Chapter 13
Forward Rate Modeling

This chapter is concerned with interest rate modeling, in which the mean reversion property plays an important role. We consider the main short rate models (Vasicek, CIR, CEV, affine models) and the computation of fixed income products, such as bond prices, in such models. Next we consider the modeling of forward rates in the HJM and BGM models, as well as in two-factor models.

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13.1 Short Term and Mean Reverting Models

Vasicek model

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vasicek [Vaš77] model, which is based on the Ornstein-Uhlenbeck process. Here, the short term interest rate process \((r_t)_{t \in \mathbb{R}^+}\) solves the equation

\[
dr_t = (a - br_t)dt + \sigma dB_t, \tag{13.1}
\]

where \(a, \sigma \in \mathbb{R}, b > 0\), and \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion, with solution
\begin{equation}
    r_t = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \quad t \in \mathbb{R}_,
\end{equation}

see Exercise 13.1. The probability distribution of \( r_t \) is Gaussian at all times \( t \), with mean
\[
    \mathbb{E}[r_t] = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}),
\]
and variance
\[
    \text{Var}[r_t] = \text{Var} \left[ \sigma \int_0^t e^{-(t-s)b} dB_s \right] = \sigma^2 \int_0^t (e^{-(t-s)b})^2 ds = \sigma^2 \int_0^t e^{-2bs} ds = \frac{\sigma^2}{2b} (1 - e^{-2bt}), \quad t \in \mathbb{R}_,
\]
\[i.e.\]
\[
    r_t \sim \mathcal{N} \left( r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}), \frac{\sigma^2}{2b} (1 - e^{-2bt}) \right), \quad t > 0.
\]

When \( b > 0 \) and in the long run, \( i.e. \) when time \( t \) is large we have
\[
    \lim_{t \to \infty} \mathbb{E}[r_t] = \frac{a}{b} \quad \text{and} \quad \lim_{t \to \infty} \text{Var}[r_t] = \frac{\sigma^2}{2b},
\]
and this distribution converges to the Gaussian \( \mathcal{N}(a/b, \sigma^2/(2b)) \) distribution, which is also the invariant (or stationary) distribution of \((r_t)_{t \in \mathbb{R}_,}\), and the process tends to revert to its long term mean \( a/b = \lim_{t \to \infty} \mathbb{E}[r_t] \) which makes the average drift vanish:
\[
    \lim_{t \to \infty} \mathbb{E}[(a - br_t)] = a - b \lim_{t \to \infty} \mathbb{E}[r_t] = 0.
\]

Figure 13.1 presents a random simulation of \( t \to r_t \) in the Vasicek model with \( r_0 = 3\% \), and shows the mean reverting property of the process with respect to \( a/b = 2.5\% \).

\* “But this long run is a misleading guide to current affairs. In the long run we are all dead.” J. M. Keynes, A Tract on Monetary Reform (1923), Ch. 3, p. 80.
Fig. 13.1: Graph of the Vasicek short rate $t \mapsto r_t$ with $a = 0.025$, $b = 1$, and $\sigma = 0.1$.

As can be checked from the simulation of Figure 13.1 the value of $r_t$ in the Vasicek model may become negative due to its Gaussian distribution. Although real interest rates can sometimes fall below zero, this can be regarded as a potential drawback of the Vasicek model.

Example - TNX yield

We consider the yield of the 10 Year Treasury Note on the Chicago Board Options Exchange (CBOE). Treasury notes usually have a maturity between one and 10 years, whereas treasury bonds have maturities beyond 10 years)

```r
library(quantmod)
getSymbols("^TNX",from="2012-01-01",to="2016-01-01",src="yahoo")
rate=Ad(\text{ts1}/\text{ts1})
chartSeries(rate,up.col="blue",theme="white")
n = sum(!is.na(rate))
```

The next Figure 13.2 displays the yield of the 10 Year Treasury Note.

Fig. 13.2: CBOE 10 Year Treasury Note yield (TNX).
Cox-Ingersoll-Ross (CIR) model

The Cox-Ingersoll-Ross (CIR) \cite{CIR85} model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear stochastic differential equation

\begin{equation}
    dr_t = \beta (\alpha - r_t) dt + \sigma \sqrt{r_t} dB_t,
\end{equation}

with \( \alpha > 0, \beta > 0, \sigma > 0 \).

The probability distribution of \( r_t \) at time \( t > 0 \) admits the noncentral Chi square probability density function given by

\begin{equation}
    f_t(x) = \frac{2\beta}{\sigma^2(1 - e^{-\beta t})} \exp \left( -\frac{2\beta(x + r_0 e^{-\beta t})}{\sigma^2(1 - e^{-\beta t})} \right) \left( \frac{x}{r_0 e^{-\beta t}} \right)^{\alpha\beta/\sigma^2-1/2} \frac{I_{2\alpha\beta/\sigma^2-1} \left( \frac{4\beta \sqrt{r_0 x e^{-\beta t}}}{\sigma^2(1 - e^{-\beta t})} \right)}{I_{\lambda}(z)}
\end{equation}

where

\begin{equation}
    I_\lambda(z) := \frac{\lambda}{2} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k! \Gamma(\lambda + k + 1)}, \quad z \in \mathbb{R},
\end{equation}

is the modified Bessel function of the first kind, cf. Corollary 24 in \cite{AL05}. Note that \( f_t(x) \) is not defined at \( x = 0 \) if \( \alpha\beta/\sigma^2 - 1/2 < 0 \), i.e. \( \sigma^2 > 2\alpha\beta \), in which case the probability distribution of \( r_t \) admits a point mass at \( x = 0 \).

On the other hand, \( r_t \) remains almost surely strictly positive under the Feller condition \( 2\alpha\beta \geq \sigma^2 \), cf. the study of the associated probability density in Lemma 4 of \cite{Fel51}.

Figure 13.3 presents a random simulation of \( t \mapsto r_t \) in the CIR model in the case \( \sigma^2 > 2\alpha\beta \), in which the process is mean reverting with respect to \( \alpha = 2.5\% \) and has a nonzero probability of hitting 0.
In large time $t$, using the asymptotics

$$I_\lambda(z) \approx_{z \to 0} \frac{1}{\Gamma(\lambda + 1)} \left( \frac{z}{2} \right)^\lambda,$$

the density (13.4) becomes the gamma density

$$f(x) = \lim_{t \to \infty} f_t(x) = \frac{1}{\Gamma(2\alpha \beta / \sigma^2)} \left( \frac{2\beta}{\sigma^2} \right)^{2\alpha \beta / \sigma^2} x^{-1+2\alpha \beta / \sigma^2} e^{-2\beta x / \sigma^2}, \quad x > 0.$$ (13.5)

with shape parameter $2\alpha \beta / \sigma^2$ and scale parameter $\sigma^2 / (2\beta)$, which is also the invariant distribution of $r_t$.

Other classical mean reverting models include the Courtadon (1982) model

$$dr_t = \beta (\alpha - r_t) dt + \sigma r_t dB_t,$$

where $\alpha, \beta, \sigma$ are nonnegative, cf. Exercise 13.4, and the exponential Vasicek model

$$dr_t = r_t (\eta - a \log r_t) dt + \sigma r_t dB_t,$$

where $a, \eta, \sigma > 0$, cf. Exercises 4.16 and 4.17.

**Constant Elasticity of Variance (CEV)**

Constant Elasticity of Variance models are designed to take into account nonconstant volatilities that can vary as a power of the underlying asset. The Marsh-Rosenfeld (1983) model

$$dr_t = (\beta r_t^{-\gamma} + \alpha r_t) dt + \sigma r_t^{\gamma/2} dB_t$$ (13.6)

where $\alpha \in \mathbb{R}$, $\beta > 0$, $\sigma > 0$, $\gamma > 0$ are constants and $\beta > 0$ is the variance (or diffusion) elasticity coefficient, covers most of the CEV models. Denoting by $v^2(r) := \sigma^2 r^\gamma$ the variance coefficient in (13.6), constant elasticity refers to the constant ratio

$$\frac{dv^2(r)/v^2(r)}{dr/r} = \frac{2 r v'(r)}{v(r)} = 2 \frac{d \log v(r)}{d \log r} = 2 \frac{d \log r^{\gamma/2}}{d \log r} = \gamma$$

between the relative change $dv(r)/v(r)$ in the variance $v(r)$ and the relative change $dr/r$ in $r$.

For $\gamma = 1$, (13.6) yields the standard CIR equation

$$dr_t = (\beta + \alpha r_t) dt + \sigma \sqrt{r_t} dB_t,$$
and for $\beta = 0$ we get the standard CEV model

$$dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t.$$  

If $\gamma = 2$ this yields the Dothan [Dot78] model

$$dr_t = \alpha r_t dt + \sigma r_t dB_t,$$

which is a version of geometric Brownian motion used for short term interest rate modeling.

**Time-dependent affine models**

The class of short rate interest rate models admits a number of generalizations (see the references quoted in the introduction of this chapter), including the class of affine models of the form

$$dr_t = (\eta(t) + \lambda(t) r_t) dt + \sqrt{\delta(t) + \gamma(t) r_t} dB_t. \quad (13.7)$$

Such models are called affine because the associated bonds can be priced using an affine PDE of the type (13.17) below, as will be seen after Proposition 13.2.

The family of affine models also includes:

i) the Ho-Lee model

$$dr_t = \theta(t) dt + \sigma dB_t,$$

where $\theta(t)$ is a deterministic function of time, as an extension of the Merton model $dr_t = \theta dt + \sigma dB_t$,

ii) the Hull-White model [HW90], cf. Section 13.1,

$$dr_t = (\theta(t) - \alpha(t) r_t) dt + \sigma(t) dB_t$$

which is itself a time-dependent extension of the Vasicek model.

**13.2 Calibration of the Vasicek model**

The Vasicek equation (13.1), *i.e.*

$$dr_t = (a - b r_t) dt + \sigma dB_t$$

can be discretized according to a discrete-time sequence $(t_k)_{k=0,1,\ldots,n} = (t_0, t_1, \ldots, t_n)$ as

$$r_{t_{k+1}} - r_{t_k} = (a - br_{t_k}) \Delta t + \sigma Z_k, \quad k \in \mathbb{N},$$
where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k \geq 0}$ is a Gaussian white noise with variance $\Delta t$, i.e. a sequence of independent, centered and identically distributed $\mathcal{N}(0, \Delta t)$ Gaussian random variables, which yields

$$r_{t_{k+1}} = r_{t_k} + (a - b r_{t_k}) \Delta t + \sigma Z_k = a \Delta t + (1 - b \Delta t) r_{t_k} + \sigma Z_k, \quad k \in \mathbb{N}.$$

Based on a set $(\tilde{r}_{t_k})_{k=0,\ldots,n}$ of market data we consider the residual

$$\sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a \Delta t - (1 - b \Delta t) \tilde{r}_{t_k})^2 \quad (13.8)$$

which represents the quadratic distance between the observed data sequence $(\tilde{r}_{t_{k+1}})_{k=0,1,\ldots,n-1}$ and its predictions $(a \Delta t + (1 - b \Delta t) \tilde{r}_{t_k})_{k=0,1,\ldots,n-1}$.

In order to minimize the residual (13.8) over $a$ and $b$ we use Ordinary Least Square (OLS) regression, and equate the following derivatives to zero. Namely, we have

$$\frac{\partial}{\partial a} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - a \Delta t - (1 - b \Delta t) \tilde{r}_{t_l})^2$$

$$= -2 \Delta t \left( -a n \Delta t + \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b \Delta t) \tilde{r}_{t_l}) \right)$$

$$= 2 a n (\Delta t)^2 - 2 \Delta t \sum_{l=0}^{n-1} \tilde{r}_{t_{l+1}} + 2 \Delta t (1 - b \Delta t) \sum_{l=0}^{n-1} \tilde{r}_{t_l}$$

$$= 0,$$

hence

$$a \Delta t = \frac{\Delta t}{n} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b \Delta t) \tilde{r}_{t_l}),$$

and

$$\frac{\partial}{\partial b} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a \Delta t - (1 - b \Delta t) \tilde{r}_{t_k})^2$$

$$= \Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left( -a \Delta t + \tilde{r}_{t_{k+1}} - (1 - b \Delta t) \tilde{r}_{t_k} \right)$$

$$= \Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left( \tilde{r}_{t_{k+1}} - (1 - b \Delta t) \tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b \Delta t) \tilde{r}_{t_l}) \right)$$
This leads to an estimate the parameters \(a\) and \(b\) respectively as empirical mean and covariance of \((\tilde{r}_t)_{k=0,1,\ldots,n}\), i.e.

\[
\hat{a} \Delta t = \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - (1 - \hat{b} \Delta t) \tilde{r}_{t_k}),
\]

and

\[
1 - \hat{b} \Delta t = \frac{\sum_{k=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{k+1}} - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_l}}{\sum_{k=0}^{n-1} (\tilde{r}_{t_k})^2 - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_l}}.
\]

This also yields

\[
\sigma^2 \Delta t = \text{Var}[\sigma Z_k] = \text{Var} [\tilde{r}_{t_{k+1}} - (1 - b \Delta t) \tilde{r}_{t_k} - a \Delta t], \quad k \in \mathbb{N},
\]

hence \(\sigma\) can be estimated as

\[
\hat{\sigma}^2 \Delta t = \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k} (1 - \hat{b} \Delta t) - \hat{a} \Delta t)^2.
\]

Defining \(\hat{r}_{t_k} := r_{t_k} - a/b, k \in \mathbb{N}\), we have

\[
\hat{r}_{t_{k+1}} = r_{t_{k+1}} - a/b
\]

\[
= r_{t_k} - a/b + (a - br_{t_k}) \Delta t + \sigma Z_k
\]

\[
= r_{t_k} - a/b - b(r_{t_k} - a/b) \Delta t + \sigma Z_k
\]

\[
= \hat{r}_{t_k} - b \hat{r}_{t_k} \Delta t + \sigma Z_k
\]

\[
= (1 - b \Delta t) \hat{r}_{t_k} + \sigma Z_k, \quad k \in \mathbb{N}.
\]

In other words, the sequence \((\hat{r}_{t_k})_{k \in \mathbb{N}}\) is modeled according to an autoregressive AR(1) time series with parameter \(1 - b \Delta t\), in which the current state
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$X_n$ of the system is expressed as the linear combination

$$X_n := \sigma Z_n + \alpha X_{n-1}, \quad n \geq 1,$$

(13.9)

which can be solved recursively as the causal series

$$X_n = \sigma Z_n + \alpha (\sigma Z_{n-1} + \alpha X_{n-2}) = \cdots = \sigma \sum_{k=0}^{\infty} \alpha^k Z_{n-k},$$

which converges when $|\alpha| < 1$, i.e. $|1 - b \Delta t| < 1$. Note that the variance of $X_n$ is given by

$$\text{Var}[X_n] = \sigma^2 \text{Var} \left[ \sum_{k=0}^{\infty} \alpha^k Z_{n-k} \right]$$

$$= \sigma^2 \Delta t \sum_{k=0}^{\infty} \alpha^{2k}$$

$$= \sigma^2 \Delta t \sum_{k=0}^{\infty} (1 - b \Delta t)^{2k}$$

$$= \frac{\sigma^2 \Delta t}{1 - (1 - b \Delta t)^2}$$

$$= \frac{\sigma^2 \Delta t}{2b \Delta t - b^2 (\Delta t)^2}$$

$$\simeq \frac{\sigma^2}{2b}, \quad [\Delta t \simeq 0],$$

which is the expected variance (13.3) of the Vasicek process in the stationary regime.

