Chapter 5
The Black-Scholes PDE

In this chapter we review the notions of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also derive the Black-Scholes partial differential equation (PDE) for self-financing portfolios, and we solve this equation using the heat kernel method.

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5.1 Continuous-Time Market Model

The construction of the risk-free asset price process \((A_t)_{t \in \mathbb{R}_+}\) admits the following equivalent formulations:

\[
\frac{A_{t+dt} - A_t}{A_t} = r dt, \quad \frac{dA_t}{A_t} = r dt, \quad A'_t = rA_t, \quad t \in \mathbb{R}_+.
\]

with the solution

\[
A_t = A_0 e^{rt}, \quad t \in \mathbb{R}_+, \tag{5.1}
\]

where \(r > 0\) is the risk-free interest rate.\(^\dagger\) The risky asset price process \((S_t)_{t \in \mathbb{R}_+}\) will be modeled using a geometric Brownian motion defined from the equation

\(^\dagger\) “Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist”, Kenneth E. Boulding, Boulding (1973), page 248.
\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+,
\]
(5.2)

see Section 4.6.

By Proposition 4.12 we have

\[ S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right), \quad t \in \mathbb{R}_+. \]

The next Figure 5.1 presents a graph of underlying market data that can be compared to the geometric Brownian motion of Figure 4.13.

Fig. 5.1: Graphs of underlying prices.

The adjusted close price \( \text{Ad}() \) is the closing price after adjustments for applicable splits and dividend distributions.

5.2 Self-Financing Portfolio Strategies

Let \( \xi_t \) and \( \eta_t \) denote the (possibly fractional) quantities invested at time \( t \) over the time period \([t, t + dt]\), respectively in the assets \( S_t \) and \( A_t \), and let
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\[ \xi_t = (\eta_t, \xi_t), \quad S_t = (A_t, S_t), \quad t \in \mathbb{R}_+, \]

denote the associated portfolio and asset price processes. The portfolio value \( V_t \) at time \( t \) is given by

\[ V_t = \xi_t \cdot S_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+. \quad (5.3) \]

Our description of portfolio strategies proceeds in four steps which correspond to different interpretations of the self-financing condition.

**Portfolio update**

The portfolio strategy \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) is self-financing if the portfolio value remains constant after updating the portfolio from \((\eta_t, \xi_t)\) to \((\eta_{t+dt}, \xi_{t+dt})\), i.e.

\[ \xi_t \cdot S_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = A_{t+dt} \eta_{t+dt} + S_{t+dt} \xi_{t+dt} = \xi_{t+dt} \cdot S_{t+dt}, \quad (5.4) \]

which is the continuous-time equivalent of the self-financing condition already encountered in the discrete setting of Chapter 2, see Definition 2.1. A major difference with the discrete-time case of Definition 2.1, however, is that the continuous-time differentials \(dS_t\) and \(d\xi_t\) do not make pathwise sense as continuous-time stochastic integrals are defined by \(L^2\) limits, cf. Proposition 4.9, or by convergence in probability.

Equivalently, Condition (5.4) can be rewritten as

\[ A_{t+dt} d\eta_t + S_{t+dt} d\xi_t = 0, \quad (5.5) \]

where

\[ d\eta := \eta_{t+dt} - \eta_t \quad \text{and} \quad d\xi := \xi_{t+dt} - \xi_t \]

denote the respective changes in portfolio allocations. Equivalently, we have

\[ A_{t+dt} (\eta_t - \eta_{t+dt}) = S_{t+dt} (\xi_{t+dt} - \xi_t). \quad (5.6) \]

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In other words, when one sells a (possibly fractional) quantity $\eta_t - \eta_{t+dt} > 0$ of the risk-free asset priced $A_{t+dt}$ at the end of the time period $[t, t+dt]$ for the total amount $A_{t+dt}(\eta_t - \eta_{t+dt})$, one should entirely spend this income to buy the corresponding quantity $\xi_{t+dt} - \xi_t > 0$ of the risky asset for the same amount $S_{t+dt}(\xi_{t+dt} - \xi_t) > 0$.

Similarly, if one sells a quantity $-d\xi_t > 0$ of the risky asset $S_{t+dt}$ between the time periods $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $-S_{t+dt}d\xi_t$, one should entirely use this income to buy a quantity $d\eta_t > 0$ of the risk-free asset for an amount $A_{t+dt}d\eta_t > 0$, i.e.

$$A_{t+dt}d\eta_t = -S_{t+dt}d\xi_t.$$

Condition (5.6) can be rewritten as

$$S_t(\xi_{t+dt} - \xi_t) + A_{t+dt}(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) = 0,$$

i.e.

$$S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) + dA_t \cdot d\eta_t = 0,$$

which yields

$$S_t d\xi_t + A_t d\eta_t + d\xi_t \cdot dS_t = 0 \tag{5.7}$$

in differential notation, as $dA_t \cdot d\eta_t = rA_t dt \cdot d\eta_t = 0$.

**Portfolio differential**

In practice, the self-financing portfolio property will be characterized by the following proposition.

**Proposition 5.1.** A portfolio allocation $(\xi_t, \eta_t)_{t \in \mathbb{R}^+}$ with price

$$V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}^+,$$

is self-financing according to (5.4) if and only if the relation

$$dV_t = \underbrace{\eta_t dA_t}_{\text{risk-free P/L}} + \underbrace{\xi_t dS_t}_{\text{risky P/L}} \tag{5.8}$$

holds.

**Proof.** We check that by Itô’s calculus we have

$$dV_t = \eta_t dA_t + A_t d\eta_t + d\eta_t \cdot dA_t + \xi_t dS_t + S_t d\xi_t + d\xi_t \cdot dS_t$$

$$= \eta_t dA_t + \xi_t dS_t + A_t d\eta_t + S_t d\xi_t + d\xi_t \cdot dS_t,$$

since

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\[(A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = dA_t \cdot d\eta_t = r A_t (dt \cdot d\eta_t) \simeq 0\]

in the sense of the Itô calculus by the Itô Table 4.1. Hence, Condition (5.7) rewrites as (5.8), which is equivalent to (5.4) and (5.5).

Let

\[\tilde{V}_t := e^{-rt} V_t \quad \text{and} \quad \tilde{S}_t := e^{-rt} S_t\]

respectively denote the discounted portfolio value and discounted risky asset prices at time \(t \geq 0\). We have

\[
d\tilde{S}_t = d(e^{-rt} S_t) = S_t d(e^{-rt}) + e^{-rt} dS_t = -re^{-rt} S_t dt + e^{-rt} dS_t = -re^{-rt} S_t dt + \mu e^{-rt} S_t dt + \sigma e^{-rt} S_t dB_t = \tilde{S}_t ((\mu - r) dt + \sigma dB_t).
\]

In the next lemma we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted profits and losses (number of risky assets \(\xi_t\) times discounted price variation \(d\tilde{S}_t\)).

