Chapter 2
Discrete-Time Model

A basic limitation of the two time step model considered in Chapter 1 is that it does not allow for trading until the end of the time period is reached. In order to be able to re-allocate the portfolio over time we need to consider a discrete-time, multistep financial model with \( N + 1 \) time instants \( t = 0, 1, \ldots, N \). The practical importance of this model lies also in its direct computer implementability.

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2.1 Discrete-Time Compounding

**Investment plan**

We invest an amount \( m \) each year in an investment plan that carries a constant interest rate \( r \). At the end of the \( N - th \) year, the value of the amount \( m \) invested at the beginning of year \( k = 1, 2, \ldots, N \) has turned into \( (1 + r)^{N-k+1}m \) and the value of the plan at the end of the \( N - th \) year becomes

\[
A_N : = m \sum_{k=1}^{N} (1 + r)^{N-k+1} \tag{2.1}
\]
\[ N. \text{Privault} \]

\[
= m \sum_{k=1}^{N} (1 + r)^k \\
= m(1 + r) \frac{(1 + r)^N - 1}{r},
\]

i.e.

\[
\frac{A_N}{m} = \frac{(1 + r)^{N+1} - (1 + r)}{r},
\]

and

\[
N + 1 = \frac{1}{1 + r} \log \left( 1 + r + \frac{r A_N}{m} \right).
\]

**Loan repayment**

At time \( t = 0 \) one borrows an amount \( A_1 := A \) over a period of \( N \) years at the constant interest rate \( r \) per year. Denoting by \( A_k \) the amount owned by the borrower at the beginning of year \( n^o \ k = 1, 2, \ldots, N \), the amount \( m \) refunded at the end of the first year can be decomposed as

\[
m = r A_1 + (m - r A_1),
\]

into \( r A_1 \) paid in interest and \( m - r A_1 \) in principal repayment, i.e. there remains

\[
A_2 = A_1 - (m - r A_1) \\
= (1 + r) A_1 - m,
\]

to be refunded. Similarly, the amount \( m \) refunded at the end of the second year can be decomposed as

\[
m = r A_2 + (m - r A_2),
\]

into \( r A_2 \) paid in interest and \( m - r A_2 \) in principal repayment, i.e. there remains

\[
A_3 = A_2 - (m - r A_2) \\
= (1 + r) A_2 - m \\
= (1 + r)((1 + r) A_1 - m) - m \\
= (1 + r)^2 A_1 - m - (1 + r)m
\]

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to be refunded. It follows that in general, at the beginning of year \( k + 1 \) there remains

\[
A_{k+1} = (1 + r)^k A_1 - m \left( 1 + (1 + r) + \cdots + (1 + r)^{k-1} \right)
\]

\[
= (1 + r)^k A_1 - m \sum_{i=0}^{k-1} (1 + r)^i
\]

\[
= (1 + r)^k A_1 + m \frac{1 - (1 + r)^k}{r}
\]

to be refunded. The repayment at the end of year \( k \) is decomposed as

\[
m = rA_k + (m - rA_k),
\]

into

\[
rA_k = m + (1 + r)^{k-1} (rA_1 - m)
\]

in interest repayment, and

\[
m - rA_k = (1 + r)^{k-1} (m - rA_1)
\]

in principal repayment, hence

\[
A_k = \frac{m - (1 + r)^{k-1} (m - rA)}{r}, \quad k = 1, 2, \ldots, N.
\]

At the beginning of year \( N + 1 \), the loan should be completely repayed, hence \( A_{N+1} = 0 \), which reads

\[
(1 + r)^N A + m \frac{1 - (1 + r)^N}{r} = 0.
\]

From this relation we deduce

\[
m = \frac{r(1 + r)^N A}{(1 + r)^N - 1} = \frac{r}{1 - (1 + r)^{-N}} A,
\]

or

\[
\frac{A}{m} = \frac{1 - (1 + r)^{-N}}{r},
\]

and

\[
N = \frac{1}{\log(1 + r)} \log \frac{m}{m - rA} = - \log(1 - rA/m) \frac{r}{\log(1 + r)},
\]

where “\( \log \)” denotes the natural logarithm “\( \ln \)”. 
Remark: One needs \( m > rA \).

In this case we also have

\[
A = \frac{m}{(1+r)^N} \frac{(1+r)^N - 1}{r} = m \frac{1 - (1+r)^{-N}}{r} = \sum_{l=1}^{N} \frac{m}{(1+r)^l},
\]

hence the first interest repayment becomes

\[
rA = mr \sum_{l=1}^{N} \frac{1}{(1+r)^l} = m \left( 1 - (1+r)^{-N} \right),
\]

and the first principal repayment is

\[
m - rA = \frac{m}{(1+r)^N}.
\]

More generally, the \( k \)-th interest repayment can be written as

\[
mr \sum_{l=1}^{N-k+1} (1+r)^{-l} = m \left( 1 - \frac{1}{(1+r)^{N-k+1}} \right),
\]

and the \( k \)-th principal repayment is

\[
\frac{m}{(1+r)^{N-k+1}}, \quad k = 1, 2, \ldots, N.
\]

Note that the sum of discounted payments at the rate \( r \) is

\[
\sum_{l=1}^{N} \frac{m}{(1+r)^l} = m \frac{1 - (1+r)^{-N}}{r} = A.
\]

2.2 Stochastic Processes

A stochastic process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is a family \((X_t)_{t \in T}\) of random variables \(X_t : \Omega \rightarrow \mathbb{R}\) indexed by a set \(T\). Examples include:

- the two-instant model: \(T = \{0, 1\}\),

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- the discrete-time model with finite horizon: \( T = \{0, 1, \ldots, N\} \),
- the discrete-time model with infinite horizon: \( T = \mathbb{N} \),
- the continuous-time model: \( T = \mathbb{R}_+ \).

For real-world examples of stochastic processes one can mention:

- the time evolution of a risky asset; in this case \( X_t \) represents the price of the asset at time \( t \in T \).

- the time evolution of a physical parameter - for example, \( X_t \) represents a temperature observed at time \( t \in T \).

In this chapter we will focus on the finite horizon discrete-time model with \( T = \{0, 1, \ldots, N\} \).

Here, the vector

\[
\tilde{S}_0 = (S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)})
\]

denotes the prices at time \( t = 0 \) of \( d + 1 \) assets numbered \( 0, 1, \ldots, d \).

The random vector

\[
\tilde{S}_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})
\]

on \( \Omega \) denotes the values at time \( t = 1, 2, \ldots, N \) of assets \( 0, 1, \ldots, d \), and forms a stochastic process \((\tilde{S}_t)_{t=0,1,\ldots,N}\).

In the sequel we assume that asset \( n^0 \) 0 is a risk-free asset (of savings account type) yielding an interest rate \( r \), i.e. we have

\[
S_t^{(0)} = (1 + r)^t S_0^{(0)}, \quad t = 0, 1, \ldots, N.
\]

**2.3 Portfolio Strategies and Arbitrage**

A portfolio strategy is a stochastic process \((\tilde{\xi}_t)_{t=1,2,\ldots,N} \subset \mathbb{R}^{d+1}\) where \( \xi_t^{(k)} \)
denotes the (possibly fractional) quantity of asset \( n^0 k \) held in the portfolio over the time interval \((t - 1, t] \), \( t = 1, 2, \ldots, N \).

