$ and $S_t$ for different values of the time to maturity $T - t$. 

The standard Lookback call option gives the right to buy the underlying asset at its historically lowest price. In this case the strike price is $m_0^T$ and the payoff is

$$C = S_T - m_0^T.$$
In Figures 9.6 and 9.7 we note that when $S_t$ is far from $M^t_0$, an increase in the underlying asset price $S_t$ clearly results into a higher call option price. On the other hand, this behavior may not be as strong when $S_t$ is close to $m^t_0$ because a decrease in the underlying asset price $S_t$ could also lead to a decrease in the value of $m^t_0$, therefore potentially increasing the option payoff.

**Fig. 9.6: Graph of the lookback call option price.**

**Fig. 9.7: Graph of the lookback call option price (2D) with $m^t_0 = 40$.**

**Black-Scholes approximation of lookback call option prices**

Letting

$$BS_c(S, K, r, \sigma, \tau) = S \Phi\left(\delta_+ \left(\frac{S}{K}\right)\right) - K e^{-r\tau} \Phi\left(\delta_- \left(\frac{S}{K}\right)\right)$$

denote the standard Black-Scholes formula for the price of a European call option, we observe that the lookback call option price satisfies

* The animation works in Acrobat Reader on the entire pdf file.
e^{-(T-t)r} \mathbb{E}^*[S_T-m^T_0 \mid \mathcal{F}_t] = \text{BS}_c(S_t,m^t_0,r,\sigma,T-t)

-S_t \sigma^2 \left( \Phi \left( -\delta^+_t \left( \frac{S_t}{m_0^t} \right) \right) - e^{-(T-t)r} \left( \frac{m^t_0}{S_t} \right)^{2r/\sigma^2} \Phi \left( \delta^-_t \left( \frac{m^t_0}{S_t} \right) \right) \right),

i.e.

e^{-(T-t)r} \mathbb{E}^*[S_T-m^T_0 \mid \mathcal{F}_t] := \text{BS}_c(S_t,m^t_0,r,\sigma,T-t) + S_t h_c \left( T-t, \frac{S_t}{m^t_0} \right)

where the correction term

\[ h_c(\tau, z) = -\frac{\sigma^2}{2r} \left( \Phi \left( -\delta^+_{\tau} (z) \right) - e^{-r\tau z - 2r/\sigma^2} \Phi \left( \delta^-_{\tau} \left( \frac{1}{z} \right) \right) \right), \quad (9.15) \]

is small when \( z = S_t/m^t_0 \) becomes large or \( \tau \) becomes small. In addition, \( h_p(\tau, z) \) is linked to \( h_c(\tau, z) \) by the relation

\[ h_c(\tau, z) = h_p(\tau, z) - \frac{\sigma^2}{2r} \left( 1 - e^{-r\tau z - 2r/\sigma^2} \right), \quad \tau \in \mathbb{R}_+, \quad z \in \mathbb{R}_+, \]

where \((z, \tau) \mapsto e^{-r\tau z - 2r/\sigma^2}\) also solves the PDE (9.13).

**Black-Scholes PDE for lookback call option prices**

By the same argument as in the proof of Proposition 9.3, the function \( f(t, x, y) \) satisfies the Black-Scholes PDE

\[ rf(t, x, y) = \frac{\partial f}{\partial t} (t, x, y) + rx \frac{\partial f}{\partial x} (t, x, y) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2} (t, x, y), \quad t, x > 0, \]

under the boundary conditions

\[
\begin{align*}
\lim_{y \to 0} f(t, x, y) &= x, \quad 0 \leq t \leq T, \quad x > 0, \quad (9.16a) \\
\frac{\partial f}{\partial y} (t, x, y)_{x=y} &= 0, \quad 0 \leq t \leq T, \quad y > 0, \quad (9.16b) \\
f(T, x, y) &= x - y, \quad 0 \leq y \leq x, \quad (9.16c)
\end{align*}
\]

and the corresponding self-financing hedging strategy is given by

\[ \xi_t = \frac{\partial f}{\partial x} (t, S_t, m_0^t), \quad t \in [0, T], \quad (9.17) \]
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which represents the quantity of the risky asset $S_t$ to be held at time $t$ in the hedging portfolio.