Example - TNX yield calibration

The next code is estimating the parameters of the Vasicek model using the 10 Year Treasury Note. yield data of Figure 13.2.

```r
ratek<-as.vector(rate)
ratekplus1 <- c(ratek[-1],0)
b <- (sum(ratek*ratekplus1) - sum(ratek)*sum(ratekplus1)/n)/(sum(ratek*ratek) - sum(ratek)*sum(ratek)/n)
a <- sum(ratekplus1)/n-b*sum(ratek)/n
sigma <- sqrt(sum((ratekplus1-b*ratek-a)^2)/n)
```

The code below is generating Vasicek random samples according to the AR(1) time series (13.9).
for (i in 1:100) {ar.sim<-arima.sim(model=list(ar=c(b)),n.start=100,n)
y=ratek[1]+a/b+sigma*ar.sim
time <- as.POSIXct(time(TNX), format = "%Y-%m-%d")
yield <- xts(x = y, order.by = time)
chartSeries(yield,up.col="blue",theme="white")
Sys.sleep(0.5)}

Fig. 13.4: Calibrated Vasicek samples.

13.3 Zero-Coupon and Coupon Bonds

A zero-coupon bond is a contract priced $P(t, T)$ at time $t < T$ to deliver the face value (or par value) $P(T, T) = 1$ at time $T$. In addition to its value at maturity, a bond may yield a periodic coupon payment at regular time intervals until the maturity date.

Fig. 13.5: Five-dollar Louisiana bond of 1875 with 7.5% biannual coupons and maturity $T = 01/01/1886$. 

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http://www.ntu.edu.sg/home/nprivault/index.html
The computation of the arbitrage price $P_0(t,T)$ of a zero-coupon bond based on an underlying short term interest rate process $(r_t)_{t \in \mathbb{R}^+}$ is a basic and important issue in interest rate modeling.

**Constant short rate**

In case the short term interest rate is a constant $r_t = r$, $t \in \mathbb{R}^+$, a standard arbitrage argument shows that the price $P(t,T)$ of the bond is given by

$$P(t,T) = e^{-r(T-t)}, \quad 0 \leq t \leq T.$$  

Indeed, if $P(t,T) > e^{-r(T-t)}$ we could issue a bond at the price $P(t,T)$ and invest this amount at the compounded risk free rate $r$, which would yield $P(t,T)e^{r(T-t)} > 1$ at time $T$.

On the other hand, if $P(t,T) < e^{-r(T-t)}$ we could borrow $P(t,T)$ at the rate $r$ to buy a bond priced $P(t,T)$. At maturity time $T$ we would receive $\$1$ and refund only $P(t,T)e^{r(T-t)} < 1$.

**Deterministic short rates**

Similarly to the above, when the short term interest rate process $(r(t))_{t \in \mathbb{R}^+}$ is a deterministic function of time, a similar argument shows that

$$P(t,T) = e^{-\int_t^T r(s)ds}, \quad 0 \leq t \leq T. \quad (13.10)$$

**Stochastic short rates**

In case $(r_t)_{t \in \mathbb{R}^+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$-adapted random process the formula (13.10) is no longer valid as it relies on future information, and we replace it with

$$P(t,T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (13.11)$$

under a risk-neutral probability measure $\mathbb{P}^*$. It is natural to write $P(t,T)$ as a conditional expectation under a martingale measure, as the use of conditional expectation helps to “filter out” the (random/unknown) future information past time $t$ contained in $\int_t^T r_s ds$. The expression (13.11) makes sense as the “best possible estimate” of the future quantity $e^{-\int_t^T r_s ds}$ in mean square sense, given information known up to time $t$. 

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Coupon bonds

Pricing bonds with non-zero coupon is not difficult since in general the amount and periodicity of coupons are deterministic.* In the case of the succession of coupon payments $c_1, c_2, \ldots, c_n$ at times $T_1, T_2, \ldots, T_n \in (t, T]$, another application of the above absence of arbitrage argument shows that the price $P_c(t, T)$ of the coupon bond with discounted coupon payments is given by

$$P_c(t, T) := \mathbb{E}^* \left[ \sum_{k=1}^n c_k e^{-\int_t^{T_k} r_s ds} \mid \mathcal{F}_t \right] + \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right]$$

$$= \sum_{k=1}^n c_k \mathbb{E}^* \left[ e^{-\int_t^{T_k} r_s ds} \mid \mathcal{F}_t \right] + P_0(t, T)$$

$$= P_0(t, T) + \sum_{k=1}^n c_k P_0(t, T_k), \quad 0 \leq t \leq T_1,$$

which represents the present value at time $t$ of future $\$1$ receipts at times $T_1, T_3, \ldots, T_n$, in addition to a terminal $\$1$ payment.

In the case of a constant coupon rate $c$ paid at regular time intervals $\tau = T_{k+1} - T_k$, $k = 0, 1, \ldots, n$, with $T_0 = t$ and constant deterministic short rate $r$, we find

$$P_c(t, T) = e^{-rn\tau} + c \sum_{k=1}^n e^{(T_k - t)r}$$

$$= e^{-rn\tau} + e^{-k\tau r}$$

$$= e^{-rn\tau} + \frac{e^{-r\tau} - e^{-r(n+1)\tau}}{1 - e^{-r\tau}}.$$

In the case of a continuous-time coupon rate $c$ we find

$$P_c(t, T) = P_0(t, T) + c \int_t^T P_0(t, u) du$$

$$= e^{-r(T-t)} + c \int_0^{T-t} e^{-ru} du$$

$$= e^{-r(T-t)} + \frac{c}{r} (1 - e^{-r(T-t)}),$$

$$= \frac{c}{r} + \frac{r - c}{r} e^{-r(T-t)}, \quad 0 \leq t \leq T,$$

* However, coupon default cannot be excluded.

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see also Figure 13.7 below.

In the sequel, we will mostly consider zero-coupon bonds priced as \( P(t, T) = P_0(t, T), \ 0 \leq t \leq T. \)

**Martingale property of discounted bond prices**

The following proposition shows that Assumption (A) of Chapter 12 is satisfied, in other words, the bond price process \( t \mapsto \tilde{P}(t, T) \) can be used as a numéraire.

**Proposition 13.1.** The discounted bond price process

\[
\tilde{P}(t, T) = e^{-\int_0^t r_s \, ds} P(t, T)
\]

is a martingale under \( \mathbb{P}^*. \)

**Proof.** By (13.11) we have

\[
\tilde{P}(t, T) = e^{-\int_0^t r_s \, ds} P(t, T)
\]

\[
= e^{-\int_0^t r_s \, ds} \mathbb{E}^* \left[ e^{-\int_t^T r_s \, ds} \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_0^t r_s \, ds} e^{-\int_t^T r_s \, ds} \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_0^T r_s \, ds} \bigg| \mathcal{F}_t \right],
\]

and this suffices to conclude since by the “tower property” (18.38) of conditional expectations, any process \( (X_t)_{t \in \mathbb{R}_+} \) of the form \( t \mapsto X_t := \mathbb{E}^*[F \mid \mathcal{F}_t], \ F \in L^1(\Omega), \) is a martingale, see also Relation (6.1). \( \square \)

**Bond pricing PDE**

We assume from now on that the underlying short rate process is solution to the stochastic differential equation

\[
dr_t = \mu(t, r_t) \, dt + \sigma(t, r_t) \, dB_t \tag{13.12}
\]

where \( (B_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion under \( \mathbb{P}^*. \) Note that specifying the dynamics of \( (r_t)_{t \in \mathbb{R}_+} \) under the historical probability measure \( \mathbb{P} \) will also lead to a notion of market price of risk (MPoR) for the modeling of short rates.

Since all solutions of stochastic differential equations such as (13.12) have the Markov property, cf. e.g. Theorem V-32 of [Pro04], the arbitrage price \( P(t, T) \) can be rewritten as a function \( F(t, r_t) \) of \( r_t \), i.e.

\[
\mathcal{O}
\]
Given that

\[ P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid r_t \right] = F(t, r_t), \]

and depends on \( r_t \) only instead of depending on all information available in \( \mathcal{F}_t \) up to time \( t \), meaning that the pricing problem can now be formulated as a search for the function \( F(t, x) \).

**Proposition 13.2.** (Bond pricing PDE). Consider a short rate \((r_t)_{t \in \mathbb{R}_+}\) modeled by a diffusion equation of the form

\[ dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dB_t. \]

The bond pricing PDE for \( P(t, T) = F(t, r_t) \) is written as

\[ x F(t, x) = \frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x), \quad (13.13) \]

\( t \in \mathbb{R}_+, \ x \in \mathbb{R} \), subject to the terminal condition

\[ F(T, x) = 1, \quad x \in \mathbb{R}. \quad (13.14) \]

**Proof.** By Itô’s formula we have

\[
d \left( e^{-\int_0^t r_s ds} P(t, T) \right) = -r_t e^{-\int_0^t r_s ds} P(t, T)dt + e^{-\int_0^t r_s ds} dP(t, T) \\
= -r_t e^{-\int_0^t r_s ds} F(t, r_t)dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\
= -r_t e^{-\int_0^t r_s ds} F(t, r_t)dt + e^{-\int_0^t r_s ds} \left[ \frac{\partial F}{\partial r_t}(t, r_t)(\mu(t, r_t)dt + \sigma(t, r_t)dB_t) \right. \\
+ e^{-\int_0^t r_s ds} \left( \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial r_t^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt \\
= e^{-\int_0^t r_s ds} \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t)dB_t \\
+ e^{-\int_0^t r_s ds} \left( -r_t F(t, r_t) + \mu(t, r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt. \quad (13.15) \]

Given that \( t \mapsto e^{-\int_0^t r_s ds} P(t, T) \) is a martingale, the above expression (13.15) should only contain terms in \( dB_t \) (cf. Corollary II-1, page 72 of [Pro04]), and all terms in \( dt \) should vanish inside (13.15). This leads to the identities

\[ xF(t, x) = \frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x), \]

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\[
\begin{aligned}
  r_t F(t, r_t) &= \mu(t, r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \\
  d \left( e^{-\int_0^t r_s ds} P(t, T) \right) &= e^{-\int_0^t r_s ds} \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t) dB_t,
\end{aligned}
\]

which recover (13.13). Condition (13.14) is due to the fact that \( P(T, T) = 1 \).

In the case of an interest rate process modeled by (13.7) we have

\[
\mu(t, x) = \eta(t) + \lambda(t) x \quad \text{and} \quad \sigma(t, x) = \sqrt{\delta(t) + \gamma(t) x},
\]

hence (13.13) yields the (time dependent) affine PDE

\[
x F(t, x) = \frac{\partial F}{\partial t}(t, x) + (\eta(t) + \lambda(t) x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} (\delta(t) + \gamma(t) x) \frac{\partial^2 F}{\partial x^2}(t, x),
\]

(13.17)

\( t \in \mathbb{R}_+, x \in \mathbb{R} \). By (13.16a), the above proposition also shows that

\[
\frac{dP(t, T)}{P(t, T)} = \frac{1}{P(t, T)} d \left( e^{\int_0^t r_s ds} e^{-\int_0^t r_s ds} P(t, T) \right) = \frac{r_t P(t, T) dt + e^{\int_0^t r_s ds} \left( e^{-\int_0^t r_s ds} P(t, T) \right)}{P(t, T)}
\]

\[
= r_t dt + \frac{1}{F(t, r_t)} \frac{\partial F}{\partial x}(t, r_t) \sigma(t, r_t) dB_t
\]

(13.18)

In the Vasicek case

\( dr_t = (a - br_t) dt + \sigma dW_t \),

the bond price takes the form

\[
F(t, r_t) = P(t, T) = e^{A(T-t) + r_tC(T-t)}
\]

where \( A(\cdot) \) and \( C(\cdot) \) are functions of time, cf. (13.25) below, and (13.18) yields

\[
\frac{dP(t, T)}{P(t, T)} = r_t dt - \frac{\sigma}{b} \left( 1 - e^{-\frac{(T-t)b}{b}} \right) dW_t,
\]

(13.19)
since \( F(t, x) = e^{A(T-t)+xC(T-t)} \).

Note that more generally, all affine short rate models as defined in Relation (13.7), including the Vasicek model, will yield a bond pricing formula of the form
\[
P(t, T) = e^{A(T-t)+r_tC(T-t)},
\]
cf. e.g. § 3.2.4. of [BM06].

**Probabilistic solution of the Vasicek PDE**

Next, we solve the PDE (13.13), written with \( \mu(t, x) = a-bx \) and \( \sigma(t, x) = \sigma \) in the Vasicek [Vaš77] model
\[
dr_t = (a-bx)dt + \sigma dB_t
\]
as
\[
\begin{cases}
  xF(t, x) = \frac{\partial F}{\partial t}(t, x) + (a-bx) \frac{\partial F}{\partial x}(t, x) + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2}(t, x), \\
  F(T, x) = 1,
\end{cases}
\]
by a direct computation of the conditional expectation
\[
F(t, r_t) = P(t, T) = \mathbb{E}^r \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right].
\]

See also Exercise 13.6 for a bond pricing formula in the CIR model.