Note that the portfolio allocation

\[
\tilde{\xi}_t = (\xi_t^{(0)}, \xi_t^{(1)}, \ldots, \xi_t^{(d)})
\]

remains constant over the interval \((t - 1, t]\) while the stock price changes from \( S_{t-1}^{(k)} \) to \( S_t^{(k)} \) over this time interval.
In other words,
\[ \xi_t^{(k)} S_{t-1}^{(k)} \]
represents the amount invested in asset no. \( k \) at the beginning of the time interval \( (t-1, t] \), and
\[ \xi_t^{(k)} S_t^{(k)} \]
represents the value of this investment at the end of the time interval \( (t-1, t] \), \( t = 1, 2, \ldots, N \).

**Self-financing portfolio strategy**

The price of the portfolio at the beginning of the time interval \( (t-1, t] \) is
\[ \bar{\xi}_t \cdot \bar{S}_{t-1} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t-1}^{(k)} , \]
when the market “opens” at time \( t-1 \), and when the market “closes” at the end of the time interval \( (t-1, t] \) it becomes
\[ \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} , \tag{2.2} \]
\( t = 1, 2, \ldots, N \).

At the beginning of the next trading period \( (t, t+1] \) the portfolio value becomes
\[ \bar{\xi}_{t+1} \cdot \bar{S}_{t} = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_{t}^{(k)} , \quad t = 0, 1, \ldots, N - 1. \tag{2.3} \]

Note that the stock price \( \bar{S}_t \) is assumed to remain constant “overnight”, i.e. from the end of \( (t-1, t] \) to the beginning of \( (t, t+1] \).

Obviously the question arises whether (2.2) should be identical to (2.3). In the sequel we will need such a consistency hypothesis, called self-financing condition, on the portfolio strategy \( \xi_t \).

**Definition 2.1.** A portfolio strategy \( (\bar{\xi}_t)_{t=1,2,\ldots,N} \) is said to be self-financing if
\[ \bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_{t} , \quad t = 1, 2, \ldots, N - 1. \tag{2.4} \]

The meaning of the self-financing condition (2.4) is simply that one cannot take any money in or out of the portfolio during the “overnight” transition period at time \( t \). In other words, at the beginning of the new trading period \( (t, t+1] \) one should re-invest the totality of the portfolio value obtained at
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the end of the interval \((t - 1, t]\).

The next figure is an illustration of the self-financing condition.

<table>
<thead>
<tr>
<th>Portfolio value</th>
<th>(\xi_t \tilde{S}_{t-1})</th>
<th>(\xi_t \tilde{S}<em>t = \xi</em>{t+1} \tilde{S}_t)</th>
<th>(\xi_{t+1} \tilde{S}_{t+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset value</td>
<td>(S_{t-1})</td>
<td>(S_t)</td>
<td>(S_{t+1})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time scale</th>
<th>(t - 1)</th>
<th>(t)</th>
<th>(t + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio allocation</td>
<td>(\xi_t)</td>
<td>(\xi_t)</td>
<td>(\xi_{t+1})</td>
</tr>
</tbody>
</table>

“Morning” “Evening” “Morning” “Evening”

Fig. 2.1: Illustration of the self-financing condition (2.4).

By (2.2) and (2.3) the self-financing condition (2.4) can be rewritten as

\[
\sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \quad t = 0, 1, \ldots, N - 1,
\]

or

\[
\sum_{k=0}^{d} (\xi_{t+1}^{(k)} - \xi_t^{(k)}) S_t^{(k)} = 0, \quad t = 0, 1, \ldots, N - 1.
\]

Note that any portfolio strategy \((\tilde{\xi}_t)_{t=1,2,\ldots,N}\) which is constant over time, i.e. \(\tilde{\xi}_t = \tilde{\xi}_{t+1}, t = 1, 2, \ldots, N - 1\), is self-financing by construction.

Here, portfolio re-allocation happens “overnight”, during which time the global portfolio value remains the same due to the self-financing condition. The portfolio allocation \(\xi_t\) remains the same throughout the day, however the portfolio value changes from morning to evening due to a change in the stock price. Also, \(\tilde{\xi}_0\) is not defined and its value is actually not needed in this framework.

Clearly the chosen unit of time may not be the day, and it can be replaced by weeks, hours, minutes, or even fractions of seconds in high-frequency trading.

We will denote by

\[ V_t := \tilde{\xi}_t \cdot \tilde{S}_t \]

the portfolio value at times \(t = 1, 2, \ldots, N\), with, by the self-financing condition (2.4),

\[ V_t = \tilde{\xi}_{t+1} \cdot \tilde{S}_t, \quad t = 0, 1, \ldots, N - 1, \]
and in particular
\[ V_0 = \tilde{\xi}_1 \cdot \tilde{S}_0 \]
at \( t = 0 \).

Discounting

My portfolio \( S_t \) grew by \( b = 5\% \) this year.
Q: Did I achieve a positive return?
A:

(a) Scenario A.

My portfolio \( S_t \) grew by \( b = 5\% \) this year.
The risk-free or inflation rate is \( r = 10\% \).
Q: Did I achieve a positive return?
A:

(b) Scenario B.

Fig. 2.2: Why apply discounting?

**Definition 2.2.** Let
\[ \tilde{X}_t := (\tilde{S}_t^{(0)}, \tilde{S}_t^{(1)}, \ldots, \tilde{S}_t^{(d)}) \]
denote the vector of discounted asset prices, defined as:
\[ \tilde{S}_t^{(i)} = \frac{1}{(1 + r)^t} S_t^{(i)}, \quad i = 0, 1, \ldots, d, \quad t = 0, 1, \ldots, N. \]

We can also write
\[ \tilde{X}_t := \frac{1}{(1 + r)^t} \tilde{S}_t, \quad t = 0, 1, \ldots, N. \]

The discounted price at time 0 of the portfolio is defined by
\[ \tilde{V}_t = \frac{1}{(1 + r)^t} V_t, \quad t = 0, 1, \ldots, N. \]

We have
\[
\tilde{V}_t = \frac{1}{(1 + r)^t} \tilde{\xi}_t \cdot \tilde{S}_t \\
= \frac{1}{(1 + r)^t} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\
= \sum_{k=0}^{d} \xi_t^{(k)} \tilde{S}_t^{(k)}
\]
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\[ = \tilde{\xi}_t \cdot \tilde{X}_t, \quad t = 1, 2, \ldots, N, \]

and

\[ \tilde{V}_0 = \tilde{\xi}_1 \cdot \tilde{X}_0 = \tilde{\xi}_1 \cdot \tilde{S}_0. \]

The effect of discounting from time \( t \) to time 0 is to divide prices by \((1 + r)^t\), making all prices comparable at time 0.

**Arbitrage**

The definition of arbitrage in discrete time follows the lines of its analog in the two-step model.