In other words, the price of the lookback call option takes the form

$$f(t, S_t, m_t) = e^{-(T-t)r} \mathbb{E}^* [S_T - m_T^T \mid F_t],$$

where the function $f(t, x, y)$ is given by

$$f(t, x, y) = x \Phi \left( \delta_t^+ \left( \frac{x}{y} \right) \right) - e^{-(T-t)r} y \Phi \left( \delta_t^- \left( \frac{x}{y} \right) \right)$$

$$+ e^{-(T-t)r} \frac{\sigma^2}{2r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \delta_t^- \left( \frac{y}{x} \right) \right) - e^{(T-t)r} \Phi \left( -\delta_t^- \left( \frac{x}{y} \right) \right)$$

$$= x - ye^{-(T-t)r} \Phi \left( \delta_t^- \left( \frac{x}{y} \right) \right) - x \left( 1 + \sigma^2 \right) \Phi \left( -\delta_t^- \left( \frac{x}{y} \right) \right)$$

$$+ ye^{-(T-t)r} \frac{\sigma^2}{2r} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \delta_t^- \left( \frac{y}{x} \right) \right).$$

Checking the boundary conditions

The boundary condition (9.16a) is explained by the fact that

$$f(t, x, 0) = e^{-(T-t)r} \mathbb{E}^* [S_T - m_0^T \mid S_t = x, m_0^t = 0]$$

$$= e^{-(T-t)r} \mathbb{E}^* [S_T \mid S_t = x, m_0^t = 0]$$

$$= e^{-(T-t)r} \mathbb{E}^*[S_T \mid S_t = x]$$

$$= e^{-(T-t)r} x, \quad x \geq 0.$$  

On the other hand, (9.16b) follows from the fact that

$$f(T, x, y) = \mathbb{E}^* [S_T - m_0^T \mid S_T = x, m_0^T = y] = x - y.$$  

We have

$$f(t, x, x) = xC(T-t),$$

with

$$C(\tau) = 1 - e^{-r \tau} \Phi (\delta_t^- (1)) - \left( 1 + \frac{\sigma^2}{2r} \right) \Phi (-\delta_t^- (1)) + e^{-r \tau} \frac{\sigma^2}{2r} \Phi (\delta_t^- (1)),$$

$\tau > 0$, hence

$$\frac{\partial f}{\partial x} (t, x, x) = C(T-t), \quad t \in [0, T],$$  

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while we also have
\[ \frac{\partial f}{\partial y} (t, x, y)_{y=x} = 0, \quad 0 \leq x \leq y. \]

**Scaling property of lookback call option prices**

We note the scaling property:

\[
\begin{align*}
   f(t, x, y) &= e^{-r(T-t)} E^* \left[ S_T - m_0^T \mid S_t = x, \ m_0^t = y \right] \\
   &= e^{-r(T-t)} E^* \left[ S_T - \min (m_0^t, m_t^T) \mid S_t = x, \ m_0^t = y \right] \\
   &= e^{-r(T-t)} x E^* \left[ \frac{S_T}{S_t} - \max \left( \frac{m_0^t}{S_t}, \frac{m_t^T}{S_t} \right) \mid S_t = x, \ m_0^t = y \right] \\
   &= e^{-r(T-t)} x E^* \left[ \frac{S_T}{x} - \max \left( \frac{y}{x}, \frac{m_t^T}{x} \right) \mid S_t = x, \ m_0^t = y \right] \\
   &= e^{-r(T-t)} x E^* \left[ (S_T - m_0^t, m_t^T) \mid S_t = 1, \ m_0^t = \frac{y}{x} \right] \\
   &= e^{-r(T-t)} x E^* \left[ S_T - m_0^T \mid S_t = 1, \ m_0^t = \frac{y}{x} \right] \\
   &= x f(t, 1, y/x) \\
   &= x g \left( T - t, \frac{1}{z} \right),
\end{align*}
\]