The following lemma is the continuous-time analog of Lemma 3.2.

**Lemma 5.2.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) be a portfolio strategy with value

\[V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}^+.
\]

The following statements are equivalent:

(i) the portfolio strategy \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) is self-financing,

(ii) the discounted portfolio value \(\tilde{V}_t\) can be written as the stochastic integral sum

\[\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}^+,
\]

of discounted profits and losses.

**Proof.** Assuming that (i) holds, the self-financing condition and (5.1)-(5.2) show that

\[
dV_t = \eta_t dA_t + \xi_t dS_t = r \eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t = r V_t dt + (\mu - r) \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}^+,
\]

hence

\[e^{-rt} dV_t = r e^{-rt} V_t dt + (\mu - r) e^{-rt} \xi_t S_t dt + \sigma e^{-rt} \xi_t S_t dB_t, \quad t \in \mathbb{R}^+,
\]
\[ d\tilde{V}_t = d\left(e^{-rt}V_t\right) \]
\[ = -re^{-rt}V_t dt + e^{-rt}dV_t \]
\[ = (\mu - r)\xi_t e^{-rt}S_t dt + \sigma \xi_t e^{-rt}S_t dB_t \]
\[ = (\mu - r)\xi_t \tilde{S}_t dt + \sigma \xi_t \tilde{S}_t dB_t \]
\[ = \xi_t d\tilde{S}_t, \quad t \in \mathbb{R}_+, \]

i.e. (5.9) holds by integrating on both sides as

\[ \tilde{V}_t - \tilde{V}_0 = \int_0^t d\tilde{V}_u = \int_0^t \xi_u d\tilde{S}_u, \quad t \in \mathbb{R}_+. \]

(ii) Conversely, if (5.9) is satisfied we have

\[ dV_t = d\left(e^{rt}\tilde{V}_t\right) \]
\[ = re^{rt}\tilde{V}_t dt + e^{rt}d\tilde{V}_t \]
\[ = re^{rt}\tilde{V}_t dt + e^{rt}\xi_t d\tilde{S}_t \]
\[ = rV_t dt + e^{rt}\xi_t \tilde{S}_t ((\mu - r) dt + \sigma dB_t) \]
\[ = r\eta_t dA_t + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \]
\[ = \eta_t dA_t + \xi_t dS_t, \]

hence the portfolio is self-financing according to Definition 5.1. \( \square \)

As a consequence of Relation (5.9), the problem of hedging a claim \( C \) with maturity \( T \) reduces to that of finding the process \( (\xi_t)_{t \in [0,T]} \) appearing in the decomposition of the discounted claim payoff \( \tilde{C} = e^{-rT}C \) as a stochastic integral:

\[ \tilde{C} = \tilde{V}_T = \tilde{V}_0 + \int_0^T \xi_t d\tilde{S}_t. \]

Note that according to (5.9), the (non-discounted) self-financing portfolio price \( V_t \) can be written as

\[ V_t = e^{rt}V_0 + (\mu - r) \int_0^t e^{(t-u)r} \xi_u S_u du + \sigma \int_0^t e^{(t-u)r} \xi_u S_u dB_u, \quad t \in \mathbb{R}_+. \]

(5.10)

5.3 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the discrete and two-step models. In the sequel we will only consider ad-
missible portfolio strategies whose total value $V_t$ remains nonnegative for all times $t \in [0, T]$.

**Definition 5.3.** A portfolio strategy $(\xi_t, \eta_t)_{t \in [0,T]}$ with price $V_t = \xi_t S_t + \eta_t A_t$, $t \in \mathbb{R}_+$, constitutes an arbitrage opportunity if all three following conditions are satisfied:

i) $V_0 \leq 0$ at time $t = 0$, \hspace{1cm} [start from a zero-cost portfolio or in debt]

ii) $V_T \geq 0$ at time $t = T$, \hspace{1cm} [finish with a nonnegative amount]

iii) $\mathbb{P}(V_T > 0) > 0$ at time $t = T$. \hspace{1cm} [profit made with non-zero probability]

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure $\mathbb{P}^*$, the return of the risky asset over the time interval $[u,t]$ equals the return of the risk-free asset given by

$$A_t = e^{(t-u)r}A_u, \quad 0 \leq u \leq t.$$

Recall that the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, i.e.

$$\mathcal{F}_t = \sigma(B_u : 0 \leq u \leq t), \quad t \in \mathbb{R}_+.$$

**Definition 5.4.** A probability measure $\mathbb{P}^*$ on $\Omega$ is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*[S_t | \mathcal{F}_u] = e^{(t-u)r}S_u, \quad 0 \leq u \leq t, \quad (5.11)$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

As in the discrete-time case, $\mathbb{P}^*$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^*[S_t | \mathcal{F}_u] > e^{(t-u)r}S_u, \quad 0 \leq u \leq t,$$

meaning that by taking risks in buying $S_t$, one could make an expected return higher than that of the risk-free asset

$$A_t = e^{(t-u)r}A_u, \quad 0 \leq u \leq t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\flat$ satisfies

$$\mathbb{E}^\flat[S_t | \mathcal{F}_u] < e^{(t-u)r}S_u, \quad 0 \leq u \leq t.$$

From the relation

$$A_t = e^{(t-u)r}A_u, \quad 0 \leq u \leq t,$$
we interpret (5.11) by saying that the expected return of the risky asset $S_t$ under $\mathbb{P}^*$ equals the return of the risk-free asset $A_t$. Recall that the discounted price $\tilde{S}_t$ of the risky asset in $\$$ at time 0 is defined by

$$\tilde{S}_t := e^{-rt} S_t = \frac{S_t}{A_t/A_0}, \quad t \in \mathbb{R}_+,$$

i.e. $A_t/A_0$ plays the role of a numéraire in the sense of Chapter 12.

**Definition 5.5.** A continuous-time process $(Z_t)_{t \in \mathbb{R}_+}$ of integrable random variables is a martingale under $\mathbb{P}$ and with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ if

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s, \quad 0 \leq s \leq t.$$ 

Note that when $(Z_t)_{t \in \mathbb{R}_+}$ is a martingale, $Z_t$ is in particular $\mathcal{F}_t$-measurable at all times $t \in \mathbb{R}_+$.

In continuous-time finance, the martingale property can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

As in the discrete-time case, the notion of martingale can be used to characterize risk-neutral probability measures as in the next proposition.

**Proposition 5.6.** The probability measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted risky asset price process $(\tilde{S}_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$.