**Definition 2.3.** A portfolio strategy \( (\tilde{\xi}_t)_{t=1,2,\ldots,N} \) constitutes an arbitrage opportunity if all three following conditions are satisfied:

i) \( V_0 \leq 0 \) at time \( t = 0 \), \[ \text{[start from a zero-cost portfolio or in debt]} \]

ii) \( V_N \geq 0 \) at time \( t = N \), \[ \text{[finish with a nonnegative amount]} \]

iii) \( \mathbb{P}(V_N > 0) > 0 \) at time \( t = N \). \[ \text{[profit made with non-zero probability]} \]

**2.4 Contingent Claims**

Recall that from Definition 1.8, a contingent claim is given by the nonnegative random payoff \( C \) of an option contract at maturity time \( t = N \). For example, in the case of the European call of Definition 0.2, the payoff \( C \) is given by \( C = (S_N^{(i)} - K)^+ \) where \( K \) is called the strike (or exercise) price. The list given below is somewhat restrictive and there exists many more option types, with new ones appearing constantly on the markets.

**Physical delivery vs cash settlement**

The cash settlement realized through the payoff \( C = (S_N^{(i)} - K)^+ \) can be replaced by the *physical delivery* of the underlying asset in exchange for the strike price \( K \). Physical delivery occurs only when \( S_N^{(i)} > K \), in which case the underlying asset can be sold at the price \( S_N^{(i)} \) by the option holder, for a profit \( S_N^{(i)} - K \). When \( S_N^{(i)} > K \), no delivery occurs and the profit is 0, which is consistent with the expression \( C = (S_N^{(i)} - K)^+ \). A similar procedure can be applied to other option contracts.

**European options**

The payoff of a European call option on the underlying asset \( n^o \ i \) with maturity \( N \) and strike price \( K \) is

\[ \diamond \]
\[ C = (S_N^{(i)} - K)^+ = \begin{cases} S_N^{(i)} - K & \text{if } S_N^{(i)} \geq K, \\ 0 & \text{if } S_N^{(i)} < K. \end{cases} \]

The *moneyness* at time \( t = 0, 1, \ldots, N \) of the European call option with strike price \( K \) on the asset \( n^o \) \( i \) is the ratio
\[ M_t^{(i)} := \frac{S_t^{(i)} - K}{S_t^{(i)}}, \quad t = 0, 1, \ldots, N. \]

The option is said to be *out of the money* (OTM) when \( M_t^{(i)} < 0 \), *in the money* (ITM) when \( M_t^{(i)} > 0 \), and *at the money* (ATM) when \( M_t^{(i)} = 0 \).

The payoff of a European put option on the underlying asset \( n^o \) \( i \) with exercise date \( N \) and strike price \( K \) is
\[ C = (K - S_N^{(i)})^+ = \begin{cases} K - S_N^{(i)} & \text{if } S_N^{(i)} \leq K, \\ 0 & \text{if } S_N^{(i)} > K. \end{cases} \]

The *moneyness* at time \( t = 0, 1, \ldots, N \) of the European put option with strike price \( K \) on the asset \( n^o \) \( i \) is the ratio
\[ M_t^{(i)} := \frac{K - S_t^{(i)}}{S_t^{(i)}}, \quad t = 0, 1, \ldots, N. \]

**Binary options**

Binary (or digital) options, also called cash-or-nothing options, are options whose payoffs are of the form
\[ C = \mathbb{1}_{[K, \infty]}(S_N^{(i)}) = \begin{cases} \$1 & \text{if } S_N^{(i)} \geq K, \\ 0 & \text{if } S_N^{(i)} < K, \end{cases} \]
for binary call options, and
\[ C = \mathbb{1}_{(-\infty, K]}(S_N^{(i)}) = \begin{cases} \$1 & \text{if } S_N^{(i)} \leq K, \\ 0 & \text{if } S_N^{(i)} > K, \end{cases} \]
for binary put options.
Asian options

The payoff of an Asian call option (also called average value option) on the underlying asset \( n^o \ i \) with exercise date \( N \) and strike price \( K \) is

\[
C = \left( \frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} - K \right)^+.
\]

The payoff of an Asian put option on the underlying asset \( n^o \ i \) with exercise date \( N \) and strike price \( K \) is

\[
C = \left( K - \frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} \right)^+.
\]

We refer to Section 10.1 for the pricing of Asian options in continuous time. It can be shown, cf. Exercise 3.8 that Asian call option prices can be upper bounded by European call option prices.

Other examples of such options include weather derivatives (based on average temperatures) and volatility derivatives (based on average volatility).

Barrier options

The payoff of a down-an-out (or knock-out) barrier call option on the underlying asset \( n^o \ i \) with exercise date \( N \), strike price \( K \) and barrier \( B \) is

\[
C = \left( S_N^{(i)} - K \right)^+ \mathbb{1}\left\{ \min_{t=0,1,...,N} S_t^{(i)} > B \right\} = \begin{cases} (S_N^{(i)} - K)^+ & \text{if } \min_{t=0,1,...,N} S_t^{(i)} > B, \\ 0 & \text{if } \min_{t=0,1,...,N} S_t^{(i)} \leq B. \end{cases}
\]

This option is also called a “Callable Bull Contract with no residual value”, in which \( B \) denotes the call price \( B \geq K \). It is also called a turbo warrant with no rebate.

The payoff of an up-and-out barrier put option on the underlying asset \( n^o \ i \) with exercise date \( N \), strike price \( K \) and barrier \( B \) is

\[
C = \left( K - S_N^{(i)} \right)^+ \mathbb{1}\left\{ \max_{t=0,1,...,N} S_t^{(i)} < B \right\} = \begin{cases} (K - S_N^{(i)})^+ & \text{if } \max_{t=0,1,...,N} S_t^{(i)} < B, \\ 0 & \text{if } \max_{t=0,1,...,N} S_t^{(i)} \geq B. \end{cases}
\]
This option is also called a “Callable Bear Contract with no residual value”, in which the call price $B$ usually satisfies $B \leq K$. See [EP06], [WC08] for the pricing of Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants. We refer the reader to Chapter 8 for the pricing and hedging of related options in continuous time. Barrier options in continuous time are priced in Section 8.5.

**Lookback options**

The payoff of a floating strike lookback call option on the underlying asset no $i$ with exercise date $N$ is

$$C = S_N^{(i)} - \min_{t=0,1,...,N} S_t^{(i)}.$$

The payoff of a floating strike lookback put option on the underlying asset no $i$ with exercise date $N$ is

$$C = \left( \max_{t=0,1,...,N} S_t^{(i)} \right) - S_N^{(i)}.$$

We refer to Section 9.1 for the pricing of lookback options in continuous time.

**Vanilla vs exotic options**

Vanilla options such as European or binary options, have a payoff $\phi(S_N^{(i)})$ that depends only on the terminal value $S_N^{(i)}$ of the underlying asset at maturity, as opposed to exotic or path-dependent options such as Asian, barrier, or lookback options, whose payoff may depend on the whole path of the underlying asset price until expiration time.
Exotic vs vanilla options
Vanilla options are called that way because:
(A) They were first used for the trading of vanilla by the Maya beginning around the 14th century.
(B) “Plain vanilla” is the most standard and common of all ice cream flavors.
(C) To meet FDA standards, pure vanilla extract must contain 13.35 ounces of vanilla beans per gallon.
(D) Sir Charles C. Vanilla, FLS, was the early discoverer of the properties of Brownian motion in asset pricing.