where
\[
g(\tau, z) := 1 - \frac{1}{z} e^{-r\tau} \Phi \left( \delta^- \left( \frac{1}{z} \right) \right) - \left( 1 + \frac{\sigma^2}{2r} \right) \Phi \left( -\delta^+ \left( \frac{1}{z} \right) \right) + \frac{\sigma^2}{2r} e^{-r\tau} z^{-2r/\sigma^2} \Phi \left( \delta^- \left( \frac{1}{z} \right) \right),
\]

with \( g(\tau, 1) = C(T-t) \), and
\[
f(t, x, y) = x g \left( T - t, \frac{x}{y} \right)
\]

and the boundary condition

\[
\begin{align}
   \frac{\partial g}{\partial z} (\tau, 1) &= 0, \quad \tau > 0, \quad (9.19a) \\
   g(0, z) &= 1 - \frac{1}{z}, \quad z \geq 1. \quad (9.19b)
\end{align}
\]
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The next Figure 9.8 shows a graph of the function $g(\tau, z)$.

![Fig. 9.8: Normalized lookback call option price.](image)

The next Figure 9.9 represents the path of the underlying asset price used in Figure 9.8.

![Fig. 9.9: Graph of the underlying market price.](image)

The next Figure 9.10 represents the corresponding underlying asset price and its running minimum.

![Fig. 9.10](image)
Next, we represent the option price as a function of time, together with the process \((S_t - m^0_t)_{t \in \mathbb{R}^+}\).

Black-Scholes approximation

Due to the relation

\[
BS_c(x, y, r, \sigma, \tau) = x \Phi \left( \delta^+ \left( \frac{x}{y} \right) \right) - ye^{-r\tau} \Phi \left( \delta^- \left( \frac{x}{y} \right) \right) \\
= xBS_c \left( 1, \frac{y}{x}, r, \sigma, \tau \right)
\]

for the standard Black-Scholes call price formula, recall that \(f(t, x, y)\) can be decomposed as

\[
f(t, x, y) = xBS_c \left( 1, \frac{y}{x}, r, \sigma, T - t \right) + xh_c \left( T - t, \frac{x}{y} \right),
\]
where \( h_c(\tau, z) \) is the function given by (9.15), \( i.e.\)
\[
f(t, x, y) = xg(T - t, \frac{x}{y}),
\]
with
\[
g(\tau, z) = BS_c\left(1, \frac{1}{z}, r, \sigma, \tau\right) + h_c(\tau, z),
\]
(9.20)
where \((x, y) \mapsto xh_c(T - t, x/y)\) also satisfies the Black-Scholes PDE (9.4), \( i.e. \) \((\tau, z) \mapsto BS_c(1, 1/z, r, \sigma, \tau)\) and \( h_c(\tau, z) \) both satisfy the PDE (9.13) under the boundary condition
\[
h_c(0, z) = 0, \quad z \geq 1.
\]
The next Figures 9.12 and 9.13 show the decomposition of \( g(t, z) \) in (9.20) and Figures 9.8-9.9 into the sum of the Black-Scholes call price function \( BS_c(1, 1/z, r, \sigma, \tau) \) and \( h(t, z) \).

Fig. 9.12: Black-Scholes call price in the decomposition (9.20) of the normalized lookback call option price \( g(\tau, z) \).
Fig. 9.13: Function $h_c(\tau, z)$ in the decomposition (9.20) of the normalized lookback call option price $g(\tau, z)$.