**Proof.** If $\mathbb{P}^*$ is a risk-neutral probability measure, we have

$$\mathbb{E}^* [\tilde{S}_t \mid \mathcal{F}_u] = \mathbb{E}^* [e^{-rt} S_t \mid \mathcal{F}_u]$$

$$= e^{-rt} \mathbb{E}^* [S_t \mid \mathcal{F}_u]$$

$$= e^{-rt} e^{(t-u)r} S_u$$

$$= e^{-ru} S_u$$

$$= \tilde{S}_u, \quad 0 \leq u \leq t,$$

hence $(\tilde{S}_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. Conversely, if $(\tilde{S}_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$ then

$$\mathbb{E}^* [S_t \mid \mathcal{F}_u] = e^{rt} \mathbb{E}^* [\tilde{S}_t \mid \mathcal{F}_u]$$

$$= e^{rt} \tilde{S}_u$$

$$= e^{(t-u)r} S_u, \quad 0 \leq u \leq t,$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral according to Definition 5.4. \hfill \square

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In the sequel we will only consider probability measures $P^*$ that are equivalent to $P$ in the sense that they have the same events of zero probability.

**Definition 5.7.** A probability measure $P^*$ on $(\Omega, F)$ is said to be equivalent to another probability measure $P$ when

$$P^*(A) = 0 \text{ if and only if } P(A) = 0, \text{ for all } A \in F. \quad (5.12)$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

**Theorem 5.8.** A market is without arbitrage opportunity if and only if it admits at least one (equivalent) risk-neutral probability measure $P^*$.


### 5.4 Market Completeness

**Definition 5.9.** A contingent claim with payoff $C$ is said to be attainable if there exists a (self-financing) portfolio strategy $(\eta_t, \xi_t)_{t \in [0, T)}$ such that at the maturity time $T$ the equality

$$V_T = \eta_T A_T + \xi_T S_T = C$$

holds (almost surely) between random variables.

When a claim with payoff $C$ is attainable, its price at time $t$ will be given by the value $V_t$ of a self-financing portfolio hedging $C$.

**Definition 5.10.** A market model is said to be complete if every contingent claim $C$ is attainable.

The next result is a continuous-time restatement of the second fundamental theorem of asset pricing.

**Theorem 5.11.** A market model without arbitrage opportunities is complete if and only if it admits only one (equivalent) risk-neutral probability measure $P^*$.


In the Black and Scholes (1973) model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete.
5.5 The Black-Scholes Formula

We start by deriving the Black-Scholes Partial Differential Equation (PDE) for the price of a self-financing portfolio. Note that the drift parameter \( \mu \) in (5.2) is absent in the PDE (5.13), and it does not appear as well in the Black-Scholes formula (5.18).

**Proposition 5.12.** Let \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) be a portfolio process such that

(i) the strategy \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) is self-financing,

(ii) the portfolio price \( V_t := \eta_t A_t + \xi_t S_t \), takes the form

\[
V_t = g(t, S_t), \quad t \in \mathbb{R}_+,
\]

for some function \( g \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+) \) of \( t \) and \( S_t \).

Then the function \( g(t, x) \) satisfies the Black-Scholes PDE

\[
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0, \quad (5.13)
\]

and \( \xi_t = \xi_t(S_t) \) is given by the partial derivative

\[
\xi_t = \xi_t(S_t) = \frac{\partial g}{\partial x}(t, S_t), \quad t \in \mathbb{R}_+. \quad (5.14)
\]

**Proof.** (i) First, we note that the self-financing condition (5.8) in Proposition 5.1 implies

\[
dV_t = \eta_t dA_t + \xi_t dS_t
\]

\[
= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t
\]

\[
= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t
\]

\[
= rg(t, S_t) dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t,
\]

\( t \in \mathbb{R}_+ \). We now rewrite (4.28) under the form of an Itô process

\[
S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,
\]

as in (4.21), by taking

\[
u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \quad t \in \mathbb{R}_+.
\]

(ii) By (4.23), the application of Itô’s formula Theorem 4.11 to \( V_t = g(t, S_t) \) leads to
The Black-Scholes PDE

\[ dV_t = dg(t, S_t) = v_t \frac{\partial g}{\partial x}(t, S_t) dt + u_t \frac{\partial g}{\partial x}(t, S_t) dB_t \]

\[ + \frac{\partial g}{\partial t}(t, S_t) dt + \frac{1}{2} |u_t|^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt \]

\[ = \frac{\partial g}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t) dt + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t) dB_t. \]

(5.16)

By respective identification of the terms in \( dB_t \) and \( dt \) in (5.15) and (5.16) we get

\[ \left\{ \begin{array}{l}
rg(t, S_t) dt + (\mu - r) \xi_t S_t dt = \frac{\partial g}{\partial t}(t, S_t) dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t) dt + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt,

\xi_t S_t \sigma dB_t = S_t \sigma \frac{\partial g}{\partial x}(t, S_t) dB_t,
\end{array} \right. \]

hence

\[ \left\{ \begin{array}{l}
rg(t, S_t) = \frac{\partial g}{\partial t}(t, S_t) + r S_t \frac{\partial g}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t),

\xi_t = \frac{\partial g}{\partial x}(t, S_t).
\end{array} \right. \]

(5.17)

The derivative giving \( \xi_t \) in (5.14) is called the **Delta** of the option price, see Proposition 5.14 below. The amount invested on the risk-free asset is

\[ \eta_t A_t = V_t - \xi_t S_t = g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t), \]

and \( \eta_t \) is given by

\[ \eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{1}{A_t} \left( g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t) \right) = \frac{1}{A_0 e^{rt}} \left( g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t) \right). \]

In the next proposition we add a terminal condition \( g(T, x) = f(x) \) to the Black-Scholes PDE in order to price a claim \( C \) of the form \( C = h(S_T) \). As in the discrete-time case, the arbitrage price \( \pi_t(C) \) at time \( t \in [0, T] \) of the claim \( C \) is defined to be the price \( V_t \) of the self-financing portfolio hedging \( C \).
Proposition 5.13. The arbitrage price \( \pi_t(C) \) at time \( t \in [0,T] \) of the (vanilla) option with payoff \( C = h(S_T) \) is given by \( \pi_t(C) = g(t,S_t) \), where the function \( g(t,x) \) is solution of the following Black-Scholes PDE:

\[
\begin{align*}
    rg(t,x) &= \frac{\partial g}{\partial t}(t,x) + rx \frac{\partial g}{\partial x}(t,x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x), \\
    g(T,x) &= h(x), \quad x > 0.
\end{align*}
\]

Market terms and data

Intrinsic value. The intrinsic value at time \( t \in [0,T] \) of the option with payoff \( C = h(S_T^{(1)}) \) is given by the immediate exercise payoff \( h(S_T^{(1)}) \). The extrinsic value at time \( t \in [0,T] \) of the option is the remaining difference \( \pi_t(C) - h(S_T^{(1)}) \) between the option price \( \pi_t(C) \) and the immediate exercise payoff \( h(S_T^{(1)}) \). In general, the option price \( \pi_t(C) \) decomposes as

\[
\pi_t(C) = \underbrace{h(S_t^{(1)})}_{\text{intrinsic value}} + \underbrace{\pi_t(C) - h(S_t^{(1)})}_{\text{extrinsic value}}, \quad t \in [0,T].
\]