Fig. 2.3: Take the Quiz.

2.5 Martingales and Conditional Expectation

Before proceeding to the definition of risk-neutral probability measures in discrete time we need to introduce more mathematical tools such as conditional expectations, filtrations, and martingales.

Conditional expectations

Clearly, the expected value of any risky asset or random variable is dependent on the amount of available information. For example, the expected return on a real estate investment typically depends on the location of this investment.

In the probabilistic framework the available information is formalized as a collection $\mathcal{G}$ of events, which may be smaller than the collection $\mathcal{F}$ of all available events, i.e. $\mathcal{G} \subset \mathcal{F}$.

The notation $\mathbb{E}[F \mid \mathcal{G}]$ represents the expected value of a random variable $F$ given (or conditionally to) the information contained in $\mathcal{G}$, and it is read “the conditional expectation of $F$ given $\mathcal{G}$”. In a certain sense, $\mathbb{E}[F \mid \mathcal{G}]$ represents the best possible estimate of $F$ in mean square sense, given the information contained in $\mathcal{G}$.

The conditional expectation satisfies the following five properties, cf. Section 18.6 for details and proofs.

* The collection $\mathcal{G}$ is also called a $\sigma$-algebra, cf. Section 18.1.
i) $\mathbb{E}[FG \mid \mathcal{G}] = G \mathbb{E}[F \mid \mathcal{G}]$ if $G$ depends only on the information contained in $\mathcal{G}$.

ii) $\mathbb{E}[G \mid \mathcal{G}] = G$ when $G$ depends only on the information contained in $\mathcal{G}$.

iii) $\mathbb{E}[\mathbb{E}[F \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[F \mid \mathcal{G}]$ if $\mathcal{G} \subset \mathcal{H}$, called the *tower property*, cf. also Relation (18.40).

iv) $\mathbb{E}[F \mid \mathcal{G}] = \mathbb{E}[F]$ when $F$ “does not depend” on the information contained in $\mathcal{G}$ or, more precisely stated, when the random variable $F$ is *independent* of the $\sigma$-algebra $\mathcal{G}$.

v) If $G$ depends only on $\mathcal{G}$ and $F$ is independent of $\mathcal{G}$, then

$$\mathbb{E}[h(F,G) \mid \mathcal{G}] = \mathbb{E}[h(F,x)]_{x=G}.$$ 

When $\mathcal{H} = \{\emptyset, \Omega\}$ is the trivial $\sigma$-algebra we have

$$\mathbb{E}[F \mid \mathcal{H}] = \mathbb{E}[F], \quad F \in L^1(\Omega).$$

See (18.40) and (18.46) for illustrations of the tower property by conditioning with respect to discrete and continuous random variables.

**Filtrations**

The total amount of “information” available on the market at times $t = 0, 1, \ldots, N$ is denoted by $\mathcal{F}_t$. We assume that

$$\mathcal{F}_t \subset \mathcal{F}_{t+1}, \quad t = 0, 1, \ldots, N - 1,$$

which means that the amount of information available on the market increases over time.

Usually, $\mathcal{F}_t$ corresponds to the knowledge of the values $S^{(i)}_0, S^{(i)}_1, \ldots, S^{(i)}_t$, $i = 1, 2, \ldots, d$, of the risky assets up to time $t$. In mathematical notation we say that $\mathcal{F}_t$ is generated by $S^{(i)}_0, S^{(i)}_1, \ldots, S^{(i)}_t$, $i = 1, 2, \ldots, d$, and we usually write

$$\mathcal{F}_t = \sigma \left( S^{(i)}_0, S^{(i)}_1, \ldots, S^{(i)}_t, \quad i = 1, 2, \ldots, d \right), \quad t = 0, 1, \ldots, N,$$

with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Example: Consider the simple random walk

$$Z_t := X_1 + X_2 + \cdots + X_t, \quad t \geq 0,$$
where \((X_t)_{t \geq 1}\) is a sequence of independent, identically distributed \((-1,1)\)-valued random variables. The filtration (or information flow) \((F_t)_{t \geq 0}\) generated by \((Z_t)_{t \geq 0}\) is given by \(F_0 = \{\emptyset, \Omega\}\), \(F_1 = \{\emptyset, \{X_1 = 1\}, \{X_1 = -1\}, \Omega\}\), and
\[
F_2 = \sigma \left( \{\emptyset, \{X_1 = 1, X_2 = 1\}, \{X_1 = 1, X_2 = -1\}, \{X_1 = -1, X_2 = 1\}, \{X_1 = -1, X_2 = -1\}, \Omega\} \right).
\]

The notation \(F_t\) is useful to represent a quantity of information available at time \(t\). Note that different agents or traders may work with different filtrations. For example, an insider may have access to a filtration \((G_t)_{t=0,1,...,N}\) which is larger than the ordinary filtration \((F_t)_{t=0,1,...,N}\) available to an ordinary agent, in the sense that
\[
F_t \subset G_t, \quad t = 0, 1, \ldots, N.
\]

The notation \(\mathbb{E}[F \mid F_t]\) represents the expected value of a random variable \(F\) given (or conditionally to) the information contained in \(F_t\). Again, \(\mathbb{E}[F \mid F_t]\) denotes the best possible estimate of \(F\) in mean square sense, given the information known up to time \(t\).

We will assume that no information is available at time \(t = 0\), which translates as
\[
\mathbb{E}[F \mid F_0] = \mathbb{E}[F]
\]
for any integrable random variable \(F\). As above, the conditional expectation with respect to \(F_t\) satisfies the following five properties:

i) \(\mathbb{E}[FG \mid F_t] = F \mathbb{E}[G \mid F_t]\) if \(F\) depends only on the information contained in \(F_t\).

ii) \(\mathbb{E}[F \mid F_t] = F\) when \(F\) depends only on the information known at time \(t\) and contained in \(F_t\).

iii) \(\mathbb{E}[\mathbb{E}[F \mid F_{t+1}] \mid F_t] = \mathbb{E}[F \mid F_t]\) if \(F_t \subset F_{t+1}\) (by the tower property, cf. also Relation (6.1) below).

iv) \(\mathbb{E}[F \mid F_t] = \mathbb{E}[F]\) when \(F\) does not depend on the information contained in \(F_t\).

v) If \(F\) depends only on \(F_t\) and \(G\) is independent of \(F_t\), then
\[
\mathbb{E}[h(F, G) \mid F_t] = \mathbb{E}[h(x, G)]_{x=F}.
\]
Note that by the tower property (iii) the process $t \mapsto \mathbb{E}[F | \mathcal{F}_t]$ is a martingale, cf. e.g. Relation (6.1) for details.

**Martingales**

A martingale is a stochastic process whose value at time $t+1$ can be estimated using conditional expectation given its value at time $t$. Recall that a process $(M_t)_{t=0,1,\ldots,N}$ is said to be $(\mathcal{F}_t)_{t=0,1,\ldots,N}$-adapted if the value of $M_t$ depends only on the information available at time $t$ in $\mathcal{F}_t$, $t = 0,1,\ldots,N$.