**Proposition 13.3.** The zero coupon bond price in the Vasicek model (13.20) can be expressed as
\[
P(t, T) = e^{A(T-t)+r_tC(T-t)}, \quad 0 \leq t \leq T,
\]
where \( A(x) \) and \( C(x) \) are functions of time to maturity given by
\[
C(x) := -\frac{1}{b} (1-e^{-bx}),
\]
and
\[
A(x) := \frac{4ab - 3\sigma^2}{4b^2} + \frac{\sigma^2 - 2ab}{2b^2} x + \frac{\sigma^2 - ab}{b^3} e^{-bx} - \frac{\sigma^2}{4b^3} e^{-2bx}
\]
\[
= - \left( \frac{a}{b} - \frac{\sigma^2}{2b^2} \right) (x + C(x)) - \frac{\sigma^2}{4b} C^2(x), \quad x \geq 0.
\]

**Proof.** Recall that in the Vasicek model, the short rate \( (r_t)_{t \in \mathbb{R}_+} \) solution of (13.20) has the expression

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\[ r_t = g(t) + \int_0^t h(t, s) dB_s = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \]

where \( g \) and \( h \) are the deterministic functions

\[ g(t) := r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}), \quad t \in \mathbb{R}_+, \]

and

\[ h(t, s) := \sigma e^{-(t-s)b}, \quad 0 \leq s \leq t, \]

are deterministic functions. Using the fact that Wiener integrals are Gaussian random variables and the Gaussian moment generating function, we have

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]
\]
\[
= \mathbb{E}^* \left[ e^{-\int_t^T (g(s) + \int_0^s h(s, u) dB_u) ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T g(s) ds \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_0^s h(s, u) dB_u ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T g(s) ds - \int_0^T \int_{\max(u, t)}^T h(s, u) dB_u \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_{\max(u, t)}^T h(s, u) dB_u ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T g(s) ds - \int_0^T \int_{\max(u, t)}^T h(s, u) dB_u \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_{\max(u, t)}^T h(s, u) dB_u ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T g(s) ds - \int_0^T \int_{\max(u, t)}^T h(s, u) dB_u \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_{\max(u, t)}^T h(s, u) dB_u ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T g(s) ds - \int_0^T \int_{\max(u, t)}^T h(s, u) dB_u \right) \mathbb{E}^* \left[ e^{-\int_t^T \int_{\max(u, t)}^T h(s, u) dB_u ds} \middle| \mathcal{F}_t \right]
\]
\[
= \exp \left( -\int_t^T (r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs})) ds - \sigma \int_0^T e^{-(s-u)b} dB_u \right)
\]
\[
\times \exp \left( \frac{\sigma^2}{2} \int_t^T \left( \int_u^T e^{-(s-u)b} ds \right)^2 du \right)
\]
\[
= \exp \left( -\int_t^T (r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs})) ds - \sigma \int_0^T e^{-(T-t)b} \int_0^t e^{-(t-u)b} dB_u \right)
\]
\[
\times \exp \left( \frac{\sigma^2}{2} \int_t^T e^{2bu} \left( \frac{e^{-bu} - e^{-bT}}{b} \right)^2 du \right)
\]
\[
= \exp \left( -\int_t^T (r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs})) ds - \sigma \int_0^T e^{-(T-t)b} \int_0^t e^{-(t-u)b} dB_u \right)
\]
\[
\times \exp \left( \frac{\sigma^2}{2} \int_t^T e^{2bu} \left( \frac{e^{-bu} - e^{-bT}}{b} \right)^2 du \right)
\]

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\[ = e^{A(T-t)+r_tC(T-t)}, \quad (13.25) \]

where \( A(x) \) and \( C(x) \) are the functions given by (13.23) and (13.24).

**Analytical solution of the Vasicek PDE**

In order to solve the PDE (13.21) analytically, we may look for a solution of the form

\[
F(t, x) = e^{A(T-t)+xC(T-t)}, \quad (13.26)
\]

where \( A(\cdot) \) and \( C(\cdot) \) are functions to be determined under the conditions \( A(0) = 0 \) and \( C(0) = 0 \). Substituting (13.26) into the PDE (13.13) with the Vasicek coefficients \( \mu(t, x) = (a-bx) \) and \( \sigma(t, x) = \sigma \) shows that

\[
x e^{A(T-t)+xC(T-t)} = -(A'(T-t) - xC'(T-t)) e^{A(T-t)+xC(T-t)} + (a-bx)C(T-t) e^{A(T-t)+xC(T-t)} + \frac{1}{2} \sigma^2 C^2(T-t) e^{A(T-t)+xC(T-t)},
\]

i.e.

\[
x = -A'(T-t) + xC'(T-t) + (a-bx)C(T-t) + \frac{1}{2} \sigma^2 C^2(T-t).
\]

By identification of terms for \( x = 0 \) and \( x \neq 0 \), this yields the system of Riccati and linear differential equations

\[
\begin{align*}
A'(s) &= aC(s) + \frac{\sigma^2}{2} C^2(s) \\
C'(s) &= 1 + bC(s),
\end{align*}
\]

which can be solved to recover the above value of \( P(t, T) = F(t, r_t) \) via

\[
C(s) = \frac{1}{b} (1 - e^{-bs})
\]

and

\[
A(t) = A(0) + \int_0^t A'(s) \, ds \\
= \int_0^t \left( aC(s) + \frac{\sigma^2}{2} C^2(s) \right) \, ds \\
= \int_0^t \left( \frac{a}{b} (1 - e^{-bs}) + \frac{\sigma^2}{2b^2} (1 - e^{-bs})^2 \right) \, ds \\
= \frac{a}{b} \int_0^t (1 - e^{-bs}) \, ds + \frac{\sigma^2}{2b^2} \int_0^t (1 - e^{-bs})^2 \, ds
\]
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\[
F(t, r_t) = \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2}t + \frac{\sigma^2 - ab}{b^3}e^{-bt} - \frac{\sigma^2}{4b^3}e^{-2bt}, \quad t \in \mathbb{R}.
\]

Vasicek bond price simulations

In this section we consider again the Vasicek model, in which the short rate \((r_t)_{t \in \mathbb{R}}\) is solution to (13.1). Figure 13.6 presents a random simulation of \(t \mapsto P(t, T)\) in the Vasicek model with \(\sigma = 10\%\), \(r_0 = 5\%\), \(b = 0.5\), and \(a = 0.025\). The graph of the corresponding deterministic zero-coupon bond price obtained for \(a = b = \sigma = 0\) is also shown on the Figure 13.6.

![Graph of t ↦ F(t, r_t) = P(t, T) vs t ↦ e^{−r_0(T−t)}.](image)

Figure 13.7 presents market price data for a coupon bond with coupon rate \(c = 6.25\%\).

![Bond price graph with maturity 01/18/08 and coupon rate 6.25%.](image)
Zero-coupon bond price and yield data

The following zero-coupon bond price data was downloaded at EMMA from the Municipal Securities Rulemaking Board.

ORANGE CNTY CALIF PENSION OBLIG CAP APPREC-TAXABLE-REF-SER A (CA)
CUSIP: 68428LBB9
Dated Date: 06/12/1996 (June 12, 1996)
Maturity Date: 09/01/2016 (September 1st, 2016)
Interest Rate: 0.0%
Principal Amount at Issuance: $26,056,000
Initial Offering Price: 19.465

```
library(quantmod)
bondprice <- read.table("bond_data_R.txt", col.names = c("Date","HighPrice","LowPrice","HighYield","LowYield","Count","Amount"))
head(bondprice)
time <- as.POSIXct(bondprice$Date, format = "%Y-%m-%d")
price <- xts(x = bondprice$HighPrice, order.by = time)
yield <- xts(x = bondprice$HighYield, order.by = time)
chartSeries(price, up.col="blue", theme="white")
chartSeries(yield, up.col="blue", theme="white")
```

<table>
<thead>
<tr>
<th>Date</th>
<th>HighPrice</th>
<th>LowPrice</th>
<th>HighYield</th>
<th>LowYield</th>
<th>Count</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2016-01-13</td>
<td>99.082</td>
<td>98.982</td>
<td>1.666</td>
<td>1.501</td>
<td>2</td>
<td>20000</td>
</tr>
<tr>
<td>2 2015-12-29</td>
<td>99.183</td>
<td>99.183</td>
<td>1.250</td>
<td>1.250</td>
<td>1</td>
<td>10000</td>
</tr>
<tr>
<td>3 2015-12-21</td>
<td>97.952</td>
<td>97.952</td>
<td>3.014</td>
<td>3.014</td>
<td>1</td>
<td>10000</td>
</tr>
<tr>
<td>4 2015-12-17</td>
<td>99.141</td>
<td>98.550</td>
<td>2.123</td>
<td>1.251</td>
<td>5</td>
<td>610000</td>
</tr>
<tr>
<td>5 2015-12-07</td>
<td>98.770</td>
<td>98.770</td>
<td>1.714</td>
<td>1.714</td>
<td>2</td>
<td>10000</td>
</tr>
<tr>
<td>6 2015-12-04</td>
<td>98.363</td>
<td>98.118</td>
<td>2.628</td>
<td>2.280</td>
<td>2</td>
<td>10000</td>
</tr>
</tbody>
</table>

![graph](image.png)

Fig. 13.8: Orange Cnty Calif bond prices.

The next Figure 13.9 plots the bond yield $y(t, T)$ defined as

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\[ y(t, T) = -\frac{\log P(t, T)}{T - t}, \quad \text{or} \quad P(t, T) = e^{-(T-t)y(t, T)}, \quad 0 \leq t \leq T. \]

![Fig. 13.9: Orange Cnty Calif bond yields.](image)

**Bond pricing in the Dothan model**

In the Dothan [Dot78] model, the short term interest rate process \( (r_t)_{t \in \mathbb{R}^+} \) is modeled according to a geometric Brownian motion

\[ dr_t = \mu r_t dt + \sigma r_t dB_t, \quad (13.27) \]

where the volatility \( \sigma > 0 \) and the drift \( \mu \in \mathbb{R} \) are constant parameters and \( (B_t)_{t \in \mathbb{R}^+} \) is a standard Brownian motion. In this model the short term interest rate \( r_t \) remains always positive, while the proportional volatility term \( \sigma r_t \) accounts for the sensitivity of the volatility of interest rate changes to the level of the rate \( r_t \).

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero-coupon bond price

\[ P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \]

For convenience of notation we let \( p = 1 - 2\mu/\sigma^2 \) and rewrite \((13.27)\) as

\[ dr_t = (1-p) \frac{\sigma^2}{2} r_t dt + \sigma r_t dB_t, \]

with solution

\[ r_t = r_0 e^{\sigma B_t - p\sigma^2 t/2}, \quad t \in \mathbb{R}^+. \quad (13.28) \]

By the Markov property of \( (r_t)_{t \in \mathbb{R}^+} \), the bond price \( P(t, T) \) is a function \( F(t, r_t) \) of \( r_t \) and time \( t \in [0, T] \):
\[ P(t,T) = F(t,r_t) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \] (13.29)

By computation of the conditional expectation (13.29) using (10.8) we easily obtain the following result, cf. Proposition 1.2 of [PP11], where the function \( \theta(v,t) \) is defined in (10.4).

**Proposition 13.4.** The zero-coupon bond price \( P(t,T) = F(t,r_t) \) is given for all \( p \in \mathbb{R} \) by

\[ F(t,x) = e^{-\sigma^2 p^2 T - T/8} \int_0^\infty \int_0^\infty e^{-ux} \exp \left( -2 \frac{(1 + z^2)}{\sigma^2} \right) \theta \left( \frac{4z \sigma^2}{\sigma^2 y}, \frac{\sigma^2 (T-t)}{4} \right) \frac{du}{u} \frac{dz}{z^{p+1}}, \quad x > 0. \tag{13.30} \]

**Proof.** By Proposition 10.1, cf. [Yor92], Proposition 2, the probability distribution of the time integral

\[ e^{-r_t w_{T-t} - \sigma^2 s/2 ds} \]

is given by

\[ P \left( \int_0^{T-t} e^{-\sigma B_s - p \sigma^2 s/2 ds} \right) = \int_{-\infty}^\infty P \left( \int_0^t e^{-\sigma B_s - p \sigma^2 s/2 ds} \right) dy \]

\[ = \frac{\sigma}{2} \int_{-\infty}^\infty e^{-p \sigma z/2 - p \sigma^2 t/8} \exp \left( -2 \frac{1 + e^\sigma z}{\sigma^2 y} \right) \theta \left( \frac{4e^\sigma z/2}{\sigma^2 y}, \frac{\sigma^2 (T-t)}{4} \right) \frac{dy}{y} \frac{dz}{z^{p+1}}, \quad y > 0, \]

where the exchange of integrals is justified by the Fubini theorem and the nonnegativity of integrands. Hence by (10.8) and (13.28) we find

\[ F(t,r_t) = P(t,T) \]

\[ = \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \exp \left( -r_t \int_t^T e^{\sigma (B_s - B_t) - \sigma^2 p(s-t)/2 ds} \right) \mid \mathcal{F}_t \right] \]

\[ = \mathbb{E}^* \left[ \exp \left( -x \int_t^T e^{\sigma (B_s - B_t) - \sigma^2 p(s-t)/2 ds} \right) \right]_{x=r_t} \]

\[ = \mathbb{E}^* \left[ \exp \left( -x \int_0^{T-t} e^{\sigma B_s - \sigma^2 ps/2 ds} \right) \right]_{x=r_t} \]

\[ = \int_0^\infty e^{-rx} \mathbb{P} \left( \int_0^{T-t} e^{\sigma B_s - \sigma^2 ps/2 ds} \right) dy \]
The zero-coupon bond price $P(t, T) = F(t, r_t)$ in the Dothan model can also be written for all $p \in \mathbb{R}$ as

$$F(t, x) = \frac{(2x)^{p/2}}{2\pi^{p/2}} \int_0^\infty u e^{-\sigma^2(p^2+u^2)t/8} \sinh(\pi u) \left| \Gamma \left( \frac{p}{2} + i \frac{u}{2} \right) \right|^2 K_{iu} \left( \frac{\sqrt{8x}}{\sigma} \right) du$$

$$+ \frac{(2x)^{p/2}}{\sigma^p} \sum_{k=0}^{\infty} \frac{2(p-2k)^+}{k!(p-k)!} e^{\sigma^2 k(p-k)/2} K_{p-2k} \left( \frac{\sqrt{8x}}{\sigma} \right), \quad x > 0, \; t > 0,$$

cf. Corollary 2.2 of [PP10], see also [PU13] for numerical computations. Zero-coupon bond prices in the Dothan model can also be computed by the conditional expression

$$\mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \right] = \int_0^\infty \mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \mid r_T = z \right] d\mathbb{P}(r_T \leq z),$$

where $r_T$ has the lognormal distribution

$$d\mathbb{P}(r_T \leq z) = d\mathbb{P}(r_0 e^{\sigma BT - \sigma^2 T/2} \leq z) = \frac{1}{z\sqrt{2\pi \sigma^2 T}} e^{-\left(\sigma^2 T/2 + \log(z/r_0)\right)^2/(2\sigma^2 T)}.$$

In Proposition 13.5 we note that the conditional Laplace transform

$$\mathbb{E} \left[ \exp \left( -\int_0^T r_t dt \right) \mid r_T = z \right]$$

cf. (13.35) above, can be computed by a closed-form integral expression based on the modified Bessel function of the second kind

$$K_{\zeta}(z) := \frac{z^\zeta}{2^{\zeta+1}} \int_0^\infty \exp \left( -u - \frac{z^2}{4u} \right) \frac{du}{u^{\zeta+1}}, \quad \zeta \in \mathbb{R}, \quad z \in \mathbb{C}, \quad (13.32)$$

cf. e.g. [Wat95] page 183, provided that the real part $\Re(z^2)$ of $z^2 \in \mathbb{C}$ is positive.