Gearing. The gearing at time \( t \in [0,T] \) of the option with payoff \( C = h(S_T) \) is defined as the ratio

\[
G_t := \frac{S_t}{\pi_t(C)} = \frac{S_t}{g(t,S_t)}, \quad t \in [0,T].
\]

Effective gearing. The effective gearing at time \( t \in [0,T] \) of the option with payoff \( C = h(S_T) \) is defined as the ratio

\[
G^e_t := G_t \xi_t = \frac{\xi_t S_t}{\pi_t(C)} = \frac{S_t}{\pi_t(C)} \frac{\partial g}{\partial x}(t,S_t) = \frac{S_t}{g(t,S_t)} \frac{\partial g}{\partial x}(t,S_t) = S_t \frac{\partial \log g}{\partial x}(t,S_t), \quad t \in [0,T].
\]

The effective gearing \( G^e_t = \xi_t S_t / \pi_t(C) \) can be interpreted as the hedge ratio, i.e. the percentage of the portfolio which is invested on the risky
asset. The ratio $G_t^* = S_t \partial \log g(t, S_t) / \partial x$ can also be interpreted as an *elasticity coefficient*.

**Break-even price.** The *break-even* price $\text{BEP}_t$ of the underlying is the value of $S$ for which the intrinsic option value $h(S)$ equals the option price $\pi_t(C)$ at time $t \in [0, T]$. For European call options it is given by

$$\text{BEP}_t := K + \pi_t(C) = K + g(t, S_t), \quad t = 0, 1, \ldots, N.$$ 

whereas for European put options it is given by

$$\text{BEP}_t := K - \pi_t(C) = K - g(t, S_t), \quad t = 0, 1, \ldots, N.$$ 

**Premium.** The option *premium* $\text{OP}_t$ can be defined as the variation required from the underlying in order to reach the break-even price, *i.e.* we have

$$\text{OP}_t := \frac{\text{BEP}_t - S_t}{S_t} = \frac{K + g(t, S_t) - S_t}{S_t}, \quad t = 0, 1, \ldots, N,$$

for European call options, and

$$\text{OP}_t := \frac{S_t - \text{BEP}_t}{S_t} = \frac{S_t + g(t, S_t) - K}{S_t}, \quad t = 0, 1, \ldots, N,$$

for European put options, see Figure 5.6 below. The term “premium” is sometimes also used to denote the arbitrage price $g(t, S_t)$ of the option.

**Example - forward contracts**

When $C = S_T - K$ is the (linear) payoff of a long forward contract, *i.e.* $f(x) = x - K$, the Black-Scholes PDE admits the easy solution

$$g(t, x) = x - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T],$$

showing that the price at time $t$ of the forward contract with payoff $C = S_T - K$ is

$$S_t - K e^{-(T-t)r}, \quad x > 0, \quad t \in [0, T].$$

In addition, the Delta of the option price is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \quad t \in [0, T],$$

which leads to a static “hedge and forget” strategy, cf. Exercise 5.5. The forward contract can be realized by the option issuer as follows:

a) At time $t$, receive the option premium $S_t - e^{-(T-t)r}K$ from the option buyer.

b) Borrow $e^{-(T-t)r}K$ from the bank, to be refunded at maturity.

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c) Buy the risky asset using the amount 
\[ S_t - e^{-(T-t)r} K + e^{-(T-t)r} K = S_t. \]

d) Hold the risky asset until maturity (do nothing, constant portfolio strategy).

e) At maturity \( T \), hand in the asset to the option holder, who gives the price \( K \) in exchange.

f) Use the amount \( K = e^{(T-t)r} e^{-(T-t)r} K \) to refund the lender of \( e^{-(T-t)r} K \) borrowed at time \( t \).

Forward contracts can be used for physical delivery, e.g. for live cattle.

For a future contract expiring at time \( T \) we take \( K = S_0 e^{rT} \) and the contract is usually quoted at time \( t \) using the forward price

\[ e^{(T-t)r} (S_t - K e^{-(T-t)r}) = e^{(T-t)r} S_t - K = e^{(T-t)r} S_t - S_0 e^{rT}, \]

or simply using \( e^{(T-t)r} S_t \). Future contracts are non-deliverable forward contracts which are “marked to market” at each time step via a cash flow exchange between the two parties, ensuring that the absolute difference \( |e^{(T-t)r} S_t - K| \) has been credited to the buyer’s account if \( e^{(T-t)r} S_t > K \), or to the seller’s account if \( e^{(T-t)r} S_t < K \).

In the case of European options, the simple “hedge and forget” constant strategy \( \xi_t = 1, \eta_t = \eta_0, t \in [0, T] \), will hedge the option only if \( S_T + \eta_0 A_T \geq S_T - K \), i.e. if \(-\eta_0 A_T \leq K \).

**Black-Scholes formula for European call options**

Recall that in the case of a European call option with strike price \( K \) the payoff function is given by \( f(x) = (x - K)^+ \) and the Black-Scholes PDE reads

\[
\begin{align*}
rg_c(t, x) &= \frac{\partial g_c(t, x)}{\partial t} + rx \frac{\partial g_c(t, x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g_c(t, x)}{\partial x^2} \\
g_c(T, x) &= (x - K)^+.
\end{align*}
\]

In Sections 5.6 and 5.7 we will prove that the solution of this PDE is given by the Black-Scholes formula

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\[ g_c(t, x) = \mathbb{B}(K, x, \sigma, r, T - t) = x\Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \]

(5.18)

with

\[ d_+(T - t) := \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T-t}}, \]

(5.19)

\[ d_-(T - t) := \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T-t}}, \]

(5.20)

where “log” denotes the natural logarithm “ln”, see Proposition 5.17 below. Here, “log” denotes the natural logarithm “ln”, and

\[ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}, \]

denotes the standard Gaussian distribution function, with

\[ d_+(T - t) = d_-(T - t) + |\sigma|\sqrt{T-t}. \]

In other words, a European call option with strike price \( K \) and maturity \( T \) is priced at time \( t \in [0, T] \) as

\[ g_c(t, S_t) = \mathbb{B}(K, S_t, \sigma, r, T - t) = S_t\Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t)), \quad t \in [0, T]. \]

The following R script is an implementation of the Black-Scholes formula for European call options in R.*

```r
BSCall <- function(S, K, r, T, sigma) {
  d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
  d2 <- d1 - sigma * sqrt(T)
  BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
  return(BSCall)
}
```

The interest in the formula (5.18) in comparison with the Cox-Ross-Rubinstein (CRR) model of Section 2.7 is that provides an analytical solution that can be evaluated in one single step, which is therefore computationally much more efficient.