**Definition 2.4.** A stochastic process $(M_t)_{t=0,1,\ldots,N}$ is called a discrete-time martingale with respect to the filtration $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ if $(M_t)_{t=0,1,\ldots,N}$ is $(\mathcal{F}_t)_{t=0,1,\ldots,N}$-adapted and satisfies the property

$$
\mathbb{E}[M_{t+1} | \mathcal{F}_t] = M_t, \quad t = 0,1,\ldots,N-1.
$$

Note that the above definition implies that $M_t \in \mathcal{F}_t$, $t = 0,1,\ldots,N$. In other words, a random process $(M_t)_{t=0,1,\ldots,N}$ is a martingale if the best possible prediction of $M_{t+1}$ in the mean square sense given $\mathcal{F}_t$ is simply $M_t$.

In discrete-time finance, the martingale property can be used to characterize risk-neutral probability measures, and for the computation of conditional expectations.

**Exercise.** Using the tower property of conditional expectations, show that Definition 2.4 can be equivalently stated by saying that

$$
\mathbb{E}[M_n | \mathcal{F}_k] = M_k, \quad 0 \leq k < n.
$$

A particular property of martingales is that their expectation is constant over time.

**Proposition 2.5.** Let $(Z_n)_{n \in \mathbb{N}}$ be a martingale. We have

$$
\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \quad n \in \mathbb{N}.
$$

**Proof.** From the tower property (18.40) we have:

$$
\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1} | \mathcal{F}_n]] = \mathbb{E}[Z_n], \quad n \in \mathbb{N},
$$

hence by induction on $n \in \mathbb{N}$ we have

$$
\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_{n-1}] = \cdots = \mathbb{E}[Z_1] = \mathbb{E}[Z_0], \quad n \in \mathbb{N}.
$$

As an example of the use of martingales, we can mention weather forecasting. If $M_t$ denotes the random temperature observed at time $t$, this process...
is a martingale when the best possible forecast of tomorrow’s temperature \( M_{t+1} \) given information known up to time \( t \) is simply today’s temperature \( M_t \), \( t = 0, 1, \ldots, N - 1 \).

**Definition 2.6.** A stochastic process \( (\xi_k)_{k \geq 1} \) is said to be predictable if \( \xi_k \) depends only on the information in \( F_{k-1} \), \( k \geq 1 \).

When \( F_0 \) simply takes the form \( F_0 = \{ \emptyset, \Omega \} \) we find that \( \xi_1 \) is a constant when \( (\xi_t)_{t=1,2,\ldots,N} \) is a predictable process. Recall that on the other hand, the process \( (S_t^{(i)})_{t=0,1,\ldots,N} \) is adapted as \( S_t^{(i)} \) depends only on the information in \( F_t \), \( t = 0, 1, \ldots, N \), \( i = 1, 2, \ldots, d \).

The discrete-time stochastic integral (2.5) will be interpreted as the sum of discounted profits and losses \( \xi_k (\tilde{S}_k^{(1)} - \tilde{S}_{k-1}^{(1)}) \), \( k = 1, 2, \ldots, t \), in a portfolio holding a quantity \( \xi_k \) of a risky asset whose price variation is \( \tilde{S}_k^{(1)} - \tilde{S}_{k-1}^{(1)} \) at time \( k = 1, 2, \ldots, t \).

An important property of martingales is that the discrete-time stochastic integral (2.5) of a predictable process is itself a martingale, see also Proposition 6.1 for the continuous-time analog of the following proposition, which will be used in the proof of Theorem 3.5 below.

In the sequel, the martingale (2.5) will be interpreted as a discounted portfolio value, in which \( \tilde{S}_k^{(1)} - \tilde{S}_{k-1}^{(1)} \) represents the increment in the discounted asset price and \( \xi_k \) is the amount invested in that asset, \( k = 1, 2, \ldots, N \).

**Proposition 2.7.** Given \( (X_k)_{k=0,1,\ldots,N} \) a martingale and \( (\xi_k)_{k=1,2,\ldots,N} \) a (bounded) predictable process, the discrete-time process \( (M_t)_{t=0,1,\ldots,N} \) defined by

\[
M_t = \sum_{k=1}^{t} \xi_k (X_k - X_{k-1}), \quad t = 0, 1, \ldots, N, \tag{2.5}
\]

is a martingale.

**Proof.** Given \( n > t \geq 0 \) we have

\[
\mathbb{E}[M_n \mid F_t] = \mathbb{E} \left[ \sum_{k=1}^{n} \xi_k (X_k - X_{k-1}) \mid F_t \right]
\]

\[
= \sum_{k=1}^{n} \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid F_t \right]
\]

\[
= \sum_{k=1}^{t} \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid F_t \right] + \sum_{k=t+1}^{n} \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid F_t \right]
\]

* See [here](http://www.ntu.edu.sg/home/nprivault/indext.html) for a related discussion of martingale strategies in a particular case.
\[ \sum_{k=1}^{t} \xi_k (X_k - X_{k-1}) + \sum_{k=t+1}^{n} \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t \right] \]

\[ = M_t + \sum_{k=t+1}^{n} \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t \right]. \]

In order to conclude to \( \mathbb{E} [M_n \mid \mathcal{F}_t] = M_t \) we need to show that

\[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t \right] = 0, \quad t + 1 \leq k \leq n. \]

First we note that when \( 0 \leq t \leq k - 1 \) we have \( \mathcal{F}_t \subset \mathcal{F}_{k-1} \), hence by the “tower property” of conditional expectations we get

\[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1} \right] \mid \mathcal{F}_t \right]. \]

Next, since the process \( (\xi_k)_{k \geq 1} \) is predictable, \( \xi_k \) depends only on the information in \( \mathcal{F}_{k-1} \), and using Property \( (ii) \) of conditional expectations we may pull out \( \xi_k \) out of the expectation since it behaves as a constant parameter given \( \mathcal{F}_{k-1}, k = 1, 2, \ldots, n \). This yields

\[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1} \right] = \xi_k \mathbb{E} \left[ X_k - X_{k-1} \mid \mathcal{F}_{k-1} \right] = 0 \]

since

\[ \mathbb{E} \left[ X_k - X_{k-1} \mid \mathcal{F}_{k-1} \right] = \mathbb{E} \left[ X_k \mid \mathcal{F}_{k-1} \right] - \mathbb{E} \left[ X_{k-1} \mid \mathcal{F}_{k-1} \right] \\
= \mathbb{E} \left[ X_k \mid \mathcal{F}_{k-1} \right] - X_{k-1} \\
= 0, \quad k = 1, 2, \ldots, N, \]

because \( (X_k)_{k=0,1,\ldots,N} \) is a martingale. We conclude that

\[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1} \right] = \xi_k \mathbb{E} \left[ X_k - X_{k-1} \mid \mathcal{F}_{k-1} \right] = 0, \]

and, more generally, that

\[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ \xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1} \right] \mid \mathcal{F}_t \right] \\
= \mathbb{E} \left[ \xi_k \mathbb{E} \left[ X_k - X_{k-1} \mid \mathcal{F}_{k-1} \right] \mid \mathcal{F}_t \right] \\
= 0, \quad k = t + 1, t + 2, \ldots, n. \]
2.6 Market Completeness and Risk-Neutral Measures

As in the two time step model, the concept of risk-neutral measures will be used to price financial claims under the absence of arbitrage hypothesis.*

**Definition 2.8.** A probability measure \( \mathbb{P}^* \) on \( \Omega \) is called a risk-neutral measure if under \( \mathbb{P}^* \), the expected return of each risky asset equals the return \( r \) of the risk-free asset, that is

\[
\mathbb{E}^* \left[ S_t^{(i)} \bigg| \mathcal{F}_t \right] = (1 + r) S_t^{(i)}, \quad t = 0, 1, \ldots, N - 1, \quad (2.6)
\]

\( i = 0, 1, \ldots, d \). Here, \( \mathbb{E}^* \) denotes the expectation under \( \mathbb{P}^* \).