**Proposition 13.5.** [PY16], Proposition 4.1. Taking $r_0 = 1$, for all $\lambda, z > 0$ we have

$$\mathbb{E} \left[ \exp \left( -\lambda \int_0^T r_s ds \mid r_T = z \right) \right] = 4e^{-\sigma^2 T/8} \frac{\pi^{3/2} \sigma^2 p(z)}{\sqrt{T}} K_1 \left( \frac{\sqrt{8\lambda} \sqrt{1 + 2\sqrt{z} \cosh \xi + z} / \sigma}{\sqrt{1 + 2\sqrt{z} \cosh \xi + z}} \right) d\xi. \quad (13.33)$$

\[\square\]
Note however that (13.33) fails for small values of $T$, and for this reason the integral can be estimated by a gamma approximation, cf. (13.34) below. Under the gamma approximation we can approximate the conditional bond price on the Dothan short rate $r_t$ as

$$
\mathbb{E} \left[ \exp \left( -\lambda \int_0^T r_t \, dt \right) \mid r_T = z \right] \simeq (1 + \lambda \theta(z))^{-\nu(z)},
$$

where the parameters $\nu(z)$ and $\theta(z)$ are determined by conditional moment fitting to a gamma distribution, as

$$
\theta(z) := \frac{\operatorname{Var}[\Lambda_T \mid S_T = z]}{\mathbb{E}[\Lambda_T \mid S_T = z]}, \quad \nu(z) := \frac{\left( \mathbb{E}[\Lambda_T \mid S_T = z] \right)^2}{\operatorname{Var}[\Lambda_T \mid S_T = z]} = \frac{\mathbb{E}[\Lambda_T \mid S_T = z]}{\theta},
$$

cf. [PY16], which yields

$$
\mathbb{E} \left[ \exp \left( -\lambda \int_0^T r_s \, ds \right) \right] \simeq \int_0^\infty (1 + \lambda \theta(z))^{-\nu(z)} \, d\mathbb{P}(r_T \leq z). \quad (13.34)
$$

Note that $\theta(z)$ is known in physics as the Fano factor which measures the dispersion of the probability distribution of $\Lambda_T$ given that $S_T = z$. Figures 13.10 shows that the stratified gamma approximation (13.34) matches the Monte Carlo estimate, while the use of the integral expressions (13.31) and (13.33) leads to numerical instabilities.

Fig. 13.10: Approximation of Dothan bond prices $t \mapsto F(t, x)$ with $\sigma = 0.3$ and $T = 10$.

Related computations for yield options in the CIR model can also be found in [PP17].

**Path integrals in option pricing**

In physics, the Feynman path integral

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\[ \psi(y, t) := \int_{x(0)=x, \ x(t)=y} Dx(\cdot) \exp \left( \frac{i}{\hbar} S(x(\cdot)) \right) \]

where \( \hbar \) is the Planck constant and \( S(x(\cdot)) \) is the action given by the Lagrangian

\[ L(x(s), \dot{x}(s), s) := \frac{1}{2} m (\dot{x}(s))^2 - V(x(s)), \]

as

\[ S(x(\cdot)) = \int_0^t L(x(s), \dot{x}(s), s) \, ds = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 - V(x(s)) \right) \, ds \]

\[ \approx \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} - V(x(t_{i-1})) \right) \Delta t_i, \]

solves the Schrödinger equation

\[ i \hbar \frac{\partial \psi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(x, t) + V(x(t)) \psi(x, t). \]

After the Wick rotation \( t \mapsto -it \), the function

\[ \phi(y, t) := \int_{x(0)=x, \ x(t)=y} Dx(\cdot) \exp \left( -\frac{1}{\hbar} S(x(\cdot)) \right) \]

where \( S(x(\cdot)) \) is the action

\[ S(x(\cdot)) := \int_0^t L(x(s), \dot{x}(s), s) \, ds = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 + V(x(s)) \right) \, ds \]

\[ \approx \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} + V(x(t_{i-1})) \right) \Delta t_i, \]

solves the heat equation

\[ \hbar \frac{\partial \phi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2}(x, t) + V(x(t)) \phi(x, t). \]

Given the action

\[ S(x(\cdot)) = \int_0^t \left( \frac{1}{2} m (\dot{x}(s))^2 + V(x(s)) \right) \, ds \]

\[ \approx \sum_{i=1}^N \left( \frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} + V(x(t_{i-1})) \right) \Delta t_i, \]

we can rewrite the Euclidean path integral as
This type of path integral computation is particularly useful for bond pricing, as (13.35) can be interpreted as the price of a bond with short term interest rate process \((r_t)_{t \in \mathbb{R}^+} := (V(B_t)))_{t \in \mathbb{R}^+}\) conditionally to the value of the endpoint \(B_t = y\), cf. (13.33) below. It can also be useful for exotic option pricing, cf. Chapter 10, and for risk management. The path integral (13.35) can be estimated either by closed-form expressions using Partial Differential Equations (PDEs) or probability densities, by approximations such as (conditional) Moment matching, or by Monte Carlo estimation, from the paths of a Brownian bridge as shown in Figure 13.11.

\begin{align*}
\phi(y, t) &= \int_{x(0) = x, \ x(t) = y} \mathcal{D}x(\cdot) \exp \left( -\frac{1}{\bar{h}} S(x(\cdot)) \right) \\
&= \int_{x(0) = x, \ x(t) = y} \mathcal{D}x(\cdot) \exp \left( -\frac{1}{2\bar{h}} \sum_{i=1}^{N} \frac{(x(t_i) - x(t_{i-1}))^2}{2\Delta t_i} - \frac{1}{\bar{h}} \sum_{i=1}^{N} V(x(t_{i-1})) \right) \\
&= \mathbb{E}^* \left[ \exp \left( -\frac{1}{\bar{h}} \int_{0}^{t} V(B_s)ds \right) \bigg| B_0 = x, B_t = y \right].
\end{align*}

Fig. 13.11: Brownian bridge.

\section*{13.4 Forward Rates}

A forward interest rate contract (or Forward Rate Agreement, FRA) gives to its holder the possibility to lock an interest rate denoted by \(f(t, T, S)\) at present time \(t\) for a loan to be delivered over a future period of time \([T, S]\), with \(t \leq T \leq S\). The rate \(f(t, T, S)\) is called a forward interest rate. When \(T = t\), the spot forward rate \(f(t, t, T)\) is coincides with the \underline{yield}, see (13.38).

Figure 13.12 presents a typical yield curve on the LIBOR (London Interbank Offered Rate) market with \(t = 07\) May 2003.
Maturity transformation, i.e., the ability to transform short term borrowing (debt with short maturities, such as deposits) into long term lending (credits with very long maturities, such as loans), is among the roles of banks. Profitability is then dependent on the difference between long rates and short rates.

Another example of market data is given in the next Figure 13.13, in which the red and blue curves refer respectively to July 21 and 22 of year 2011.

Long maturities usually correspond to higher rates as they carry an increased risk. The dip observed with short maturities can correspond to a lower motivation to lend/invest in the short term.
Forward rates from bond prices

Let us determine the arbitrage or “fair” value of the forward interest rate \( f(t, T, S) \) by implementing the Forward Rate Agreement using the instruments available in the market, which are bonds priced at \( P(t, T) \) for various maturity dates \( T > t \).

The loan can be realized using the available instruments (here, bonds) on the market, by proceeding in two steps:

1) At time \( t \), borrow the amount \( P(t, S) \) by issuing (or short selling) one bond with maturity \( S \), which means refunding $1 at time \( S \).

2) Since the money is only needed at time \( T \), the rational investor will invest the amount \( P(t, S) \) over the period \([t, T]\) by buying a (possibly fractional) quantity \( P(t, S)/P(t, T) \) of a bond with maturity \( T \) priced \( P(t, T) \) at time \( t \). This will yield the amount

\[
$1 \times \frac{P(t, S)}{P(t, T)}
\]

at time \( T > 0 \).

As a consequence, the investor will actually receive \( P(t, S)/P(t, T) \) at time \( T \), to refund $1 at time \( S \).

The corresponding forward rate \( f(t, T, S) \) is then given by the relation

\[
\frac{P(t, S)}{P(t, T)} \exp ((S - T)f(t, T, S)) = 1, \quad 0 \leq t \leq T \leq S, \quad (13.36)
\]

where we used exponential compounding, which leads to the following definition (13.37).

**Definition 13.6.** The forward rate \( f(t, T, S) \) at time \( t \) for a loan on \([T, S]\) is given by

\[
f(t, T, S) = \frac{\log P(t, T) - \log P(t, S)}{S - T}. \quad (13.37)
\]

The spot forward rate \( f(t, t, T) \) coincides with the yield \( y(t, T) \), with
Forward Rate Modeling

\[
f(t, t, T) = y(t, T) = -\frac{\log P(t, T)}{T - t}, \quad \text{or} \quad P(t, T) = e^{-(T-t)f(t,t,T)},
\]
\[0 \leq t \leq T.\]  

**Proposition 13.7.** The instantaneous forward rate \( f(t, T) = f(t, T, T) \) is defined by taking the limit of \( f(t, T, S) \) as \( S \searrow T \), satisfies

\[
f(t, T) := \lim_{S \searrow T} f(t, T, S) = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}. \tag{13.39}
\]

**Proof.** We have

\[
f(t, T) := \lim_{S \searrow T} f(t, T, S) \\
= -\lim_{S \searrow T} \frac{\log P(t, S) - \log P(t, T)}{S - T} \\
= -\lim_{\varepsilon \searrow 0} \frac{\log P(t, T + \varepsilon) - \log P(t, T)}{\varepsilon} \\
= -\frac{\partial \log P(t, T)}{\partial T} \\
= -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}.
\]

The above equation (13.39) can be viewed as a differential equation to be solved for \( \log P(t, T) \) under the initial condition \( P(T, T) = 1 \), which yields the following proposition.

**Proposition 13.8.** We have

\[
P(t, T) = \exp \left( -\int_t^T f(t, s) ds \right), \quad 0 \leq t \leq T. \tag{13.40}
\]

**Proof.** We check that

\[
\log P(t, T) = \log P(t, T) - \log P(t, t) = \int_t^T \frac{\partial \log P(t, s)}{\partial s} ds = -\int_t^T f(t, s) ds.
\]

Proposition 13.8 also shows that

\[
f(t, t, t) = f(t, t) \\
= \frac{\partial}{\partial T} \int_t^T f(t, s) ds \bigg|_{T=t}
\]
\[ \frac{\partial}{\partial T} \log P(t, T) \big|_{T=t} = - \frac{1}{P(t, T)} \frac{\partial}{\partial T} P(t, T) \big|_{T=t} = - \frac{\partial}{\partial T} E^* \left[ e^{-\int_t^T r_s \, ds} \bigg| F_t \right] \big|_{T=t} = E^* \left[ r_t e^{-\int_t^T r_s \, ds} \bigg| F_t \right] \big|_{T=t} = E^*[r_t \mid F_t] = r_t, \]

i.e. the short rate \( r_t \) can be recovered from the instantaneous forward rate as

\[ r_t = f(t, t) = \lim_{T\searrow t} f(t, T). \]

As a consequence of (13.36) and (13.40) the forward rate \( f(t, T, S) \), \( 0 \leq t \leq T \leq S \), can be recovered from (13.37) and the instantaneous forward rate \( f(t, s) \), as:

\[ f(t, T, S) = \frac{\log P(t, T) - \log P(t, S)}{S - T} = - \frac{1}{S - T} \left( \int_t^T f(t, s) \, ds - \int_t^S f(t, s) \, ds \right) = \frac{1}{S - T} \int_t^S f(t, s) \, ds, \quad 0 \leq t \leq T < S. \quad (13.41) \]

In particular, the spot forward rate or yield \( f(t, t, T) \) can be written as

\[ f(t, t, T) = - \frac{\log P(t, T)}{T - t} = \frac{1}{T - t} \int_t^T f(t, s) \, ds, \quad 0 \leq t < T. \quad (13.42) \]

Differentiation with respect to \( T \) of the above relation shows that the yield \( f(t, t, T) \) and the instantaneous forward rate \( f(t, s) \) are linked by the relation

\[ \frac{\partial f}{\partial T}(t, t, T) = - \frac{1}{(T - t)^2} \int_t^T f(t, s) \, ds + \frac{1}{T - t} f(t, T), \quad 0 \leq t < T, \]

from which it follows that
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\[
f(t, T) = \frac{1}{T-t} \int_t^T f(t, s) ds + (T-t) \frac{\partial f}{\partial T}(t, t, T)
\]
\[
= f(t, t, T) + (T-t) \frac{\partial f}{\partial T}(t, t, T), \quad 0 \leq t < T.
\]

Forward Vasicek rates

In this section we consider the Vasicek model, in which the short rate process is the solution (13.2) of (13.1) as illustrated in Figure 13.1.

In the Vasicek model, the forward rate is given by

\[
f(t, T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S-T}
\]
\[
= -r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))
\]
\[
= -\frac{\sigma^2 - 2ab}{2b^2}
\]
\[
- \frac{1}{S-T} \left( \left( \frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3} \right) (e^{-(S-t)b} - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (e^{-2(S-t)b} - e^{-2(T-t)b}) \right),
\]
and the spot forward rate, or yield, satisfies

\[
f(t, t, T) = -\log P(t, T) = -\frac{r_t C(T-t) + A(T-t)}{T-t}
\]
\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \left( \frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3} \right) (1 - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (1 - e^{-2(T-t)b}) \right),
\]
with the mean

\[
\mathbb{E}[f(t, t, T)]
\]
\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \left( \frac{\mathbb{E}[r_t]}{b} + \frac{\sigma^2 - ab}{b^3} \right) (1 - e^{-(T-t)b}) - \frac{\sigma^2}{4b^3} (1 - e^{-2(T-t)b}) \right)
\]
\[
= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t} \left( \frac{r_0}{b} e^{-bt} + \frac{a}{b^2} (1 - e^{-bt}) + \frac{\sigma^2 - ab}{b^3} (1 - e^{-(T-t)b}) \right)
\]
\[
- \frac{\sigma^2}{4b^3(T-t)} (1 - e^{-2(T-t)b}).
\]

In this model, the forward rate \( t \mapsto f(t, T, S) \) can be represented as in Figure 13.14, with \( a = 0.06, b = 0.1, \sigma = 0.1 \) and \( r_0 = \%1 \):
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Note that the forward rate curve $t \mapsto f(t, T, S)$ appears flat for small values of $t$, i.e. longer rates are more stable, while shorter rates show higher volatility or risk. Similar features can be observed in Figure 13.15 for the instantaneous short rate given by

$$f(t, T) := -\frac{\partial \log P(t, T)}{\partial T}$$

$$= r_t e^{-b(T-t)} + \frac{a}{b} (1 - e^{-b(T-t)}) - \frac{\sigma^2}{2b^2} (1 - e^{-b(T-t)})^2,$$

from which the relation $\lim_{T \searrow t} f(t, T) = r_t$ can be easily recovered. We can also evaluate the mean

$$\mathbb{E}[f(t, T)] = \mathbb{E}[r_t] e^{-b(T-t)} + \frac{a}{b} (1 - e^{-b(T-t)}) - \frac{\sigma^2}{2b^2} (1 - e^{-b(T-t)})^2$$

$$= \left( r_0 e^{-bT} + \frac{a}{b} (e^{-bT} - e^{-bT}) \right) + \frac{a}{b} (1 - e^{-bT}) - \frac{\sigma^2}{2b^2} (1 - e^{-bT})^2.$$

The instantaneous forward rate $t \mapsto f(t, T)$ can be represented as in Figure 13.15, with $a = 0.06$, $b = 0.1$, $\sigma = 0.1$ and $r_0 = \%1$:

Fig. 13.14: Forward rate process $t \mapsto f(t, T, S)$.

Fig. 13.15: Instantaneous forward rate process $t \mapsto f(t, T)$. 