One can easily check that \( g_c(t, 0) = 0 \) for all \( t \in [0, T] \) and

* Download the corresponding IPython notebook that can be run here.

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\[
\lim_{t \to T} d_+(T - t) = \lim_{t \to T} d_-(T - t) = \begin{cases} 
+\infty, & x > K, \\
\frac{1}{2}, & x = K, \\
-\infty, & x < K,
\end{cases}
\]

which allows us to recover the boundary condition

\[
g_c(T, x) = \lim_{t \to \infty} g_c(t, x) = \begin{cases} 
x \Phi(+\infty) - K \Phi(+\infty) = x - K, & x > K \\
\frac{x}{2} - \frac{K}{2} = 0, & x = K \\
x \Phi(-\infty) - K \Phi(-\infty) = 0, & x < K
\end{cases} = (x - K)^+
\]

at \( t = T \). Similarly, we can check that

\[
\lim_{T \to \infty} d_-(T - t) = \begin{cases} 
+\infty, & r > \frac{\sigma^2}{2}, \\
\frac{1}{2}, & r = \frac{\sigma^2}{2}, \\
-\infty, & r < \frac{\sigma^2}{2},
\end{cases}
\]

and \( \lim_{T \to \infty} d_+(T - t) = +\infty \), hence

\[
\lim_{T \to \infty} \text{Bl}(K, S_t, \sigma, r, T - t) = S_t, \quad t \in \mathbb{R}_+.
\]

Figure 5.3 presents an interactive graph of the Black-Scholes call price map, \textit{i.e.} the solution

\[
(t, x) \mapsto g_c(t, x) = x \Phi(d_+(T - t)) - K e^{-(T-t)r} \Phi(d_-(T - t))
\]

of the Black-Scholes PDE for a call option.
Proposition 5.14. The Black-Scholes Delta of the European call option is given by
\[ \xi_t = \xi_t(S_t) = \frac{\partial g_c}{\partial x}(t, S_t) = \Phi(d_+(T - t)) \in [0, 1], \] (5.21)
where \( d_+(T - t) \) is given by (5.19).

Proof. By (5.18) we have
\[
\frac{\partial g_c}{\partial x}(t, x) = \frac{\partial}{\partial x} \left( x \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \right) \tag{5.22}
\]
\[ -Ke^{-(T-t)r} \frac{\partial}{\partial x} \Phi \left( \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \]
\[ = \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \]
\[ + x \frac{\partial}{\partial x} \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \]
\[ -Ke^{-(T-t)r} \frac{\partial}{\partial x} \Phi \left( \frac{\log(x/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \]
\[ = \Phi \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \]
\[ + \frac{1}{|\sigma|\sqrt{2\pi(T - t)}} \exp \left( -\frac{1}{2} \left( \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right)^2 \right) \]

* Right-click on the figure for interaction and “Full Screen Multimedia” view.
Δ \left( \log \left( \frac{x}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) (T - t) \right)
\frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp \left( -\frac{1}{2} \left( \log \left( \frac{x}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right)^2 \right)
\Delta = \Phi \left( \frac{\log \left( \frac{x}{K} \right) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T-t}} \right).
\]

The following R script is an implementation of the Black-Scholes Delta for European call options in R.

```r
Delta <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
 Delta = pnorm(d1)
 Delta}
```

In Figure 5.4 we plot the Delta of the European call option as a function of the underlying and of time to maturity.

![Fig. 5.4: Delta of a European call option with strike price $K = 100$.](image)

The Gamma of the European call option is defined as the second derivative of the option price with respect to the underlying, which gives

\[
\gamma_t = \frac{1}{S_t \sigma \sqrt{T-t}} \Phi'(d_+(T-t))
\]

\[
\gamma_t = \frac{1}{S_t \sigma \sqrt{2\pi(T-t)}} \exp \left( -\frac{1}{2} \left( \log \left( \frac{S_t}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right)^2 \right)
\]

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\[ \geq 0. \]

In particular, a positive value of \( \gamma_t \) implies that the Delta \( \xi_t = \xi_t(S_t) \) should increase when the underlying asset price \( S_t \) increases.

In Figure 5.5 we plot the (truncated) value of the Gamma of a European call option as a function of the underlying and of time to maturity.

![Fig. 5.5: Gamma of a European call option with strike price \( K = 100 \).](image)

As Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the underlying risky asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 5.5.

<table>
<thead>
<tr>
<th>Option price ( g(t, S_t) )</th>
<th>Call option</th>
<th>Put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta ( \frac{\partial g}{\partial S}(t, S_t) )</td>
<td>( S_t \Phi(d_+(T-t)) - K e^{-\gamma(T-t)} \Phi(d_-(T-t)) )</td>
<td>( K e^{-\gamma(T-t)} \Phi(d_+(T-t)) - S_t \Phi(d_-(T-t)) )</td>
</tr>
<tr>
<td>Gamma ( \frac{\partial^2 g}{\partial S^2}(t, S_t) )</td>
<td>( \Phi(d_+(T-t)) )</td>
<td>( -\Phi(-d_+(T-t)) )</td>
</tr>
<tr>
<td>Vega ( \frac{\partial g}{\partial \sigma}(t, S_t) )</td>
<td>( S_t \sqrt{T-t} \Phi'(d_+(T-t)) )</td>
<td>( -\Phi(-d_+(T-t)) )</td>
</tr>
<tr>
<td>Theta ( \frac{\partial g}{\partial t}(t, S_t) )</td>
<td>( -\frac{S_t \Phi(d_+(T-t))}{2\sqrt{T-t}} - \frac{K e^{-\gamma(T-t)}}{2\sqrt{T-t}} \Phi(d_-(T-t)) )</td>
<td>( \frac{S_t \Phi(d_+(T-t))}{2\sqrt{T-t}} + \frac{K e^{-\gamma(T-t)}}{2\sqrt{T-t}} \Phi(-d_+(T-t)) )</td>
</tr>
<tr>
<td>Rho ( \frac{\partial g}{\partial \gamma}(t, S_t) )</td>
<td>( K(T-t) e^{-\gamma(T-t)} \Phi(d_-(T-t)) )</td>
<td>( -K(T-t) e^{-\gamma(T-t)} \Phi(-d_+(T-t)) )</td>
</tr>
</tbody>
</table>

Table 5.1: Black-Scholes Greeks (Wikipedia).
N. Privault

![Figure 5.6: Warrant terms and data.]