We also note that

$$E^* \left[ M_0^T - m_0^T \mid S_0 = x \right] = x - xe^{-(T-t)r} \Phi \left( \frac{\delta_{\tau - t}}{2r} \right)$$

$$+ xe^{-(T-t)r} \Phi \left( -\frac{\delta_{\tau - t}}{2r} \right) + xe^{-(T-t)r} \frac{\sigma^2}{2r} \Phi \left( \frac{\delta_{\tau - t}}{2r} \right)$$

$$- xe^{-(T-t)r} \Phi \left( -\frac{\delta_{\tau - t}}{2r} \right) - x$$

$$= x \left( 1 + \frac{\sigma^2}{2r} \right) \left( \Phi \left( \frac{\delta_{\tau - t}}{2r} \right) - \Phi \left( -\frac{\delta_{\tau - t}}{2r} \right) \right)$$

$$+ xe^{-(T-t)r} \left( \frac{\sigma^2}{2r} - 1 \right) \left( \Phi \left( \frac{\delta_{\tau - t}}{2r} \right) - \Phi \left( -\frac{\delta_{\tau - t}}{2r} \right) \right).$$

9.5 Hedging Lookback Call Options

In this section we compute hedging strategies for lookback options by application of the Delta hedging formula (9.17). See Bermin (1998), § 2.6.1, page 29, for another approach to the following result using the Clark-Ocone formula. Here we use (9.17) instead, cf. Proposition 4.6 of Khatib and Privault (2003).

**Proposition 9.7.** The hedging strategy of the lookback call option is given by

$$\xi_t = \Phi \left( \frac{\delta_{\tau - t}}{m_0^t} \right) - \frac{\sigma^2}{2r} \Phi \left( -\frac{\delta_{\tau - t}}{m_0^t} \right)$$

$$+ xe^{-(T-t)r} \left( \frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \left( \frac{\sigma^2}{2r} - 1 \right) \Phi \left( \frac{\delta_{\tau - t}}{m_0^t} \right), \quad t \in [0, T].$$

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Proof. By (9.17), we need to differentiate

\[ f(t, x, y) = BS_c(x, y, r, \sigma, T - t) + xh_c(T - t, \frac{x}{y}) \]

with respect to the variable \( x \), where

\[ h_c(\tau, z) = -\frac{\sigma^2}{2r} \left( \Phi(-\delta_+^\tau (z)) - e^{-r\tau} z^{-2r/\sigma^2} \Phi \left( \delta_-^\tau \left( \frac{1}{z} \right) \right) \right) \]

is given by (9.15) First, we note that the relation

\[ \frac{\partial}{\partial x} BS_c(x, y, r, \sigma, \tau) = \Phi \left( \delta_+^\tau \left( \frac{x}{y} \right) \right) \]

is known, cf. Propositions 5.14 and 6.11. Next, we have

\[ \frac{\partial}{\partial x} \left( xh_c(\tau, \frac{x}{y}) \right) = h_c(\tau, \frac{x}{y}) + x \frac{\partial h_c}{\partial z} \left( \tau, \frac{x}{y} \right), \]

and

\[ \frac{\partial h_c}{\partial z}(\tau, z) = -\frac{\sigma^2}{2r} \left( \frac{\partial}{\partial x} \left( \Phi(-\delta_+^\tau (z)) \right) - e^{-r\tau} z^{-2r/\sigma^2} \frac{\partial}{\partial z} \left( \Phi \left( \delta_-^\tau \left( \frac{1}{z} \right) \right) \right) \right) \]

\[ = -\frac{\sigma^2}{2r} \left( \frac{2r}{\sigma^2} e^{-r\tau} z^{-1-2r/\sigma^2} \Phi \left( \delta_-^\tau \left( \frac{1}{z} \right) \right) \right) \]

\[ = -\frac{\sigma}{2rz\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \delta_+^\tau (z) \right)^2 \right) \]

\[ - e^{-r\tau} z^{-2r/\sigma^2} \frac{2r}{\sigma^2} \exp \left( -\frac{1}{2} \left( \delta_-^\tau \left( \frac{1}{z} \right) \right)^2 + \frac{2r}{\sigma^2} e^{-r\tau} z^{-1-2r/\sigma^2} \Phi \left( \delta_-^\tau \left( \frac{1}{z} \right) \right) \right). \]