This version: January 16, 2019
http://www.ntu.edu.sg/home/nprivault/index.html
Yield curve data

We refer to Chapter III-12 of [Cha14] on the R package “YieldCurve” [Gui15] for the following code and further details on yield curve and interest rate modeling using R.

```r
install.packages("YieldCurve")
require(YieldCurve)
data(FedYieldCurve)
first(FedYieldCurve,'3 month')
last(FedYieldCurve,'3 month')
mat.Fed=c(0.25,0.5,1,2,3,5,7,10)
n=50
plot(mat.Fed, FedYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest rates values")
title(main=paste("Federal Reserve yield curve observed at",time(FedYieldCurve[n], sep=" ")))
grid()
```

The next Figure 13.16 is plotted using this code* which is adapted from [http://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R](http://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R)

![Federal Reserve yield curves from 1982 to 2012.](image)

European Central Bank (ECB) data can be similarly obtained.

* Click to open or download.
The next Figure 13.17 represents the output of the above script.

Fig. 13.17: European Central Bank yield curves.*

Decreasing yield curves can occur when central banks attempt to limit inflation by tightening interest rates. In the next section we turn to the modeling of the market curves observed in Figure 13.17.

**LIBOR (London Interbank Offered) Rates**

Recall that the forward rate \( f(t, T, S) \), \( 0 \leq t \leq T \leq S \), is defined using exponential compounding, from the relation

\[
f(t, T, S) = -\frac{\log P(t, S) - \log P(t, T)}{S - T}.
\]

(13.45)

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (13.45). Other types of LIBOR rates include EURIBOR (European Interbank Offered Rates), HIBOR (Hong Kong Interbank Offered Rates), SHIBOR (Shanghai Interbank Offered Rates), SI-

* The animation works in Acrobat Reader on the entire pdf file.
BOR (Singapore Interbank Offered Rates), TIBOR (Tokyo Interbank Offered Rates), etc.

The forward LIBOR \( L(t, T, S) \) for a loan on \([T, S]\) is defined using linear compounding, \textit{i.e.} by replacing (13.45) with the relation

\[
1 + (S - T)L(t, T, S) = \frac{P(t, T)}{P(t, S)}, \quad 0 \leq t \leq T,
\]

which yields the following definition.

**Definition 13.9.** The forward LIBOR rate \( L(t, T, S) \) at time \( t \) for a loan on \([T, S]\) is given by

\[
L(t, T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right), \quad 0 \leq t \leq T < S. \tag{13.46}
\]

Note that (13.46) above yields the same formula for the (LIBOR) instantaneous forward rate

\[
L(t, T) = \lim_{S \searrow T} L(t, T, S)
\]

\[
= \lim_{S \searrow T} \frac{P(t, S) - P(t, T)}{(S - T) P(t, S)}
\]

\[
= \lim_{\varepsilon \searrow 0} \frac{P(t, T + \varepsilon) - P(t, T)}{\varepsilon P(t, T + \varepsilon)}
\]

\[
= \frac{1}{P(t, T)} \lim_{\varepsilon \searrow 0} \frac{P(t, T + \varepsilon) - P(t, T)}{\varepsilon}
\]

\[
= - \frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}
\]

\[
= - \frac{\partial \log P(t, T)}{\partial T},
\]

as in (13.39).

In addition, Relation (13.46) shows that the LIBOR rate can be viewed as a forward price \( \hat{X}_t = X_t / N_t \) with numéraire \( N_t = (S - T)P(t, S) \) and \( X_t = P(t, T) - P(t, S) \), according to Relation (12.7) of Chapter 12. As a consequence, from Proposition 12.4, the LIBOR rate \( (L(t, T, S))_{t \in [T, S]} \) is a martingale under the forward measure \( \hat{P} \) defined by

\[
\frac{d\hat{P}}{d\hat{P}^*} = \frac{1}{P(0, S)} e^{-\int_0^S r_t dt}.
\]
13.5 Forward Swap Rates

The first interest rate swap occurred in 1981 between the World Bank, which was interested in borrowing German Marks and Swiss Francs, and IBM, which already had large amounts of those currencies but needed to borrow U.S. dollars.

The vanilla interest rate swap makes it possible to exchange a sequence of variable forward rates \( f(T, T_k, T_{k+1}) \), \( k = 1, 2, \ldots, n - 1 \), against a fixed rate \( \kappa \) over a time period \([T_1, T_n]\). Over the succession of time intervals \([T_1, T_2], [T_2, T_3], \ldots, [T_{n-1}, T_n]\) defining a tenor structure, see Section 14.1 for details, the accumulation of such exchanges will generate a cumulative discounted cash flow

\[
\left( \sum_{k=1}^{n-1} e^{-\int_{T_k}^{T_{k+1}} r_s ds} \left( e^{(T_{k+1}-T_k)} f(T, T_k, T_{k+1}) - 1 \right) \right)
- \left( \sum_{k=1}^{n-1} \left( e^{(T_{k+1}-T_k)} - 1 \right) e^{-\int_{T_k}^{T_{k+1}} r_s ds} \right)
= \sum_{k=1}^{n-1} e^{-\int_{T_k}^{T_{k+1}} r_s ds} \left( e^{(T_{k+1}-T_k)} f(T, T_k, T_{k+1}) - e^{(T_{k+1}-T_k)} \kappa \right),
\]

at time \( T = T_0 \), in which we used simple (or linear) interest rate compounding.

This corresponds to a payer swap in which the swap holder receives the floating leg and pays the fixed leg \( \kappa \), whereas the holder of a seller swap receives the fixed leg \( \kappa \) and pays the floating leg.

The above cash flow is used to make the contract fair, and it can be priced at time \( t \) as

\[
\mathbb{E}^* \left[ \sum_{k=1}^{n-1} e^{-\int_{T_k}^{T_{k+1}} r_s ds} \left( e^{(T_{k+1}-T_k)} f(T, T_k, T_{k+1}) - e^{(T_{k+1}-T_k)} \kappa \right) \mid \mathcal{F}_T \right]
= \sum_{k=1}^{n-1} \left( e^{(T_{k+1}-T_k)} f(T, T_k, T_{k+1}) - e^{(T_{k+1}-T_k)} \kappa \right) \mathbb{E}^* \left[ e^{-\int_{T_k}^{T_{k+1}} r_s ds} \mid \mathcal{F}_T \right]
= \sum_{k=1}^{n-1} P(T, T_{k+1}) \left( e^{(T_{k+1}-T_k)} f(T, T_k, T_{k+1}) - e^{(T_{k+1}-T_k)} \kappa \right).
\]

The swap rate \( S(T, T_1, T_n) \) is by definition the value of the rate \( \kappa \) that makes the contract fair by making this cash flow vanish.
In the sequel we will replace exponential compounding with simple linear compounding. In this case, the discounted cash flow becomes

\[
\left( \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} f(T, T_k, T_{k+1}) \right) - \left( \sum_{k=1}^{n-1} \kappa(T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} \right)
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} (f(T, T_k, T_{k+1}) - \kappa),
\]

at time \( T = T_0 \), in which we used simple (or linear) interest rate compounding. This cash flow is used to make the contract fair, and it can be priced at time \( T \) as

\[
\mathbb{E}^* \left[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} (f(T, T_k, T_{k+1}) - \kappa) \bigg| \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) (f(T, T_k, T_{k+1}) - \kappa) \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) (f(T, T_k, T_{k+1}) - \kappa).
\]

The swap rate \( S(T, T_1, T_n) \) is by definition the value of the break-even rate \( \kappa \) that makes the contract fair by making this cash flow vanish, \( i.e. \)

\[
\sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) (f(T, T_k, T_{k+1}) - \kappa) = 0. \quad (13.47)
\]

The next Proposition 13.10 makes use of the annuity numéraire

\[
P(T, T_1, T_n) := \mathbb{E}^* \left[ \sum_{k=1}^{n-1} (T_{k+1} - T_k) e^{-\int_T^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) \mathbb{E}^* \left[ e^{-\int_T^{T_{k+1}} r_s ds} \bigg| \mathcal{F}_T \right]
\]

\[
= \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}), \quad 0 \leq T \leq T_1,
\]

which represents the present value at time \( T \) of future $1 receipts at times \( T_1, T_2, \ldots, T_n \), weighted by the lengths \( T_{k+1} - T_k \) of the time intervals \( (T_k, T_{k+1}] \), \( k = 1, 2, \ldots, n - 1 \).
The time intervals \((T_{k+1} - T_k)_{k=1,2,\ldots,n-1}\) in the definition (13.48) of the annuity numéraire can be replaced by coupon payments \((c_{k+1})_{k=1,2,\ldots,n-1}\) occurring at times \((T_{k+1})_{k=1,2,\ldots,n-1}\), in which case the annuity numéraire becomes

\[
P(T, T_1, T_n) := \mathbb{E}^* \left[ \sum_{k=1}^{n-1} c_{k+1} e^{-\int_T^{T_{k+1}} r_s \, ds} \big| \mathcal{F}_T \right] (13.49)
\]

which represents the value at time \(T\) of the future coupon payments discounted according to the bond prices \(P(T, T_{k+1})_{k=1,2,\ldots,n-1}\). This expression can also be used to define *amortizing swaps* in which the value of the notional decreases over time, or *accreting swaps* in which the value of the notional increases over time.

**LIBOR swap rates**

The LIBOR swap rate \(S(T, T_1, T_n)\) is defined by the same relation as (13.47) with the forward rate \(f(t, T_k, T_{k+1})\) replaced with the LIBOR rate \(L(t, T_k, T_{k+1})\), i.e.

\[
\sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) (L(T, T_k, T_{k+1}) - S(T, T_1, T_n)) = 0. \quad (13.50)
\]

**Proposition 13.10.** The LIBOR swap rate \(S(T, T_1, T_n)\) is given by

\[
S(T, T_1, T_n) = \frac{1}{P(T, T_1, T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) L(T, T_k, T_{k+1}). \quad (13.51)
\]

**Proof.** By definition, \(S(T, T_1, T_n)\) is the (fixed) break-even rate over \([T_1, T_n]\) that will be agreed in exchange for the family of forward LIBOR rates \(L(T, T_k, T_{k+1}), k = 1, 2, \ldots, n - 1\), and it solves (13.50) i.e.

\[
0 = \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) L(T, T_k, T_{k+1})
\]
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\[-S(T, T_1, T_n) \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) \]

\[= \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(T, T_{k+1}) L(T, T_k, T_{k+1}) - P(T, T_1, T_n) S(T, T_1, T_n), \]

which shows (13.51) by solving for \( S(T, T_1, T_n) \).

\[\square\]

Using the Definition 13.46, of LIBOR rates we obtain the next corollary.

**Corollary 13.11.** The LIBOR swap rate \( S(t, T_1, T_n) \) is given by

\[ S(t, T_1, T_n) = \frac{P(t, T_1) - P(t, T_n)}{P(t, T_1, T_n)}, \quad 0 \leq t \leq T_1. \quad (13.52) \]

**Proof.** By (13.51), (13.46) and a telescoping summation argument we have

\[ S(t, T_1, T_n) = \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} (T_{k+1} - T_k) P(t, T_{k+1}) L(t, T_k, T_{k+1}) \]

\[= \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} P(t, T_{k+1}) \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right) \]

\[= \frac{1}{P(t, T_1, T_n)} \sum_{k=1}^{n-1} (P(t, T_k) - P(t, T_{k+1})) \]

\[= \frac{P(t, T_1) - P(t, T_n)}{P(t, T_1, T_n)}. \quad (13.53) \]

\[\square\]

By (13.52), the bond prices \( P(t, T_1) \) can be recovered from the values of the forward swap rates \( S(t, T_1, T_n) \).

Clearly, a simple expression for the swap rate such as that of Corollary 13.11 cannot be obtained using the standard (i.e. non-LIBOR) rates defined in (13.45). Similarly, it will not be available for amortizing or accreting swaps because the telescoping summation argument does not apply to the expression (13.49) of the annuity numéraire.

When \( n = 2 \), the swap rate \( S(t, T_1, T_2) \) coincides with the forward rate \( L(t, T_1, T_2) \):

\[ S(t, T_1, T_2) = L(t, T_1, T_2). \quad (13.54) \]

Similarly to the case of LIBOR rates, Relation (13.52) shows that the LIBOR swap rate can be viewed as a forward price with (annuity) numéraire \( N_t = \emptyset \).
\( P(t, T_1, T_n) \) and \( X_t = P(t, T_1) - P(t, T_n) \). Consequently the LIBOR swap rate \((S(t, T_1, T_n)_{t \in [T, S]}\) is a martingale under the forward measure \( \hat{P} \) defined from (12.1) by
\[
\frac{d\hat{P}}{dP^*} = \frac{P(T_1, T_1, T_n)}{P(0, T_1, T_n)} e^{-\int_0^{T_1} r_t dt}.
\]

### 13.6 The HJM Model

From the beginning of this chapter we have started with the modeling of the short rate \((r_t)_{t \in \mathbb{R}_+}\), followed by its consequences on the pricing of bonds \(P(t, T)\) and on the expressions of the forward rates \(f(t, T, S)\) and \(L(t, T, S)\).

In this section we choose a different starting point and consider the problem of directly modeling the instantaneous forward rate \(f(t, T)\). The graph given in Figure 13.18 presents a possible random evolution of a forward interest rate curve using the Musiela convention, i.e. we will write
\[
g(x) = f(t, t + x) = f(t, T),
\]
under the substitution \( x = T - t, \ x \geq 0 \), and represent a sample of the instantaneous forward curve \( x \mapsto f(t, t + x) \) for each \( t \in \mathbb{R}_+ \).

![Stochastic process of forward curves.](http://www.ntu.edu.sg/home/nprivault/indext.html)  
**Fig. 13.18:** Stochastic process of forward curves.

In the Heath-Jarrow-Morton (HJM) model, the instantaneous forward rate \(f(t, T)\) is modeled under \( P^* \) by a stochastic differential equation of the form
\[
d_t f(t, T) = \alpha(t, T) dt + \sigma(t, T) dB_t, \quad 0 \leq t \leq T,
\]
(13.55)
where \( t \mapsto \alpha(t, T) \) and \( t \mapsto \sigma(t, T), \ 0 \leq t \leq T, \) are allowed to be random (adapted) processes. In the above equation, the date \( T \) is fixed and the differential \( d_t \) is with respect to \( t \).
Under basic Markovianity assumptions, a HJM model with deterministic coefficients \( \alpha(t, T) \) and \( \sigma(t, T) \) will yield a short rate process \((r_t)_{t \in \mathbb{R}_+}\) of the form
\[
dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,
\]
cf. § 6.6 of [Pri12], which is the Hull-White model [HW90], with explicit solution
\[
r_t = r_{s\text{e}} - r_{t\text{b}}(\tau)\,d\tau + w_{t\text{e}} - r_{t\text{u}}(\tau)\,d\tau + a(u)\,du + w_{t\text{u}}\sigma(u)e^{-r_{t\text{u}}(\tau)\,d\tau}dB_u,
\]
\(0 \leq s \leq t\).