The R package bizdays (requires to install QuantLib) can be used to compute calendar time vs business time to maturity

```r
install.packages("bizdays")
library(bizdays)
load_quantlib_calendars('HongKong', from='2018-01-01', to='2018-12-31')
load_quantlib_calendars('Singapore', from='2018-01-01', to='2018-12-31')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/HongKong')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/Singapore')
```

**Black-Scholes analysis for European put options**

Similarly, in the case of a European put option with strike price $K$ the payoff function is given by $f(x) = (K - x)^+$ and the Black-Scholes PDE reads

$$\begin{cases}
    rg_p(t, x) &= \frac{\partial g_p}{\partial t}(t, x) + rx\frac{\partial g_p}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 g_p}{\partial x^2}(t, x), \\
    g_p(T, x) &= (K - x)^+,
\end{cases}$$

with explicit solution

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\[
g_p(t, x) = K e^{-(T-t)r} \Phi(-d_-(T-t)) - x \Phi(-d_+(T-t)), \tag{5.23}
\]

with

\[
d_+(T-t) = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{5.24}
\]

\[
d_-(T-t) = \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{5.25}
\]
as illustrated in Figure 5.7. In other words, a European put option with strike price \(K\) and maturity \(T\) is priced at time \(t \in [0, T]\) as

\[
g_p(t, S_t) = K e^{-(T-t)r} \Phi(-d_-(T-t)) - S_t \Phi(-d_+(T-t)),
\]

\(t \in [0, T]\). One can easily check that \(g_p(t, 0) = K e^{-(T-t)r}\) and \(g_p(t, \infty) = 0\) for all \(t \in [0, T]\) and the boundary condition

\[
g_p(T, x) = \begin{cases} K \Phi(\infty) - x \Phi(\infty) = K - x, & x < K \\ \frac{K - x}{2} = 0, & x = K \end{cases} = (K - x)^+ \]

at \(t = T\). Similarly, we can check that

\[
\lim_{T \to \infty} d_-(T-t) = \begin{cases} +\infty, & r > \frac{\sigma^2}{2}, \\ \frac{1}{2}, & r = \frac{\sigma^2}{2}, \\ -\infty, & r < \frac{\sigma^2}{2}, \end{cases}
\]

and \(\lim_{T \to \infty} d_+(T-t) = +\infty\), hence

\[
\lim_{T \to \infty} \text{Bl}_p(K, S_t, \sigma, r, T-t) = 0, \quad t \in \mathbb{R}_+.
\]

The following script is an implementation of the Black-Scholes formula for European put options in R.

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http://www.ntu.edu.sg/home/nprivault/index.html
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 = d1 - sigma * sqrt(T)
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1)
BSPut}

Fig. 5.7: Graph of the Black-Scholes put price function with strike price $K = 100$.

Note that the call-put parity relation

$$g(t, S_t) = x - Ke^{-(T-t)r} = g_c(t, S_t) - g_p(t, S_t), \quad 0 \leq t \leq T, \quad (5.26)$$

is also satisfied from (5.18) and (5.23).

**Numerical examples**

In Figure 5.8 we consider the historical stock price of HSBC Holdings (0005.HK) over one year:

Fig. 5.8: Graph of the stock price of HSBC Holdings.

* Right-click on the figure for interaction and “Full Screen Multimedia” view.
Consider the call option issued by Societe Generale on 31 December 2008 with strike price \( K = $63.704 \), maturity \( T = \) October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 warrants, cf. page 7. The next graph gives the time evolution of the Black-Scholes portfolio price

\[
t \mapsto g_c(t, S_t)
\]
driven by the market price \( t \mapsto S_t \) of the underlying risky asset as given in Figure 5.8, in which the number of days is counted from the origin and not from maturity.

![Path of the Black-Scholes price for a call option on HSBC.](image_url)

Fig. 5.9: Path of the Black-Scholes price for a call option on HSBC.

As a consequence of Proposition 5.14, in the Black-Scholes model the amount invested in the risky asset is

\[
S_t \xi_t = S_t \Phi(d_+(T - t)) = S_t \Phi \left( \frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \geq 0,
\]

which is always positive, i.e. there is no short selling, and the amount invested on the risk-free asset is

\[
\eta_t A_t = -K e^{-(T-t)r} \Phi \left( \frac{\log(S_t/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}} \right) \leq 0,
\]

which is always negative, i.e. we are constantly borrowing money, as noted in Figure 5.10.

A comparison of Figure 5.10 with market data can be found in Figures 7.11 and 7.12 below.

**Cash settlement.** In the case of a cash settlement, the option issuer will satisfy the option contract by selling \( \xi_T = 1 \) stock at the price \( S_T = $83 \), refund the \( K = $63 \) risk-free investment, and hand in the remaining amount \( C = (S_T - K)^+ = 83 - 63 = $20 \) to the option holder.
**Physical delivery.** In the case of physical delivery of the underlying asset, the option issuer will deliver $\xi_T = 1$ stock to the option holder in exchange for $K = \$63$, which will be used together with the portfolio value to refund the risk-free loan.

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price $K = \$77.667$, maturity $T =$ October 05, 2009, and entitlement ratio 92.593, cf. page 7. In the next Figure 5.11, the number of days is counted from the origin and not from maturity.

The *Delta* of the Black-Scholes put option is obtained by differentiation of the call-put parity relation (5.26) as

\[
\xi_t = -(1 - \Phi(d_+(T-t))) = -\Phi(-d_+(T-t)) \in [-1,1],
\]

and the amount invested on the risky asset is

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\[-S_t \Phi(d_+(T - t)) = -S_t \Phi\left(-\frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}}\right) \leq 0,\]

i.e. there is always short selling, and the amount invested on the risk-free asset is

\[Ke^{-(T-t)r} \Phi\left(-\frac{\log(S_t/K) + (r - \sigma^2/2)(T - t)}{|\sigma|\sqrt{T - t}}\right) \geq 0,\]

which is always positive, i.e. we are constantly investing on the risk-free asset.

![Diagram of Black-Scholes price, risky investment, and riskless investment over time.](Fig. 5.12: Time evolution of the hedging portfolio for a put option on HSBC.)

In the above example the put option finished out of the money (OTM), so that no cash settlement or physical delivery occurs. A comparison of Figure 5.10 with market data can be found in Figures 7.13 and 7.14 below.

5.6 The Heat Equation

In this section we focus on the heat equation

\[
\frac{\partial g}{\partial t}(t, y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t, y)
\]

which is used to model the diffusion of heat over time through solids. Here, the data of \(g(x, t)\) represents the temperature measured at time \(t\) and point \(x\). We refer the reader to Widder (1975) for a complete treatment of this topic.

In the next proposition we notice that the solution \(f(t, x)\) of the Black-Scholes PDE can be transformed into a solution \(g(t, y)\) of the simpler heat equation by a change of variable and a time inversion \(t \mapsto T - t\) on the interval \([0, T]\), so that the terminal condition at time \(T\) in the Black-Scholes
equation (5.28) becomes an initial condition at time $t = 0$ in the heat equation (5.31).