Next, we note that

\[ e^{-\left( \delta_-^\tau (1/z) \right)^2/2} = \exp \left( -\frac{1}{2} \left( \delta_-^\tau (z) \right)^2 - \frac{1}{2} \left( \frac{4r^2}{\sigma^2} \tau - \frac{4r}{\sigma} \delta_-^\tau (z) \sqrt{\tau} \right) \right) \]

\[ = e^{-\frac{1}{2} \left( \delta_-^\tau (z) \right)^2} \exp \left( -\frac{1}{2} \left( \frac{4r^2}{\sigma^2} \tau - \frac{4r}{\sigma^2} \left( \log z + (r + \frac{1}{2} \sigma^2) \tau \right) \right) \right) \]

\[ = e^{-\frac{1}{2} \left( \delta_+^\tau (z) \right)^2} \exp \left( -\frac{2r^2}{\sigma^2} \tau + \frac{2r}{\sigma^2} \log z + \frac{2r^2}{\sigma^2} \tau + r\tau \right) \]

\[ = e^{r\tau z \frac{2r}{\sigma^2}} e^{-\left( \delta_+^\tau (z) \right)^2/2} \]

as in the proof of Proposition 5.14, hence

\[ \frac{\partial h_c}{\partial z} \left( \tau, \frac{x}{y} \right) = -e^{-r\tau} z^{-1-2r/\sigma^2} \Phi \left( \delta_-^\tau \left( \frac{1}{z} \right) \right), \]
and
\[
\frac{\partial}{\partial x} \left( x h_c \left( \frac{\tau}{x}, \frac{y}{y} \right) \right) = h_c \left( \frac{\tau}{x}, \frac{y}{y} \right) - e^{-r\tau} \left( \frac{y}{x} \right)^{2r/\sigma^2} \Phi \left( \frac{\tau}{x} \right),
\]
which concludes the proof. □

We note that at maturity \( t = T \) the hedging strategy satisfies
\[
\xi_T = \begin{cases} 
1 & \text{if } M^T_0 < S_T, \\
\frac{1}{2} - \frac{\sigma^2}{4r} + \frac{1}{2} \left( \frac{\sigma^2}{2r} - 1 \right) = 0 & \text{if } M^T_0 = S_T.
\end{cases}
\]

In Figure 9.14 we represent the Delta of the lookback call option, as given by (9.21).

---

Fig. 9.14: Delta of the lookback call option with \( r = 2\% \) and \( \sigma = 0.41. \)

The above scaling procedure can be applied to the Delta as well, by noting that \( \xi_t \) can be written as
\[
\xi_t = \zeta \left( t, \frac{S_t}{m_0^t} \right),
\]
where the function \( \zeta(t, z) \) is given by
\[
\zeta(t, z) = \Phi \left( \delta^{T-t}(z) \right) - \frac{\sigma^2}{2r} \Phi \left( -\delta^{T-t}(z) \right) + e^{-(T-t)r} z^{-2r/\sigma^2} \left( \frac{\sigma^2}{2r} - 1 \right) \Phi \left( \delta^{T-t} \left( \frac{1}{z} \right) \right),
\]
t \( \in [0, T], \) \( z \in [0, 1]. \) The graph of the function \( \zeta(t, x) \) is given in Figure 9.15.

* The animation works in Acrobat Reader on the entire pdf file.
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Fig. 9.15: Rescaled portfolio strategy for the lookback call option.