Since \( S_t^{(i)} \in \mathcal{F}_t \), denoting by

\[ R_{t+1}^{(i)} := \frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}} \]

the return of asset \( n^o \) \( i \) over the time interval \((t, t + 1]\), \( t = 0, 1, \ldots, N - 1 \), Relation (2.6) can be rewritten as

\[
\mathbb{E}^* \left[ \frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}} \bigg| \mathcal{F}_t \right] = r, \quad t = 0, 1, \ldots, N - 1,
\]

which means that the average of the return \((S_{t+1}^{(i)} - S_t^{(i)})/S_t^{(i)}\) of asset \( n^o \) \( i \) under the risk-neutral probability measure \( \mathbb{P}^* \) is equal to the risk-free rate \( r \).

In other words, taking risks under \( \mathbb{P}^* \) by buying the risky asset \( n^o \) \( i \) has a neutral effect, as the expected return is that of the risk-free asset. The measure \( \mathbb{P}^\# \) would yield a positive risk premium if we had

\[
\mathbb{E}^\# \left[ S_{t+1}^{(i)} \bigg| \mathcal{F}_t \right] = (1 + \tilde{r}) S_t^{(i)}, \quad t = 0, 1, \ldots, N - 1,
\]

with \( \tilde{r} > r \), and a negative risk premium if \( \tilde{r} < r \).

In the next proposition we reformulate the definition of risk-neutral probability measure using the notion of martingale.

**Proposition 2.9.** A probability measure \( \mathbb{P}^* \) on \( \Omega \) is a risk-neutral measure if and only if the discounted price process

\[
\tilde{S}_t^{(i)} := \frac{S_t^{(i)}}{(1 + r)^t}, \quad t = 0, 1, \ldots, n
\]

is a martingale under \( \mathbb{P}^* \), i.e.*

* See also the Efficient Market Hypothesis.
\[ \mathbb{E}^* \left[ \tilde{S}_{t+1}^{(i)} \mid \mathcal{F}_t \right] = \tilde{S}_t^{(i)}, \quad t = 0, 1, \ldots, N - 1, \tag{2.7} \]

\( i = 0, 1, \ldots, d. \)

**Proof.** It suffices to check that by the relation \( S_t^{(i)} = (1 + r)^t \tilde{S}_t^{(i)}, \) Condition (2.6) can be rewritten as

\[ (1 + r)^{t+1} \mathbb{E}^* \left[ \tilde{S}_{t+1}^{(i)} \mid \mathcal{F}_t \right] = (1 + r)(1 + r)^t \tilde{S}_t^{(i)}, \]

\( i = 1, 2, \ldots, d, \) which is clearly equivalent to (2.7) after division by \( (1 + r)^t, \) \( t = 0, 1, \ldots, N - 1. \) \( \square \)

Note that, as a consequence of Propositions 2.5 and 2.9, the discounted price process \( \tilde{S}_t^{(i)} := S_t^{(i)}/(1 + r)^t, t = 0, 1, \ldots, n, \) has constant expectation under the risk-neutral probability measure \( \mathbb{P}^*, i.e.

\[ \mathbb{E}^* \left[ \tilde{S}_t^{(i)} \right] = \tilde{S}_0^{(i)}, \quad t = 1, 2, \ldots, N, \]

for \( i = 0, 1, \ldots, d. \)

In the sequel we will only consider probability measures \( \mathbb{P}^* \) that are equivalent to \( \mathbb{P} \) in the sense that they have the share the same events of zero probability.

**Definition 2.10.** A probability measure \( \mathbb{P}^* \) on \((\Omega, \mathcal{F})\) is said to be equivalent to another probability measure \( \mathbb{P} \) when

\[ \mathbb{P}^*(A) = 0 \text{ if and only if } \mathbb{P}(A) = 0, \text{ for all } A \in \mathcal{F}. \tag{2.8} \]

Next, we restate the first fundamental theorem of asset pricing in discrete time, which can be used to check for the existence of arbitrage opportunities.

**Theorem 2.11.** A market is without arbitrage opportunity if and only if it admits at least one (equivalent) risk-neutral measure.

**Proof.** See [HK79] and Theorem 5.17 of [FS04]. \( \square \)

Next, we turn to the notion of market completeness, starting with the definition of attainability for a contingent claim.

**Definition 2.12.** A contingent claim with payoff \( C \) is said to be attainable (at time \( N \)) if there exists a self-financing portfolio strategy \((\bar{\xi}_t)_{t=1,2,\ldots,N}\) such that

\[ C = \bar{\xi}_N \cdot \bar{S}_N, \quad \mathbb{P} - a.s. \tag{2.9} \]

In case \((\bar{\xi}_t)_{t=1,2,\ldots,N}\) is a portfolio that attains the claim \( C \) at time \( N, i.e. if (2.9) is satisfied, we also say that \((\bar{\xi}_t)_{t=1,2,\ldots,N}\) hedges the claim \( C \). In case (2.9) is replaced by the condition

\[ \bar{\xi}_N \cdot \bar{S}_N \geq C, \]
we talk of super-hedging.

When a self-financing portfolio \( (\xi_t)_{t=1,2,\ldots,N} \) hedges a claim \( C \), the arbitrage price \( \pi_t(C) \) of the claim at time \( t \) is given by the value

\[
\pi_t(C) = \xi_t \cdot S_t
\]
of the portfolio at time \( t = 0,1,\ldots,N \). Recall that arbitrage prices can be used to ensure that financial derivatives are “marked” at their fair value (mark to market). Note that at time \( t = N \) we have

\[
\pi_N(C) = \xi_N \cdot S_N = C,
\]
i.e. since exercise of the claim occurs at time \( N \), the price \( \pi_N(C) \) of the claim equals the value \( C \) of the payoff.

**Definition 2.13.** A market model is said to be complete if every contingent claim is attainable.

The next result can be viewed as the second fundamental theorem of asset pricing.

**Theorem 2.14.** A market model without arbitrage opportunities is complete if and only if it admits only one (equivalent) risk-neutral measure.