The HJM condition

How to “encode” absence of arbitrage in the defining HJM Equation (13.55) is an important question. Recall that under absence of arbitrage, the bond price \(P(t, T)\) has been constructed as
\[
P(t, T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \mid \mathcal{F}_t\right] = \exp\left(-\int_t^T f(t, s) ds\right),\quad (13.56)
\]
cf. Proposition 13.8, hence the discounted bond price process is given by
\[
t \mapsto \exp\left(-\int_0^t r_s ds\right) P(t, T) = \exp\left(-\int_0^t r_s ds - \int_t^T f(t, s) ds\right)\quad (13.57)
\]
is a martingale under \(\mathbb{P}^*\) by Proposition 13.1 and Relation (13.40) in Proposition 13.8. This shows that \(\mathbb{P}^*\) is a risk-neutral probability measure, and by the first fundamental Theorem 5.8 of asset pricing we conclude that the market is without arbitrage opportunities.

**Proposition 13.12. (HJM Condition [HJM92]).** Under the condition
\[
\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s) ds, \quad t \in [0, T],\quad (13.58)
\]
which is known as the HJM absence of arbitrage condition, the discounted bond price process (13.57) is a martingale, and the probability measure \(\mathbb{P}^*\) is risk-neutral.

**Proof.** Consider the spot forward rate, or yield, given from (13.42) as
\[
f(t, t, T) = \frac{1}{T-t} \int_t^T f(t, s) ds,
\]
and consider the process \((X_t)_{t \in [0, T]}\) defined as
\[
\bigcirc
\]
\[ X_t := \int_t^T f(t,s) ds = -\log P(t,T), \quad 0 \leq t \leq T, \]
such that \( P(t,T) = e^{X_t} \), with the relation
\[
f(t,t,T) = \frac{1}{T-t} \int_t^T f(t,s) ds = \frac{X_t}{T-t}, \quad 0 \leq t \leq T, \tag{13.59}
\]
where the dynamics of \( t \mapsto f(t,s) \) is given by (13.55). We note that when \( f(t,s) = g(t)h(s) \) is a smooth function which satisfies the separation of variables property we have the relation
\[
d_t \left( \int_t^T g(t)h(s) ds \right) = d_t \left( g(t) \int_t^T h(s) ds \right)
= \int_t^T h(s) ds g(t) + g(t) d_t \int_t^T h(s) ds
= g'(t) \left( \int_t^T h(s) ds \right) dt - g(t) h(t) dt,
\]
which extends to \( f(t,s) \) as
\[
d_t \int_t^T f(t,s) ds = -f(t,t) dt + \int_t^T d_t f(t,s) ds = -r_t dt + \int_t^T d_t f(t,s) ds,
\]
which can be seen as a form of the **Leibniz integral rule**. Therefore we have
\[
d_t X_t = d_t \int_t^T f(t,s) ds
= -f(t,t) dt + \int_t^T d_t f(t,s) ds
= -f(t,t) dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t
= -r_t dt + \left( \int_t^T \alpha(t,s) ds \right) dt + \left( \int_t^T \sigma(t,s) ds \right) dB_t,
\]

hence we have
\[
|d_t X_t|^2 = \left( \int_t^T \sigma(t,s) ds \right)^2 dt.
\]

Hence by Itô’s calculus we have
\[
d_t P(t,T) = d_t e^{-X_t}
= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} (d_t X_t)^2
= -e^{-X_t} d_t X_t + \frac{1}{2} e^{-X_t} \left( \int_t^T \sigma(t,s) ds \right)^2 dt
\]
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\[
= -e^{-X_t} \left( -r_t dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t \right) \\
+ \frac{1}{2} e^{-X_t} \left( \int_t^T \sigma(t,s) ds \right)^2 dt,
\]
and the discounted bond price satisfies

\[
d_t \left( \exp \left( -\int_0^t r_s ds \right) P(t,T) \right) \\
= -r_t \exp \left( -\int_0^t r_s ds - X_t \right) dt + \exp \left( -\int_0^t r_s ds \right) d_t P(t,T) \\
= -r_t \exp \left( -\int_0^t r_s ds - X_t \right) dt - \exp \left( -\int_0^t r_s ds - X_t \right) d_t X_t \\
+ \frac{1}{2} \exp \left( -\int_0^t r_s ds - X_t \right) \left( \int_t^T \sigma(t,s) ds \right)^2 dt \\
= -r_t \exp \left( -\int_0^t r_s ds - X_t \right) dt \\
- \exp \left( -\int_0^t r_s ds - X_t \right) \left( -r_t dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t \right) \\
+ \frac{1}{2} \exp \left( -\int_0^t r_s ds - X_t \right) \left( \int_t^T \sigma(t,s) ds \right)^2 dt \\
= - \exp \left( -\int_0^t r_s ds - X_t \right) \int_t^T \sigma(t,s) ds dB_t \\
- \exp \left( -\int_0^t r_s ds - X_t \right) \left( \int_t^T \alpha(t,s) ds dt - \frac{1}{2} \left( \int_t^T \sigma(t,s) ds \right)^2 \right) dt.
\]

Thus, the discounted bond price process

\[ t \mapsto \exp \left( -\int_0^t r_s ds \right) P(t,T) \]

will be a martingale provided that

\[
\int_t^T \alpha(t,s) ds - \frac{1}{2} \left( \int_t^T \sigma(t,s) ds \right)^2 = 0, \quad 0 \leq t \leq T. \quad (13.60)
\]

Differentiating the above relation with respect to \( T \), we get

\[
\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds,
\]
which is in fact equivalent to (13.60).

\[ \square \]
**Forward HJM rates**

The HJM coefficients in the Vasicek model are in fact deterministic, and taking \( a = 0 \) we have

\[
d_t f(t, T) = \sigma^2 e^{-(T-t)b} \int_t^T e^{(t-s)b} ds dt + \sigma e^{-(T-t)b} dB_t,
\]

i.e.

\[
\alpha(t, T) = \sigma^2 e^{-(T-t)b} \int_t^T e^{(t-s)b} ds = \sigma^2 e^{-(T-t)b} \frac{1 - e^{-(T-t)b}}{b},
\]

and \( \sigma(t, T) = \sigma e^{-(T-t)b} \), and the HJM condition reads

\[
\alpha(t, T) = \sigma^2 e^{-(T-t)b} \int_t^T e^{(t-s)b} ds = \sigma(t, T) \int_t^T \sigma(t, s) ds. \tag{13.61}
\]

Random simulations of the Vasicek instantaneous forward rates are provided in Figures 13.19 and 13.20.

---

**Fig. 13.19:** Forward instantaneous curve \((t, x) \mapsto f(t, t + x)\) in the Vasicek model.*
Fig. 13.20: Forward instantaneous curve $x \mapsto f(0, x)$ in the Vasicek model.*

For $x = 0$ the first “slice” of this surface is actually the short rate Vasicek process $r_t = f(t, t) = f(t, t + 0)$ which is represented in Figure 13.21 using another discretization.

Fig. 13.21: Short term interest rate curve $t \mapsto r_t$ in the Vasicek model.

13.7 Modeling Issues

Nelson-Siegel parametrization of forward rates

In the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by 4 coefficients $z_1, z_2, z_3, z_4$, as

$$g(x) = z_1 + (z_2 + z_3 x) e^{-xz_4}, \quad x \geq 0.$$  

An example of a graph obtained by the Nelson-Siegel parametrization is given in Figure 13.22, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

* The animation works in Acrobat Reader on the entire pdf file.
Svensson parametrization of forward rates

The Svensson parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen via 6 parameters $z_1$, $z_2$, $z_3$, $z_4$, $z_5$, $z_6$ as

$$g(x) = z_1 + (z_2 + z_3 x) e^{-xz_4} + z_5 x e^{-xz_6}, \quad x \geq 0.$$ 

A typical graph of a Svensson parametrization is given in Figure 13.23, for $z_1 = 6.6$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = 1$. 

Figure 13.24 presents a fit of the market data of Figure 13.12 using a Svensson curve.
Vasicek parametrization

In the Vasicek model we have

\[
\frac{\partial f}{\partial T}(t, T) = \left( -br_t + a - \frac{\sigma^2}{b}e^{-(T-t)b} \right)e^{-(T-t)b},
\]

and one can check that the sign of the derivatives of \( f \) can only change once at most. As a consequence, the possible forward curves in the Vasicek model are limited to one change of “regime” per curve, as illustrated in Figure 13.25 for various values of \( r_t \), and in Figure 13.26.

Fig. 13.25: Graphs of forward rates with \( b = 0.16, a/b = 0.04, r_0 = 2\% , \sigma = 4.5\% \).

The next figure is also using the parameters \( b = 0.16, a/b = 0.04, r_0 = 2\% , \) and \( \sigma = 4.5\% \).
Fig. 13.26: Forward instantaneous curve \((t, x) \mapsto f(t, t + x)\) in the Vasicek model.

One may think of constructing an instantaneous rate process taking values in the Svensson space, however this type of modelization is not consistent with absence of arbitrage, and it can be proved that the HJM curves cannot live in the Nelson-Siegel or Svensson spaces, cf. §3.5 of [Bjo04b]. In other words, it can be shown that the forward yield curves produced by the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space. In addition, the Vasicek yield curves do not appear to correctly model the market forward curves cf. also Figure 13.12 above.

Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at \(t = 0\), cf. e.g. § 8.2 of [Pri12].

**Fitting the Nelson-Siegel and Svensson models to yield curve data**

Recall that in the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by four coefficients \(z_1, z_2, z_3, z_4\), as

\[
f(t, t + x) = z_1 + (z_2 + z_3x)e^{-xz_4}, \quad x \geq 0.
\]  

(13.62)

Taking \(x = T - t\), the yield \(f(t, t, T)\) is given as

\[
f(t, t, T) = \frac{1}{T - t} \int_t^T f(t, s)ds
\]

\[
= \frac{1}{x} \int_0^x f(t, t + y)dy
\]

\[
= z_1 + \frac{z_2}{x} \int_0^x e^{-yz_4}dy + \frac{z_3}{x} \int_0^x y e^{-yz_4}dy
\]

\[
= z_1 + z_2 \frac{1 - e^{-xz_4}}{xz_4} + z_3 \frac{1 - e^{-xz_4} + xe^{-xz_4}}{xz_4}.
\]

The expression (13.62) can be represented in the parametrization

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\[ f(t, t + x) = z_1 + (z_2 + z_3 x) e^{-xz_4} = \beta_0 + \beta_1 e^{-x/\lambda} + \frac{\beta_2}{\lambda} x e^{-x/\lambda}, \quad x \geq 0, \]

cf. [Cha14], with \( \beta_0 = z_1 \), \( \beta_1 = z_2 \), \( \beta_2 = z_3 / z_4 \), \( \lambda = 1 / z_4 \).

```r
require(YieldCurve)
data(ECBYieldCurve)
mat.ECB<-c(3/12, 0.5, 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30)
first(ECBYieldCurve, '1 month')
Nelson.Siegel(first(ECBYieldCurve, '1 month'), mat.ECB)

for (n in seq(from=70, to=290, by=10)) {
  ECB.NS <- Nelson.Siegel(ECBYieldCurve[n,], mat.ECB)
  ECB.S <- Svensson(ECBYieldCurve[n,], mat.ECB)
  ECB.NS.yield.curve <- NSrates(ECB.NS, mat.ECB)
  ECB.S.yield.curve <- Srates(ECB.S, mat.ECB, "Spot")

  plot(mat.ECB, as.numeric(ECBYieldCurve[n,]), type="o", lty=1, col=1,ylab="Interest rates", xlab="Maturity in years", ylim=c(3.2,4.8))
  lines(mat.ECB, as.numeric(ECB.NS.yield.curve), type="l", lty=3,col=2,lwd=2)
  lines(mat.ECB, as.numeric(ECB.S.yield.curve), type="l", lty=2,col=6,lwd=2)
  title(main=paste("ECB yield curve observed at",time(ECBYieldCurve[n], sep=" "),"vs fitted yield curve"))
  legend("bottomright", legend=c("ECB data","Nelson-Siegel","Svensson"),col=c(1,2,6), lty=1, bg='gray90')
  grid()
  Sys.sleep(2.5)}
```

Fig. 13.27: ECB data vs fitted yield curve.*

**The correlation problem and a two-factor model**

The correlation problem is another issue of concern when using the affine models considered so far. Let us compare three bond price simulations with

* The animation works in Acrobat Reader on the entire pdf file.
maturity $T_1 = 10$, $T_2 = 20$, and $T_3 = 30$ based on the same Brownian path, as given in Figure 13.28. Clearly, the bond prices $F(r_t, T_1) = P(t, T_1)$ and $F(r_t, T_2) = P(t, T_2)$ with maturities $T_1$ and $T_2$ are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp \left( A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1)) \right),$$

(13.63)

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

![Graph of $t \mapsto P(t, T_1)$](image)

**Fig. 13.28:** Graph of $t \mapsto P(t, T_1)$.

In affine short rates models, by (13.63), log $P(t, T_1)$ and log $P(t, T_2)$ are linked by the linear relationship

$$\log P(t, T_2) = \log P(t, T_1) + A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1)) \frac{\log P(t, T_1) - C(t, T_1)}{A(t, T_1)}$$

$$= \left( 1 + \frac{C(t, T_2) - C(t, T_1)}{A(t, T_1)} \right) \log P(t, T_1)$$

$$+ A(t, T_2) - A(t, T_1) - (C(t, T_2) - C(t, T_1)) \frac{C(t, T_1)}{A(t, T_1)}$$

with constant coefficients, which yields the perfect (positive or negative) correlation

$$\text{Cor}(\log P(t, T_1), \log P(t, T_2)) = \pm 1,$$

depending on the sign of the coefficient $1 + (C(t, T_2) - C(t, T_1)) / A(t, T_1)$, cf. § 8.3 of [Pri12],

A solution to the correlation problem is to consider a two-factor model based on two control processes $(X_t)_{t \in \mathbb{R}^+}, (Y_t)_{t \in \mathbb{R}^+}$ which are solution of
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\[
\begin{align*}
\left\{ 
   dX_t &= \mu_1(t, X_t) dt + \sigma_1(t, X_t) dB^{(1)}_t, \\
   dY_t &= \mu_2(t, Y_t) dt + \sigma_2(t, Y_t) dB^{(2)}_t,
\end{align*}
\] (13.64)

where \((B^{(1)}_t)_{t \in \mathbb{R}^+}, (B^{(2)}_t)_{t \in \mathbb{R}^+}\) have correlated Brownian motion with

\[
\text{Cov}(B^{(1)}_s, B^{(2)}_t) = \rho \min(s, t), \quad s, t \in \mathbb{R}^+,
\] (13.65)

and

\[
\text{d}B^{(1)}_t \cdot \text{d}B^{(2)}_t = \rho \text{d}t,
\] (13.66)

for some correlation parameter \(\rho \in [-1, 1]\). In practice, \((B^{(1)}_t)_{t \in \mathbb{R}^+}\) and \((B^{(2)}_t)_{t \in \mathbb{R}^+}\) can be constructed from two independent Brownian motions \((W^{(1)}_t)_{t \in \mathbb{R}^+}\) and \((W^{(2)}_t)_{t \in \mathbb{R}^+}\), by letting

\[
\begin{align*}
   B^{(1)}_t &= W^{(1)}_t, \\
   B^{(2)}_t &= \rho W^{(1)}_t + \sqrt{1 - \rho^2} W^{(2)}_t,
\end{align*}
\]

and Relations (13.65) and (13.66) are easily satisfied from this construction.