Similar calculations using (9.6) can be carried out for other types of lookback options, such as options on extrema and partial lookback options, cf. Khatib (2003). As a consequence of (9.23) we have

\[
e^{-(T-t)r} \mathbb{E}^* \left[ S_T - m_T^0 \mid \mathcal{F}_t \right] = S_t \Phi \left( \delta_{+}^{T-t} \left( \frac{S_t}{m_t^0} \right) \right) - m_t^0 e^{-(T-t)r} \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{m_t^0} \right) \right) + e^{-(T-t)r} S_t \frac{\sigma^2}{2r} \left( \frac{S_t}{m_t^0} \right)^{-2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_t^0}{S_t} \right) \right) - S_t \frac{\sigma^2}{2r} \Phi \left( -\delta_{+}^{T-t} \left( \frac{S_t}{m_t^0} \right) \right)
\]

\[
= \xi_t S_t - m_t^0 e^{-(T-t)r} \left( \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{m_t^0} \right) \right) + \left( \frac{S_t}{m_t^0} \right)^{1-2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_t^0}{S_t} \right) \right) \right),
\]

and the quantity of the risk-free asset \( e^{rt} \) in the portfolio is given by

\[
\eta_t = -m_t^0 e^{-rT} \left( \Phi \left( \delta_{-}^{T-t} \left( \frac{S_t}{m_t^0} \right) \right) + \left( \frac{S_t}{m_t^0} \right)^{1-2r/\sigma^2} \Phi \left( \delta_{-}^{T-t} \left( \frac{m_t^0}{S_t} \right) \right) \right) \leq 0,
\]

so that the portfolio value \( V_t \) at time \( t \) satisfies

\[
V_t = \xi_t S_t + \eta_t e^{rt}, \quad t \in \mathbb{R}_+,
\]

and one has to constantly borrow from the risk-free account in order to hedge the lookback option.

Exercises

Exercise 9.1
Exercise 9.2 Let \((B_t)_{t \in \mathbb{R}_+}\) denote a standard Brownian motion.

a) Compute the expected value
\[
\mathbb{E} \left[ \max_{t \in [0,1]} S_t \right] = \mathbb{E} \left[ e^{\sigma \max_{t \in [0,1]} (B_t - t/2)} \right].
\]

b) Compute the “optimal exercise” price
\[
\mathbb{E} \left[ e^{-\sigma^2 T/2} \mathbb{E}^* \left[ S_T - \min_{t \in [0,T]} S_t \right] \right]
\]
of a lookback call option on \(S_T\) with maturity \(T\).

Exercise 9.3 Consider a risky asset whose price \(S_t\) is given by
\[
dS_t = \sigma S_t dB_t + \sigma^2 S_t dt / 2,
\]
where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion.

a) Give the probability distribution (distribution function and probability density function) of the minimum \(\min_{t \in [0,T]} B_t\) over the interval \([0,T]\)?

b) Compute the price value
\[
e^{-\sigma^2 T/2} \mathbb{E}^* \left[ S_T - \min_{t \in [0,T]} S_t \right]
\]
of a lookback call option on \(S_T\) with maturity \(T\).

Exercise 9.4 Compute the hedging strategy of the lookback put option priced in Proposition 9.1.
Exercise 9.5  Dassios and Lim (2019) The digital drawdown call option with qualifying period pays a unit amount when the drawdown period reaches one unit of time, if this happens before fixed maturity $T$, but only if the size of drawdown at this stopping time is larger than a prespecified $K$. This provides an insurance against a prolonged drawdown, if the drawdown amount is large. Specifically, the digital drawdown call option is priced as

$$\mathbb{E}^*\left[ e^{-r\tau}\mathbb{1}_{\{\tau \leq T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \geq K\}}\right],$$

where $M_0^t := \max_{u \in [0,t]} S_u$, $U_t := t - \sup\{0 \leq u \leq t : M_0^t = S_u\}$, and $\tau := \inf\{t \in \mathbb{R}_+ : U_t = 1\}$. Write the price of the drawdown option as a triple integral using the joint probability density function $f_{(\tau, S_\tau, M_\tau)}(t, x, y)$ of $(\tau, S_\tau, M_\tau)$ under the risk-neutral probability measure $\mathbb{P}^*$.

Exercise 9.6 Check by hand calculation that the solution $f(t, x, y)$ of the lookback put option pricing PDE satisfies the boundary condition

$$\frac{\partial f}{\partial y}(t, x, y)_{x=y} = 0.$$