**Proposition 5.15.** Assume that $f(t, x)$ solves the Black-Scholes PDE

$$
\begin{cases}
rf(t, x) = \frac{\partial f}{\partial t} (t, x) + rx \frac{\partial f}{\partial x} (t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} (t, x), \\
f(T, x) = (x - K)^+,
\end{cases}
$$

with terminal condition $h(x) = (x - K)^+, x > 0$. Then the function $g(t, y)$ defined by

$$g(t, y) = e^{rt} f \left( T - t, e^{\sigma y + (\sigma^2/2 - r)t} \right)$$

solves the heat equation (5.32) with initial condition

$$g(0, y) = h(e^{\sigma y}), \quad y \in \mathbb{R},$$

i.e.

$$
\begin{cases}
\frac{\partial g}{\partial t} (t, y) = \frac{1}{2} \frac{\partial^2 g}{\partial y^2} (t, y) \\
g(0, y) = h(e^{\sigma y}).
\end{cases}
$$

Proposition 5.15 will be proved in Section 5.7.

We can check by a direct calculation that the Gaussian probability density function $g(t, y) := e^{-y^2/(2t)}/\sqrt{2\pi t}$ solves the heat equation (5.27), as follows:

$$
\frac{\partial g}{\partial t} (t, y) = \frac{\partial}{\partial t} \left( \frac{e^{-y^2/(2t)}}{\sqrt{2\pi t}} \right) \\
= - \frac{e^{-y^2/(2t)}}{2t^{3/2} \sqrt{2\pi}} + \frac{y^2 e^{-y^2/(2t)}}{2t^2 \sqrt{2\pi}} \\
= \left( -\frac{1}{2t} + \frac{y^2}{2t^2} \right) g(t, y),
$$

and

$$
\frac{1}{2} \frac{\partial^2 g}{\partial y^2} (t, y) = - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{ye^{-y^2/(2t)}}{t \sqrt{2\pi t}} \right) \\
= - \frac{e^{-y^2/(2t)}}{2t \sqrt{2\pi t}} + \frac{y^2 e^{-y^2/(2t)}}{2t^2 \sqrt{2\pi}} \\
= \left( -\frac{1}{2t} + \frac{y^2}{2t^2} \right) g(t, y), \quad t \in \mathbb{R}_+, \quad y \in \mathbb{R}.
Fig. 5.13: Time-dependent solution of the heat equation.*

In Section 5.7 this equation will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

**Proposition 5.16.** The heat equation

\[
\begin{align*}
\frac{\partial g}{\partial t}(t, y) &= \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y) \\
g(0, y) &= \psi(y)
\end{align*}
\]

with initial condition

\[ g(0, y) = \psi(y) \]

has the solution

\[ g(t, y) = \int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}, \quad y \in \mathbb{R}, \quad t > 0. \]

*The animation works in Acrobat Reader on the entire pdf file.*
On the other hand it can be checked that at time $t = 0$,

$$\lim_{t \to 0} \int_{-\infty}^\infty \psi(z) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} = \lim_{t \to 0} \int_{-\infty}^\infty \psi(y + z) e^{-z^2/(2t)} \frac{dz}{\sqrt{2\pi t}} = \psi(y),$$

$y \in \mathbb{R}$. □

Let us provide a second proof of Proposition 5.16 using stochastic calculus and Brownian motion. Note that under the change of variable $x = z - y$ we have

$$g(t, y) = \int_{-\infty}^\infty \psi(z) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} = \int_{-\infty}^\infty \psi(y + x) e^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} = \mathbb{E}[\psi(y + B_t)] = \mathbb{E}[\psi(y - B_t)],$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $B_t \sim \mathcal{N}(0, t)$, $t \in \mathbb{R}_+$. Applying Itô’s formula we have

$$\mathbb{E}[\psi(y - B_t)] = \psi(y) - \mathbb{E} \left[ \int_0^t \psi'(y - B_s)dB_s \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^t \psi''(y - B_s)ds \right]$$

$$= \psi(y) + \frac{1}{2} \int_0^t \mathbb{E} \left[ \psi''(y - B_s) \right] ds$$

$$= \psi(y) + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial y^2} \mathbb{E} \left[ \psi(y - B_s) \right] ds,$$

since the expectation of the stochastic integral is zero. Hence

$$\frac{\partial g}{\partial t}(t, y) = \frac{\partial}{\partial t} \mathbb{E}[\psi(y - B_t)]$$

$$= \frac{1}{2} \frac{\partial^2}{\partial y^2} \mathbb{E} \left[ \psi(y - B_t) \right]$$

$$= \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y),$$

Concerning the initial condition we check that

$$g(0, y) = \mathbb{E}[\psi(y - B_0)] = \mathbb{E}[\psi(y)] = \psi(y).$$
The expression $g(t, y) = \mathbb{E}[\psi(y - B_t)]$ provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion. Namely, when $
abla_y(y) := 1_{[-\varepsilon, \varepsilon]}(y)$, we find that

$$ g_{\varepsilon}(t, y) = \mathbb{E}[\psi_{\varepsilon}(y - B_t)] = \mathbb{E}[1_{[-\varepsilon, \varepsilon]}(y - B_t)] = \mathbb{P}(y - B_t \in [-\varepsilon, \varepsilon]) = \mathbb{P}(y - \varepsilon \leq B_t \leq y + \varepsilon) $$

represents the probability of finding $B_t$ within a neighborhood $[y - \varepsilon, y + \varepsilon]$ of the point $y \in \mathbb{R}$.

### 5.7 Solution of the Black-Scholes PDE

In this section we solve the Black-Scholes PDE by the kernel method of Section 5.6 and a change of variables. This solution method uses the change of variables (5.29) of Proposition 5.15 and a time inversion from which the terminal condition at time $T$ in the Black-Scholes equation becomes an initial condition at time $t = 0$ in the heat equation.

Next, we state the proof Proposition 5.15.

**Proof.** Letting $s = T - t$ and $x = e^{\sigma y + (\sigma^2/2 - r)t}$, we have

$$ \frac{\partial g}{\partial t}(t, y) = re^{rt}f(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) - e^{rt} \frac{\partial f}{\partial s}(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) $$

$$ + \left( \frac{\sigma^2}{2} - r \right) e^{rt} e^{\sigma y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) $$

$$ = re^{rt}f(T - t, x) - e^{rt} \frac{\partial f}{\partial s}(T - t, x) + \left( \frac{\sigma^2}{2} - r \right) e^{rt} x \frac{\partial f}{\partial x}(T - t, x) $$

$$ = \frac{1}{2} e^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T - t, x) + \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T - t, x), \quad (5.34) $$

where on the last step we used the Black-Scholes PDE. On the other hand we have

$$ \frac{\partial g}{\partial y}(t, y) = \sigma e^{rt} e^{\sigma y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) $$

and

$$ \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, y) = \frac{\sigma^2}{2} e^{rt} e^{\sigma y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) $$

$$ + \frac{\sigma^2}{2} e^{2rt} e^{2\sigma y + 2(\sigma^2/2 - r)t} \frac{\partial^2 f}{\partial x^2}(T - t, e^{\sigma y + (\sigma^2/2 - r)t}) $$
\[ \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T - t, x) + \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T - t, x). \quad (5.35) \]

We conclude by comparing (5.34) with (5.35), which shows that \( g(t, x) \) solves the heat equation (5.32) with initial condition

\[ g(0, y) = f(T, e^{\sigma y}) = h(e^{\sigma y}). \]

In the next proposition we recover the Black-Scholes formula (5.18) by solving the PDE (5.28). The Black-Scholes formula will also be recovered by a probabilistic argument via the computation of an expected value in Proposition 6.4.