**Proof.** See [HK79] and Theorem 5.38 of [FS04]. \( \square \)

### 2.7 The Cox-Ross-Rubinstein (CRR) Market Model

We consider the discrete-time Cox-Ross-Rubinstein model [CRR79] with \( N + 1 \) time instants \( t = 0,1,\ldots,N \) and \( d = 1 \) risky asset, also called the binomial model. The price \( S_t^{(0)} \) of the risk-free asset evolves as

\[
S_t^{(0)} = S_0^{(0)} (1 + r)^t, \quad t = 0,1,\ldots,N.
\]

Let the return of the risky asset \( S_t^{(1)} \) be defined as

\[
R_t := \frac{S_t^{(1)} - S_{t-1}^{(1)}}{S_{t-1}^{(1)}}, \quad t = 1,2,\ldots,N.
\]

In the CRR model the return \( R_t \) is random and allowed to take only two values \( a \) and \( b \) at each time step, i.e.

\[
R_t \in \{a,b\}, \quad t = 1,2,\ldots,N,
\]

with \(-1 < a < b\). That means, the evolution of \( S_{t-1}^{(1)} \) to \( S_t^{(1)} \) is random and given by
\[ S_t^{(1)} = \begin{cases} (1 + b)S_{t-1}^{(1)} & \text{if } R_t = b \\ (1 + a)S_{t-1}^{(1)} & \text{if } R_t = a \end{cases} = (1 + R_t)S_{t-1}^{(1)}, \quad t = 1, \ldots, N, \]

and

\[ S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t} (1 + R_k), \quad t = 0, 1, \ldots, N. \]

Note that the price process \( (S_t^{(1)})_{t=0,1,\ldots,N} \) evolves on a binary recombining (or binomial) tree of the following type:

\[ S_2 = S_0(1 + b)^2 \]

\[ S_1 = S_0(1 + b) \]

\[ S_0 \]

\[ S_1 = S_0(1 + a) \]

\[ S_2 = S_0(1 + a)(1 + b) \]

\[ S_2 = S_0(1 + a)^2. \]

The discounted asset price is

\[ \tilde{S}_t^{(1)} = \frac{S_t^{(1)}}{(1 + r)^t}, \quad t = 0, 1, \ldots, N, \]

with

\[ \tilde{S}_t^{(1)} = \begin{cases} \frac{1 + b}{1 + r} \tilde{S}_{t-1}^{(1)} & \text{if } R_t = b \\ \frac{1 + a}{1 + r} \tilde{S}_{t-1}^{(1)} & \text{if } R_t = a \end{cases} = \frac{1 + R_t}{1 + r} \tilde{S}_{t-1}^{(1)}, \quad t = 1, \ldots, N, \]

and

\[ \tilde{S}_t^{(1)} = \frac{S_0^{(1)}}{(1 + r)^t} \prod_{k=1}^{t} (1 + R_k) = S_0^{(1)} \prod_{k=1}^{t} \frac{1 + R_k}{1 + r}. \]

In this model the discounted value at time \( t \) of the portfolio is given by

\[ \tilde{\xi}_t \cdot \tilde{X}_t = \xi_t^{(0)} S_0^{(0)} + \xi_t^{(1)} \tilde{S}_t^{(1)}, \quad t = 1, 2, \ldots, N. \]

* Download the corresponding IPython notebook1 and IPython notebook2 that can be run here.
The information \( \mathcal{F}_t \) known in the market up to time \( t \) is given by the knowledge of \( S^{(1)}_1, S^{(1)}_2, \ldots, S^{(1)}_t \), which is equivalent to the knowledge of \( \tilde{S}^{(1)}_1, \tilde{S}^{(1)}_2, \ldots, \tilde{S}^{(1)}_t \) or \( R_1, R_2, \ldots, R_t \), i.e. we write

\[
\mathcal{F}_t = \sigma(S^{(1)}_1, S^{(1)}_2, \ldots, S^{(1)}_t) = \sigma(\tilde{S}^{(1)}_1, \tilde{S}^{(1)}_2, \ldots, \tilde{S}^{(1)}_t) = \sigma(R_1, R_2, \ldots, R_t),
\]

\( t = 0, 1, \ldots, N \), where, as a convention, \( S_0 \) is a constant and \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) contains no information.

**Theorem 2.15.** The CRR model is without arbitrage opportunities if and only if \( a < r < b \). In this case the market is complete and the (equivalent) risk-neutral probability measure \( \mathbb{P}^* \) is given by

\[
\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a},
\]

\( t = 0, 1, \ldots, N-1 \). In particular, \( (R_1, R_2, \ldots, R_N) \) forms a sequence of independent and identically distributed (i.i.d.) random variables under \( \mathbb{P}^* \), with

\[
p^* := \mathbb{P}^*(R_t = b) = \frac{r-a}{b-a} \quad \text{and} \quad q^* := \mathbb{P}^*(R_t = a) = \frac{b-r}{b-a},
\]

\( t = 1, 2, \ldots, N \).

**Proof.** In order to check for arbitrage opportunities we may use Theorem 2.11 and look for a risk-neutral measure \( \mathbb{P}^* \). According to the definition of a risk-neutral measure this probability \( \mathbb{P}^* \) should satisfy Condition (2.6), i.e.

\[
\mathbb{E}^*[S^{(1)}_{t+1} \mid \mathcal{F}_t] = (1+r)S^{(1)}_t, \quad t = 0, 1, \ldots, N-1.
\]

Rewriting \( \mathbb{E}^*[S^{(1)}_{t+1} \mid \mathcal{F}_t] \) as

\[
\mathbb{E}^*[S^{(1)}_{t+1} \mid \mathcal{F}_t] = \mathbb{E}^*[S^{(1)}_{t+1} \mid S^{(1)}_t] = (1+a)S^{(1)}_t \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + (1+b)S^{(1)}_t \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t),
\]

it follows that any risk-neutral measure \( \mathbb{P}^* \) should satisfy the equations

\[
\begin{cases}
(1+b)S^{(1)}_t \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + (1+a)S^{(1)}_t \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = (1+r)S^{(1)}_t \\
\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1,
\end{cases}
\]

i.e.

\[
\begin{cases}
b \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + a \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = r \\
\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1,
\end{cases}
\]

with solution
\[
\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a},
\]

for \(t = 0, 1, \ldots, N - 1\). Since the values of \(\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t)\) and \(\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t)\) computed in (2.10) are non random, they are independent of the information contained in \(\mathcal{F}_t\). As a consequence, under \(\mathbb{P}^*\), the random variable \(R_{t+1}\) is independent of the information \(\mathcal{F}_t\) up to time \(t\), which is generated by \(R_1, R_2, \ldots, R_t\), hence the sequence of random variables \((R_t)_{t=0,1,\ldots,N}\) is made of independent random variables under \(\mathbb{P}^*\), and by (2.10) we have

\[
\mathbb{P}^*(R_{t+1} = b) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a) = \frac{b-r}{b-a}.
\]

Clearly, \(\mathbb{P}^*\) can be equivalent to \(\mathbb{P}\) only if \(r-a > 0\) and \(b-r > 0\). In this case the solution \(\mathbb{P}^*\) of the problem is unique by construction, hence the market is complete by Theorem 2.14. □

As a consequence of Proposition 2.9, letting \(p^* := (r-a)/(b-a)\), when \((\epsilon_1, \ldots, \epsilon_n) \in \{a,b\}^N\) we have

\[
\mathbb{P}^*(R_1 = \epsilon_1, \ldots, R_N = \epsilon_n) = (p^*)^l(1-p^*)^{N-l},
\]

where \(l\), resp. \(N - l\), denotes the number of times the term “\(b\)”, resp. “\(a\)”, appears in the sequence \((\epsilon_1, \ldots, \epsilon_N) \in \{a,b\}^N\).