In two-factor models one chooses to build the short term interest rate \(r_t\) via

\[
r_t := X_t + Y_t, \quad t \in \mathbb{R}^+.
\]

By the previous standard arbitrage arguments we define the price of a bond with maturity \(T\) as

\[
P(t, T) := \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid X_t, Y_t \right] = \mathbb{E}^* \left[ \exp \left( - \int_t^T (X_s + Y_s) ds \right) \mid X_t, Y_t \right] = F(t, X_t, Y_t),
\] (13.67)

since the couple \((X_t, Y_t)_{t \in \mathbb{R}^+}\) is Markovian. Applying the Itô formula with two variables to

\[
t \mapsto F(t, X_t, Y_t) = P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right],
\]

and using the fact that the discounted process
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\[ t \mapsto e^{-\int_0^t r_s \, ds} P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_0^T r_s \, ds \right) \mid \mathcal{F}_t \right] \]

is an \( \mathcal{F}_t \)-martingale under \( \mathbb{P}^* \), we can derive a PDE

\[
-(x + y) F(t, x, y) + \mu_1(t, x) \frac{\partial F}{\partial x}(t, x, y) + \mu_2(t, y) \frac{\partial F}{\partial y}(t, x, y) \\
+ \frac{1}{2} \sigma_1^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x, y) + \frac{1}{2} \sigma_2^2(t, y) \frac{\partial^2 F}{\partial y^2}(t, x, y) \\
+ \rho \sigma_1(t, x) \sigma_2(t, y) \frac{\partial^2 F}{\partial x \partial y}(t, x, y) + \frac{\partial F}{\partial t}(t, x, y) = 0, \quad (13.68)
\]

on \( \mathbb{R}^2 \) for the bond price \( P(t, T) \). In the Vasicek model

\[
\begin{aligned}
\begin{cases}
  dX_t &= -a X_t \, dt + \sigma dB_{t}^{(1)}(t), \\
  dY_t &= -b Y_t \, dt + \eta dB_{t}^{(2)}(t),
\end{cases}
\end{aligned}
\]

this yields the solution \( F(t, x, y) \) of (13.68) as

\[
P(t, T) = F(t, X_t, Y_t) = F_1(t, X_t) F_2(t, Y_t) \exp (\rho U(t, T)), \quad (13.69)
\]

where \( F_1(t, X_t) \) and \( F_2(t, Y_t) \) are the bond prices associated to \( X_t \) and \( Y_t \) in the Vasicek model, and

\[
U(t, T) := \frac{\sigma \eta}{ab} \left( T - t + \frac{e^{-(T-t)a} - 1}{a} + \frac{e^{-(T-t)b} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a + b} \right)
\]

is a correlation term which vanishes when \( (B_{t}^{(1)})_{t \in \mathbb{R}_+} \) and \( (B_{t}^{(2)})_{t \in \mathbb{R}_+} \) are independent, i.e. when \( \rho = 0 \), cf [BM06], Chapter 4, Appendix A, and § 8.4 of [Pri12].

Partial differentiation of \( \log P(t, T) \) with respect to \( T \) leads to the instantaneous forward rate

\[
f(t, T) = f_1(t, T) + f_2(t, T) - \rho \frac{\sigma \eta}{ab} (1 - e^{-(T-t)a}) (1 - e^{-(T-t)b}), \quad (13.70)
\]

where \( f_1(t, T), f_2(t, T) \) are the instantaneous forward rates corresponding to \( X_t \) and \( Y_t \) respectively, cf. § 8.4 of [Pri12].

An example of a forward rate curve obtained in this way is given in Figure 13.29.
Next, in Figure 13.30 we present a graph of the evolution of forward curves in a two-factor model.

13.8 The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as nonpositivity of interest rates in Vasicek model, and lack of closed-form solutions in more complex models. The BGM [BGM97] model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as caps and swaptions on the LIBOR market.

In the BGM model we consider two bond prices $P(t, T_1), P(t, T_2)$ with maturities $T_1, T_2$ and the forward measure

$$\frac{dP_2}{dP^*_2} = e^{-\int_0^{T_2} r_s ds} \frac{P(0, T_2)}{P(0, T_2)} ,$$
with numéraire $P(t, T_2)$, cf. (12.6). The forward LIBOR rate $L(t, T_1, T_2)$ is modeled as a geometric Brownian motion under $P_2$, i.e.

$$\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = \gamma_1(t)dB_t^{(2)}, \quad (13.71)$$

$0 \leq t \leq T_1$, $i = 1, 2, \ldots, n - 1$, for some deterministic function $\gamma_1(t)$, with solution

$$L(u, T_1, T_2) = L(t, T_1, T_2) \exp \left( \int_t^u \gamma_1(s)dB_s^{(2)} - \frac{1}{2} \int_t^u |\gamma_1(s)|^2ds \right),$$

i.e. for $u = T_1$,

$$L(T_1, T_1, T_2) = L(t, T_1, T_2) \exp \left( \int_t^{T_1} \gamma_1(s)dB_s^{(2)} - \frac{1}{2} \int_t^{T_1} |\gamma_1(s)|^2ds \right).$$

Since $L(t, T_1, T_2)$ is a geometric Brownian motion under $P_2$, standard caplets can be priced at time $t \in [0, T_1]$ from the Black-Scholes formula.

The following Graph 13.31 summarizes the notions introduced in this chapter.
Forward Rate Modeling

Fig. 13.31: Roadmap of stochastic interest rate modeling.
Exercises

Exercise 13.1 Show that the solution of the equation
\[ dr_t = (a - br_t)dt + \sigma dB_t, \quad (13.72) \]
where \( a, \sigma \in \mathbb{R}, b > 0, \) is
\[ r_t = r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \quad t \in \mathbb{R}_+. \] (13.73)

Exercise 13.2 Consider a tenor structure \( \{ T_1, T_2 \} \) and a bond with maturity \( T_2 \) and price given at time \( t \in [0, T_2] \) by
\[ P(t, T_2) = \exp \left( - \int_t^{T_2} f(t, s) ds \right), \quad t \in [0, T_2], \]
where the instantaneous yield curve \( f(t, s) \) is parametrized as
\[ f(t, s) = r_1 \mathbb{1}_{[0, T_1]}(s) + r_2 \mathbb{1}_{[T_1, T_2]}(s), \quad s \in [t, T_2]. \]
Find a formula to estimate the values of \( r_1 \) and \( r_2 \) from the data of \( P(0, T_2) \) and \( P(T_1, T_2) \).

Same question for when \( f(t, s) \) is parametrized as
\[ f(t, s) = r_1 s \mathbb{1}_{[0, T_1]}(s) + (r_1 T_1 + r_2 (s - T_1)) \mathbb{1}_{[T_1, T_2]}(s), \quad s \in [t, T_2]. \]

Exercise 13.3 Let \( (B_t)_{t \in \mathbb{R}_+} \) denote a standard Brownian motion started at 0 under the risk-neutral probability measure \( \mathbb{P}^* \). We consider a short term interest rate process \( (r_t)_{t \in \mathbb{R}_+} \) in a Ho-Lee model with constant deterministic volatility, defined by
\[ dr_t = adt + \sigma dB_t, \]
where \( a \in \mathbb{R} \) and \( \sigma > 0 \). Let \( P(t, T) \) will denote the arbitrage price of a zero-coupon bond in this model:
\[ P(t, T) = \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \] (13.74)

a) State the bond pricing PDE satisfied by the function \( F(t, x) \) defined via
\[ F(t, x) := \mathbb{E}^* \left[ \exp \left( - \int_t^T r_s ds \right) \mid r_t = x \right], \quad 0 \leq t \leq T. \]
b) Compute the arbitrage price \( F(t, r_t) = P(t, T) \) from its expression (13.74) as a conditional expectation.

**Hint.** One may use the integration by parts argument

\[
\int_t^T B_s ds = TB_T - tB_t - \int_t^T s dB_s \\
= (T - t)B_t + T(B_T - B_t) - \int_t^T s dB_s \\
= (T - t)B_t + \int_t^T (T - s) dB_s,
\]

and the Laplace transform identity \( \mathbb{E}[e^{\lambda X}] = e^{\lambda^2 \eta^2 / 2} \) for \( X \sim \mathcal{N}(0, \eta^2) \).

c) Check that the function \( F(t, x) \) computed in question (b) does satisfy the PDE derived in question (a).

d) Compute the forward rate \( f(t, T, S) \) in this model.

From now on we let \( a = 0 \).

e) Compute the instantaneous forward rate \( f(t, T) \) in this model.

f) Derive the stochastic equation satisfied by the instantaneous forward rate \( f(t, T) \).

g) Check that the HJM absence of arbitrage condition is satisfied in this equation.

**Exercise 13.4** Consider the Courtadon (1982) model

\[
 dr_t = \beta(\alpha - r_t) dt + \sigma r_t dB_t, \tag{13.75}
\]

where \( \alpha, \beta, \sigma \) are nonnegative. Show that the solution of (13.75) is given by

\[
 r_t = \alpha \beta \int_0^t \frac{S_t}{S_u} du + r_0 S_t, \quad t \in \mathbb{R}_+,
\]

where \((S_t)_{t \in \mathbb{R}_+}\) is the geometric Brownian motion solution of \( dS_t = -\beta S_t dt + \sigma S_t dB_t \) with \( S_0 = 1 \).

**Exercise 13.5** Consider the Marsh-Rosenfeld interest rate model \( dr_t = (\beta r_t^{1-\gamma} + \alpha r_t) dt + \sigma r_t^{\gamma / 2} dB_t \). Given that the discounted bond price process is a martingale, derive the bond pricing PDE satisfied by the function \( F(t, x) \) such that

\[
 F(t, r_t) = P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \bigg| \mathcal{F}_t \right] = \mathbb{E}^* \left[ e^{-\int_t^T r_s ds} \bigg| r_t \right].
\]

**Exercise 13.6** Consider the CIR process \((r_t)_{t \in \mathbb{R}_+}\) solution of
\[ dr_t = -ar_t dt + \sigma \sqrt{r_t} dB_t, \]

where \( a, \sigma > 0 \) are constants \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion started at 0.

a) Write down the bond pricing PDE for the function \( F(t, x) \) given by

\[ F(t, x) := \mathbb{E}^* \left[ \exp \left( -\int_t^T r_s ds \right) \mid r_t = x \right], \quad 0 \leq t \leq T. \]

*Hint:* Use Itô calculus and the fact that the discounted bond price is a martingale.

b) Show that the PDE of Question (a) admits a solution of the form

\[ F(t, x) = e^{A(T-t)+xC(T-t)} \]

where the functions \( A(s) \) and \( C(s) \) satisfy ordinary differential equations to be also written down together with the values of \( A(0) \) and \( C(0) \).

Exercise 13.7 Convertible bonds. Consider an underlying stock price process \((S_t)_{t \in \mathbb{R}_+}\) given by

\[ dS_t = rS_t dt + \sigma S_t dB_t^{(1)}, \]

and a short term interest rate process \((r_t)_{t \in \mathbb{R}_+}\) given by

\[ dr_t = \gamma(t, r_t) dt + \eta(t, r_t) dB_t^{(2)}, \]

where \((B_t^{(1)})_{t \in \mathbb{R}_+}\) and \((B_t^{(2)})_{t \in \mathbb{R}_+}\) are two correlated Brownian motions under the risk-neutral probability measure \( \mathbb{P}^* \), with \( dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt \). A convertible bond is a corporate bond that can be exchanged into a quantity \( \alpha > 0 \) of the underlying company’s stock \( S_\tau \) at a future time \( \tau \), whichever has a higher value, where \( \alpha \) is a conversion rate.

a) Find the payoff of the convertible bond at time \( \tau \).

b) Rewrite the convertible bond payoff at time \( \tau \) as the linear combination of \( P(\tau, T) \) and a call option payoff on \( S_\tau \), whose strike price is to be determined.

c) Write down the corporate bond price at time \( t \in [0, \tau] \) as a function \( C(t, S_t, r_t) \) of the underlying asset price and interest rate, using a discounted conditional expectation, and show that the discounted corporate bond price

\[ e^{-\int_0^t r_s ds} C(t, S_t, r_t), \quad t \in [0, \tau], \]

is a martingale.

d) Write down \( d \left( e^{-\int_0^t r_s ds} C(t, S_t, r_t) \right) \) using the Itô formula and derive the pricing PDE satisfied by the function \( C(t, x, y) \) together with its terminal condition.
e) Taking the bond price $P(t, T)$ as a numéraire, price the convertible bond as a European option with strike price $K = 1$ on an underlying asset priced $Z_t := S_t / P(t, T)$, $t \in [0, \tau]$ under the forward measure $\tilde{\mathbb{E}}_T$.

f) Assuming the bond price dynamics $dP(t, T) = r_t P(t, T) dt + \sigma_B(t) P(t, T) dB_t$, determine the dynamics of the process $(Z_t)_{t \in \mathbb{R}^+}$ under the forward measure $\tilde{\mathbb{E}}_T$.

g) Assuming that $(Z_t)_{t \in \mathbb{R}^+}$ can be modeled as a geometric Brownian motion, price the corporate bond option using the Black-Scholes formula.

Exercise 13.8  Given $(B_t)_{t \in \mathbb{R}^+}$ a standard Brownian motion, consider a HJM model given by

$$dt f(t, T) = \frac{\sigma^2}{2} T(T^2 - t^2) dt + \sigma T dB_t.$$  (13.76)

a) Show that the HJM condition is satisfied by (13.76).

b) Compute $f(t, T)$ by solving (13.76).

Hint: We have $f(t, T) = f(0, T) + \int_0^t ds f(s, T) = \cdots$

c) Compute the short rate $r_t = f(t, t)$ from the result of Question (b).

d) Show that the short rate $r_t$ satisfies a stochastic differential equation of the form

$$dr_t = \eta(t) dt + (r_t - f(0, t)) \psi(t) dt + \xi(t) dB_t,$$

where $\eta(t)$, $\psi(t)$, $\xi(t)$ are deterministic functions to be determined.

Exercise 13.9  Let $(r_t)_{t \in \mathbb{R}^+}$ denote a short term interest rate process. For any $T > 0$, let $P(t, T)$ denote the price at time $t \in [0, T]$ of a zero-coupon bond defined by the stochastic differential equation

$$\frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma_t^T dB_t, \quad 0 \leq t \leq T,$$  (13.77)

under the terminal condition $P(T, T) = 1$, where $(\sigma_t^T)_{t \in [0,T]}$ is an adapted process. We define the forward measure $\mathbb{P}_T$ by

$$\mathbb{E}^* \left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \bigg| \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)} e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T.$$

Recall that

$$B_t^T := B_t - \int_0^t \sigma_s^T ds, \quad 0 \leq t \leq T,$$

is a standard Brownian motion under $\mathbb{P}_T$.

a) Solve the stochastic differential equation (13.77).
b) Derive the stochastic differential equation satisfied by the discounted bond price process

\[ t \mapsto e^{-\int_0^t r_s \, ds} P(t, T), \quad 0 \leq t \leq T, \]

and show that it is a martingale.

c) Show that

\[ \mathbb{E}^* \left[ e^{-\int_0^T r_s \, ds} \bigg| \mathcal{F}_t \right] = e^{-\int_0^t r_s \, ds} P(t, T), \quad 0 \leq t \leq T. \]

d) Show that

\[ P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s \, ds} \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \]

e) Compute \( P(t, S)/P(t, T) \), \( 0 \leq t \leq T \), show that it is a martingale under \( \mathbb{P}_T \) and that

\[ P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \int_t^T (\sigma^S_s - \sigma^T_s) dB^T_s - \frac{1}{2} \int_t^T (\sigma^S_s - \sigma^T_s)^2 \, ds \right). \]

f) Assuming that \( (\sigma^T_t)_{t \in [0, T]} \) and \( (\sigma^S_t)_{t \in [0, S]} \) are deterministic functions, compute the price

\[ \mathbb{E}^* \left[ e^{-\int_t^T r_s \, ds} (P(T, S) - \kappa)^+ \bigg| \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T \left[ (P(T, S) - \kappa)^+ \bigg| \mathcal{F}_t \right] \]

of a bond option with strike price \( \kappa \).

Recall that if \( X \) is a centered Gaussian random variable with mean \( m_t \) and variance \( \nu_t^2 \) given \( \mathcal{F}_t \), we have

\[ \mathbb{E}[(e^X - K)^+ | \mathcal{F}_t] = e^{m_t + \nu_t^2/2} \Phi \left( \frac{\nu_t}{2} + \frac{1}{\nu_t} (m_t + \nu_t^2/2 - \log K) \right) \]

\[ -K \Phi \left( -\frac{\nu_t}{2} + \frac{1}{\nu_t} (m_t + \nu_t^2/2 - \log K) \right) \]

where \( \Phi(x), x \in \mathbb{R} \), denotes the Gaussian cumulative distribution function.

Exercise 13.10 (Exercise 4.15 continued). Bridge model. Assume that the price \( P(t, T) \) of a zero-coupon bond is modeled as

\[ P(t, T) = e^{-\mu(T-t) + X^T_t}, \quad t \in [0, T], \]

where \( \mu > 0 \).
Forward Rate Modeling

a) Show that the terminal condition \( P(T, T) = 1 \) is satisfied.

b) Compute the forward rate

\[
f(t, T, S) = -\frac{1}{S - T} (\log P(t, S) - \log P(t, T)).
\]

c) Compute the instantaneous forward rate

\[
f(t, T) = -\lim_{S \searrow T} \frac{1}{S - T} (\log P(t, S) - \log P(t, T)).
\]

d) Show that the limit \( \lim_{T \searrow t} f(t, T) \) does not exist in \( L^2(\Omega) \).

e) Show that \( P(t, T) \) satisfies the stochastic differential equation

\[
\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + \frac{1}{2} \sigma^2 dt - \frac{\log P(t, T)}{T - t} dt, \quad t \in [0, T].
\]

f) Show, using the results of Exercise 13.9-(d), that

\[
P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} \Big| \mathcal{F}_t \right],
\]

where \( (r_t^T)_{t \in [0, T]} \) is a stochastic process to be determined.

g) Compute the conditional density

\[
\mathbb{E}^* \left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \Big| \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)} e^{-\int_0^t r_s^T ds}
\]

of the forward measure \( \mathbb{P}_T \) with respect to \( \mathbb{P}^* \).

h) Show that the process

\[
\bar{B}_t := B_t - \sigma t, \quad 0 \leq t \leq T,
\]

is a standard Brownian motion under \( \mathbb{P}_T \).

i) Compute the dynamics of \( X_t^S \) and \( P(t, S) \) under \( \mathbb{P}_T \).

Hint: Show that

\[
-\mu(S - T) + \sigma(S - T) \int_0^t \frac{1}{S - s} dB_s = \frac{S - T}{S - t} \log P(t, S).
\]

j) Compute the bond option price

\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} (P(T, S) - K)^+ \Big| \mathcal{F}_t \right] = P(t, T) \mathbb{E}_T \left[ (P(T, S) - K)^+ \Big| \mathcal{F}_t \right],
\]

\( 0 \leq t < T < S \).
Exercise 13.11 (Exercise 4.18 continued). Write down the bond pricing PDE for the function

\[ F(t, x) = \mathbb{E}^* \left[ e^{-\int_0^T r_s ds} \mid r_t = x \right] \]

and show that in case \( \alpha = 0 \) the corresponding bond price \( P(t, T) \) equals

\[ P(t, T) = e^{-r_t B(T-t)}, \quad 0 \leq t \leq T, \]

where

\[ B(x) = \frac{2(e^\gamma x - 1)}{2\gamma + (\beta + \gamma)(e^\gamma x - 1)}, \]

with \( \gamma = \sqrt{\beta^2 + 2\sigma^2} \).

Exercise 13.12 Consider a short rate process \((r_t)_{t \in \mathbb{R}^+} \) of the form \( r_t = h(t) + X_t \), where \( h(t) \) is a deterministic function and \((X_t)_{t \in \mathbb{R}^+} \) is a Vasicek process started at \( X_0 = 0 \).

a) Compute the price \( P(0, T) \) at time \( t = 0 \) of a bond with maturity \( T \), using \( h(t) \) and the function \( A(T) \) defined in (13.24) for the pricing of Vasicek bonds.

b) Show how the function \( h(t) \) can be estimated from the market data of the initial instantaneous forward rate curve \( f(0, t) \).

Exercise 13.13

a) Given two LIBOR spot rates \( L(t, t, T) \) and \( L(t, t, S) \), compute the corresponding LIBOR forward rate \( L(t, T, S) \).

b) Assuming that \( L(t, t, T) = 2\% \), \( L(t, t, S) = 2.5\% \) and \( t = 0 \), \( T = 1 \), \( S = 2T = 2 \), would you buy a LIBOR forward contract over \([T, 2T]\) with rate \( L(0, T, 2T) \) if \( L(T, T, 2T) \) remained at \( L(T, T, 2T) = L(0, 0, T) = 2\% \)?

Exercise 13.14 Consider a zero-coupon bond with prices \( P(1, 2) = 91.74\% \) and \( P(0, 2) = 83.40\% \) at times \( t = 0 \) and \( t = 1 \).

a) Compute the corresponding yields \( y_{0,1} \), \( y_{0,2} \) and \( y_{1,2} \) at times \( t = 0 \) and \( t = 1 \).

b) Assume that \$0.1 coupons are paid at times \( t = 1 \) and \( t = 2 \). Price the corresponding coupon bond at times \( t = 0 \) and \( t = 1 \) using the yields \( y_0 \) and \( y_1 \).

Exercise 13.15 We build a discretization \((r_{tk})_{k \in \mathbb{N}} \) of the Vasicek process solution of \( dr_t = (a - br_t)dt + \sigma dB_t \) along the binomial tree
Writing $\Delta r_1 := r_{t_1} - r_0$ and $\Delta r_2 := r_{t_2} - r_{t_1}$, find the probabilities $p(r_0)$, $q(r_0)$, $p(r_1)$, $q(r_1)$ so that

$$E[\Delta r_1] = (a - br_0)\Delta t \quad \text{and} \quad E[\Delta r_2] = (a - br_{t_1})\Delta t.$$ 

Exercise 13.16 Black-Derman-Toy model. Consider a two-step interest rate model in which the short term interest rate $r_0$ on $[0, 1]$ can turn into two possible values $r_1^u = r_0 e^{\mu \Delta t + \sigma \sqrt{\Delta t}}$ and $r_1^d = r_0 e^{\mu \Delta t - \sigma \sqrt{\Delta t}}$ on $[1, 2]$ with equal probabilities 1/2 at time $\Delta t = 1$ year and $\sigma = 22\%$ per year, and a zero-coupon bonds with prices $P(0, 1)$ and $P(0, 2)$ at time $t = 0$.

a) Write down the value of $P(1, 2)$ using $r_1^u$ and $r_1^d$.
b) Write down the value of $P(0, 2)$ using $r_1^u$, $r_1^d$ and $r_0$.
c) Estimate the value of $r_0$ from the market price $P(0, 1) = 91.74$.
d) Estimatethe values of $r_1^u$ and $r_1^d$ from the market price $P(0, 2) = 83.40$.

Exercise 13.17 Consider a yield curve $(f(t, t, T))_{0 \leq t \leq T}$ and a bond paying coupons $c_1, c_2, \ldots, c_n$ at times $T_1, T_2, \ldots, T_n$ until maturity $T_n$, and priced as

$$P(t, T_n) = \sum_{k=1}^{n} c_k e^{-(T_k-t)f(t,t,T_k)}, \quad 0 \leq t \leq T_1,$$

where $c_n$ is inclusive of the last coupon payment and the nominal $\$1$ value of the bond. Let $\hat{f}(t, t, T_n)$ denote the compounded yield to maturity defined by equating

$$P(t, T_n) = \sum_{k=1}^{n} c_k e^{-(T_k-t)\hat{f}(t,t,T_n)}, \quad 0 \leq t \leq T_1,$$

i.e. $\hat{f}(t, t, T_n)$ solves the equation

$$F(t, \hat{f}(t,t,T_n)) = P(t, T_n), \quad 0 \leq t \leq T_1,$$

with

$$F(t, x) := \sum_{k=1}^{n} c_k e^{-(T_k-t)x}, \quad 0 \leq t \leq T_1.$$
The bond duration $D(t, T_n)$ is the relative sensitivity of $P(t, T_n)$ with respect to $\tilde{f}(t, t, T_n)$ defined as

$$D(t, T_n) := -\frac{1}{P(t, T_n)} \frac{\partial F}{\partial x}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.$$ 

The bond convexity $C(t, T_n)$ is defined as

$$C(t, T_n) := \frac{1}{P(t, T_n)} \frac{\partial^2 F}{\partial x^2}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.$$ 

a) Compute the bond duration in case $n = 1$.

b) Show that the bond duration $D(t, T_n)$ can be interpreted as an average of times to maturity weighted by the respective discounted bond payoffs.

c) Show that the bond convexity $C(t, T_n)$ satisfies

$$C(t, T_n) = (D(t, T_n))^2 + (S(t, T_n))^2,$$

where $S(t, T_n)$ measures the dispersion of the duration of the bond payoffs around the portfolio duration $D(t, T_n)$.

d) Consider now the zero-coupon yield defined as

$$f_\alpha(t, t, T_n) := -\frac{1}{\alpha(T_n - t)} \log P(t, t + \alpha(T_n - t)),$$

where $\alpha \in (0, 1)$. Compute the bond duration associated to the yield $f_\alpha(t, t, T_n)$ in affine bond pricing models of the form

$$P(t, T) = e^{A(T-t)+rtB(T-t)}, \quad 0 \leq t \leq T.$$ 

e) [Wu00] Compute the bond duration associated to the yield $f_\alpha(t, t, T_n)$ in the Vasicek model in which $B(T-t) := (1 - e^{-(T-t)b})/b, 0 \leq t \leq T$.

Exercise 13.18  Stochastic string model [SCS01]. Consider an instantaneous forward rate $f(t, x)$ solution of

$$dtf(t, x) = \alpha x^2 dt + \sigma dt B(t, x), \quad (13.79)$$

with a flat initial curve $f(0, x) = r$, where $x$ represents the time to maturity, and $(B(t, x))_{(t,x)\in\mathbb{R}_+^2}$ is a standard Brownian sheet with covariance

$$\mathbb{E}[B(s, x)B(t, y)] = \min(s, t) \times \min(x, y), \quad s, t, x, y \in \mathbb{R}_+,$$

and initial conditions $B(t, 0) = B(0, x) = 0$ for all $t, x \in \mathbb{R}_+$.

a) Solve the equation (13.79) for $f(t, x)$.
b) Compute the short term interest rate $r_t = f(t, 0)$.
c) Compute the value at time $t \in [0, T]$ of the bond price

$$P(t, T) = \exp \left( - \int_0^{T-t} f(t, x) \, dx \right)$$

with maturity $T$.

d) Compute the variance $\mathbb{E} \left( \left( \int_0^{T-t} B(t, x) \, dx \right)^2 \right)$ of the centered Gaussian random variable $\int_0^{T-t} B(t, x) \, dx$.

e) Compute the expected value $\mathbb{E}^* [P(t, T)]$.

f) Find the value of $\alpha$ such that the discounted bond price

$$e^{-rt} P(t, T) = \exp \left( -rT - \frac{\alpha}{3} (T - t)^3 \right) - \sigma \int_0^{T-t} B(t, x) \, dx, \quad t \in [0, T].$$

satisfies $e^{-rt} \mathbb{E}^* [P(t, T)] = e^{-rT}$.

g) Compute the bond option price $\mathbb{E}^* \left[ \exp \left( - \int_0^T r_s \, ds \right) (P(T, S) - K)^+ \right]$ by the Black-Scholes formula, knowing that

$$\mathbb{E}[x e^{m+X} - K]^+ = x e^{m + \frac{v^2}{2}} \Phi(v + (m + \log(x/K))/v) - K \Phi((m + \log(x/K))/v),$$

when $X \sim \mathcal{N}(0, v^2)$ is a centered Gaussian random variable with variance $v^2$.

Exercise 13.19 (Exercise 13.10 continued).

a) Compute the forward rate

$$f(t, T, S) = - \frac{1}{S-T} (\log P(t, S) - \log P(t, T)).$$

b) Compute the instantaneous forward rate

$$f(t, T) = - \lim_{S \downarrow T} \frac{1}{S-T} (\log P(t, S) - \log P(t, T)).$$

c) Show that the limit $\lim_{T \downarrow t} f(t, T)$ does not exist in $L^2(\Omega)$.

d) Show that $P(t, T)$ satisfies the stochastic differential equation

$$\frac{dP(t, T)}{P(t, T)} = \sigma dB_t + \frac{1}{2} \sigma^2 dt - \frac{\log P(t, T)}{T-t} dt, \quad t \in [0, T].$$

e) Show, using the results of Exercise 13.9-(c), that

$$P(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_s \, ds} \bigg| \mathcal{F}_t \right],$$
where \((r_t^T)_{t \in [0,T]}\) is a stochastic process to be determined.
f) Compute the conditional density
\[
\mathbb{E}^* \left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \middle| \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)} e^{-\int_0^t r_s^T ds}
\]
of the forward measure \(\mathbb{P}_T\) with respect to \(\mathbb{P}^*\).
g) Show that the process
\[
\tilde{B}_t := B_t - \sigma t,
\]
\[0 \leq t \leq T,
\]
is a standard Brownian motion under \(\mathbb{P}_T\).
h) Compute the dynamics of \(X^S_t\) and \(P(t,S)\) under \(\mathbb{P}_T\).

*Hint:* Show that
\[
-\mu(S-T) + \sigma(S-T) \int_0^t \frac{1}{s-t} dB_s = \frac{S-T}{S-t} \log P(t,S).
\]
i) Compute the bond option price
\[
\mathbb{E}^* \left[ e^{-\int_t^T r_s^T ds} (P(T,S) - K)^+ \middle| \mathcal{F}_t \right] = P(t,T) \mathbb{E}_T \left[ (P(T,S) - K)^+ \middle| \mathcal{F}_t \right],
\]
\[0 \leq t < T < S.\]