**Exercises**

**Exercise 2.1**

Today I went to the Furong Peak Mall. After exiting the Poon Way MTR station I was met by a friendly investment consultant from NTRC Input, who recommended that I subscribe to the following investment plan. The plan requires to invest $2,550 per year over the first 10 years. No contribution is required from year 11 until year 20, and the total projected surrender value is $30,835 at maturity \(N = 20\). The plan also includes a death benefit which is not considered here.

<table>
<thead>
<tr>
<th>Year</th>
<th>Total Premiums</th>
<th>Guaranteed ($S)</th>
<th>Non-Guaranteed ($S)</th>
<th>Total ($S)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Paid To-date ($S)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2,550</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>5,100</td>
<td>2,460</td>
<td>140</td>
<td>2,600</td>
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<td>3</td>
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<td>4,240</td>
<td>240</td>
<td>4,480</td>
</tr>
<tr>
<td>4</td>
<td>10,200</td>
<td>6,040</td>
<td>366</td>
<td>6,406</td>
</tr>
<tr>
<td>5</td>
<td>12,750</td>
<td>8,500</td>
<td>518</td>
<td>9,018</td>
</tr>
<tr>
<td>10</td>
<td>25,499</td>
<td>19,440</td>
<td>1,735</td>
<td>21,175</td>
</tr>
<tr>
<td>15</td>
<td>25,499</td>
<td>22,240</td>
<td>3,787</td>
<td>26,027</td>
</tr>
<tr>
<td>20</td>
<td>25,499</td>
<td>24,000</td>
<td>6,835</td>
<td>30,835</td>
</tr>
</tbody>
</table>

Table 2.1: NTRC Input investment plan.
Discrete-Time Model

Compute the constant interest rate over 20 years corresponding to this investment plan.

Exercise 2.2 Today I went to the East Mall. After exiting the Bukit Kecil MTR station I was met by a friendly investment consultant from Avenda Insurance, who suggested that I subscribe to the following investment plan. The plan requires me to invest $3,581 per year over the first 10 years. No contribution is required from year 11 until year 20, and the total projected surrender value is $50,862 at maturity $N = 20$. The plan also includes a death benefit which is not considered here.

<table>
<thead>
<tr>
<th>Year</th>
<th>Total Premiums</th>
<th>Guaranteed ($)</th>
<th>Non-Guaranteed ($)</th>
<th>Total ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3,581</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>7,527</td>
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<tr>
<td>10</td>
<td>35,801</td>
<td>19,111</td>
<td>1,482</td>
<td>20,593</td>
</tr>
<tr>
<td>15</td>
<td>35,801</td>
<td>29,046</td>
<td>3,444</td>
<td>32,490</td>
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<tr>
<td>20</td>
<td>35,801</td>
<td>43,500</td>
<td>7,362</td>
<td>50,862</td>
</tr>
</tbody>
</table>

Table 2.2: Avenda Insurance investment plan.

Using the following graph, compute the constant interest rate over 20 years corresponding to this investment.

Fig. 2.4: Graph of the function $x \mapsto \frac{((1 + x)^{21} - (1 + x)^{11})}{x}$.

Exercise 2.3 Consider a two-period trinomial market model $(S_t)_{t=0,1,2}$ with $r = 0$ and three possible return rates $R_t \in \{1, 0, 1\}$. Show that $\mathbb{P}^*$ given by

$$\mathbb{P}^*(R_t = -1) := \frac{1}{4}, \quad \mathbb{P}^*(R_t = 0) := \frac{1}{2}, \quad \mathbb{P}^*(R_t = 1) := \frac{1}{4}$$
is risk-neutral.

Exercise 2.4 We consider a risk-free asset priced as $S_k^{(0)} = S_0^{(0)}$, $k = 0, 1, \ldots, N$, where the risk-free rate is $r = 0$, and a risky asset $S^{(1)}$ whose returns $R_k := \frac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}$, $k = 1, 2, \ldots, N$, form a sequence of independent identically distributed random variables taking three values $-b < 0 < b$ at each time step, with

\[ p^* := \mathbb{P}^*(R_k = b) > 0, \quad \theta^* := \mathbb{P}^*(R_k = 0) > 0, \quad q^* := \mathbb{P}^*(R_k = -b) > 0, \]

$k = 1, 2, \ldots, N$. The information known to the market up to time $k$ is denoted by $\mathcal{F}_k$.

a) Determine all possible risk-neutral measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in terms of the parameter $\theta^* \in (0, 1)$.

b) Assume that the conditional variance

\[ \text{Var}^* \left[ \frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \middle| \mathcal{F}_k \right] = \sigma^2 > 0, \quad k = 0, 1, \ldots, N - 1, \quad (2.12) \]

of the asset return is constant and equal to $\sigma^2$. Show that this condition determines a unique risk-neutral measure $\mathbb{P}_\sigma^*$ to be written explicitly, under a certain condition on $b$ and $\sigma$.

Exercise 2.5 We consider the discrete-time Cox-Ross-Rubinstein model with $N + 1$ time instants $t = 0, 1, \ldots, N$, and the price $\pi_t$ of the risk-free asset evolves as $\pi_t = \pi_0 (1 + r)^t$, $t = 0, 1, \ldots, N$. The evolution of $S_{t-1}$ to $S_t$ is given by

\[ S_t = \begin{cases} (1 + b)S_{t-1} \\ (1 + a)S_{t-1} \end{cases} \]

with $-1 < a < r < b$. The return of the risky asset $S$ is defined as

\[ R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \quad t = 1, 2, \ldots, N, \]

and $\mathcal{F}_t$ is generated by $R_1, R_2, \ldots, R_t$, $t = 1, 2, \ldots, N$.

a) What are the possible values of $R_t$?

b) Show that, under the probability measure $\mathbb{P}^*$ defined by

\[ p^* = \mathbb{P}^*(R_{t+1} = a | \mathcal{F}_t) = \frac{b - r}{b - a}, \quad q^* = \mathbb{P}^*(R_{t+1} = b | \mathcal{F}_t) = \frac{r - a}{b - a}, \]

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Discrete-Time Model

t = 0, 1, \ldots, N - 1, the expected return $E^*\left[R_{t+1} \mid \mathcal{F}_t\right]$ of $S$ is equal to the return $r$ of the risk-free asset.

c) Show that under $P^*$ the process $(S_t)_{t=0,1,\ldots,N}$ satisfies

$$E^*[S_{t+k} \mid \mathcal{F}_t] = (1 + r)^k S_t, \quad t = 0, 1, \ldots, N - k, \quad k = 0, 1, \ldots, N.$$