**Proposition 5.17.** When \( h(x) = (x - K)^+ \), the solution of the Black-Scholes PDE (5.28) is given by

\[ f(t, x) = x \Phi(d_+(T - t)) - K e^{-(T - t)r} \Phi(d_-(T - t)), \quad x > 0, \]

where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}, \]

and

\[
\begin{align*}
  d_+(T - t) &:= \log(x/K) + (r + \sigma^2/2)(T - t) / |\sigma|\sqrt{T - t}, \\
  d_-(T - t) &:= \log(x/K) + (r - \sigma^2/2)(T - t) / |\sigma|\sqrt{T - t},
\end{align*}
\]

\( x > 0. \)

**Proof.** By inversion of (5.29) with \( s = T - t \) and \( x = e^{\sigma y + (\sigma^2/2 - r)t} \) we get

\[ f(s, x) = e^{-(T - s)r} g \left( T - s, \frac{-(\sigma^2/2 - r)(T - s) + \log x}{|\sigma|} \right). \]

Hence using the solution (5.33) and Relation (5.30) we get

\[
\begin{align*}
  f(t, x) &= e^{-(T - t)r} g \left( T - t, \frac{-(\sigma^2/2 - r)(T - t) + \log x}{|\sigma|} \right) \\
  &= e^{-(T - t)r} \int_{-\infty}^{\infty} \psi \left( \frac{-(\sigma^2/2 - r)(T - t) + \log x}{|\sigma|} + z \right) e^{-z^2/(2(T - t))} \frac{dz}{\sqrt{2\pi(T - t)}} \\
  &= e^{-(T - t)r} \int_{-\infty}^{\infty} h(x e^{\sigma y - (\sigma^2/2 - r)(T - t)}) e^{-z^2/(2(T - t))} \frac{dz}{\sqrt{2\pi(T - t)}} \\
  &= e^{-(T - t)r} \int_{-\infty}^{\infty} \left( x e^{\sigma y - (\sigma^2/2 - r)(T - t)} - K \right) e^{-z^2/(2(T - t))} \frac{dz}{\sqrt{2\pi(T - t)}}.
\end{align*}
\]

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\[= e^{-(T-t)r} \times \int_{-\infty}^{\infty} \frac{z e^{\text{re}^{\sigma^2/2(T-t)}}}{\sqrt{2\pi (T-t)}} \frac{dz}{\sqrt{2\pi (T-t)}}\]

= \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} e^{|\sigma z^2/2(T-t)}} e^{-z^2/2(T-t)} \frac{dz}{\sqrt{2\pi (T-t)}}

= \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} e^{|\sigma z^2/2(T-t)}} e^{-z^2/2(T-t)} \frac{dz}{\sqrt{2\pi (T-t)}}

= \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} e^{|\sigma z^2/2(T-t)}} e^{-z^2/2(T-t)} \frac{dz}{\sqrt{2\pi (T-t)}}

= \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} e^{|\sigma z^2/2(T-t)}} e^{-z^2/2(T-t)} \frac{dz}{\sqrt{2\pi (T-t)}}

\[= x \left( 1 - \Phi \left( -d_+ (T-t) \right) \right) - Ke^{-(T-t)r} \Phi \left( -d_- (T-t) \right)\]

where we used the relation

\[1 - \Phi(a) = \Phi(-a), \quad a \in \mathbb{R}.\]

\[\square\]

Exercises

Exercise 5.1 Consider a risky asset price \( (S_t)_{t \in \mathbb{R}} \) modeled in the Cox et al. (1985) (CIR) model as

\[dS_t = \beta(\alpha - S_t)dt + \sigma \sqrt{S_t} dB_t, \quad \alpha, \beta, \sigma > 0,\]

and let \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) be a portfolio strategy whose price \( V_t := \eta_t A_t + \xi_t S_t \), takes the form \( V_t = g(t, S_t), \ t \in \mathbb{R}_+ \). Based on the self-financing condition \( \cdots \)
written as

\[ dV_t = rV_t dt - r \xi_t S_t dt + \beta (\alpha - S_t) \xi_t S_t dt + \sigma \xi_t S_t dB_t, \quad t \in \mathbb{R}_+, \quad (5.36) \]

derive the PDE satisfied by the function \( g(t, x) \) using the Itô formula.

Exercise 5.2 Black-Scholes PDE with dividends. Consider an underlying asset price process \((S_t)_{t \in \mathbb{R}_+}\) modeled as

\[ dS_t = (\mu - \delta) S_t dt + \sigma S_t dB_t, \]

where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion and \(\delta > 0\) is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails a drop in the stock price by the same amount occurring generally on the ex-dividend date, on which the purchase of the security no longer entitles the investor to the dividend amount. The list of investors entitled to dividend payment is consolidated on the date of record, and payment is made on the payable date.

```r
library(quantmod)
getSymbols("0005.HK", from="2010-01-01", to=Sys.Date(), src="yahoo")
getDividends("0005.HK", from="2010-01-01", to=Sys.Date(), src="yahoo")
```

a) Assuming that the portfolio with value \( V_t \) at time \( t \) is self-financing and that dividends are continuously reinvested, write down the portfolio variation \( dV_t \).

b) Assuming that the portfolio value \( V_t \) takes the form \( V_t = g(t, S_t) \) at time \( t \), derive the Black-Scholes PDE for the function \( g(t, x) \) with its terminal condition.

c) Compute the price at time \( t \in [0, T] \) of a European call option with strike price \( K \) by solving the corresponding Black-Scholes PDE.

Exercise 5.3 Power option. (Exercise 3.10 continued).

a) Solve the Black-Scholes PDE

\[ rg(x, t) = \frac{\partial g}{\partial t}(x, t) + rx \frac{\partial g}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(x, t) \quad (5.37) \]

with terminal condition \( g(x, T) = x^2, x > 0 \).

*Hint:* Try a solution of the form \( g(x, t) = x^2 f(t) \), and find \( f(t) \).

b) Find the respective quantities \( \xi_t \) and \( \eta_t \) of the risky asset \( S_t \) and risk-free asset \( A_t = e^{rt} \) in the portfolio with value

\[ V_t = g(S_t, t) = \xi_t S_t + \eta_t A_t \]
hedging the contract with payoff $S_T^2$ at maturity.

Exercise 5.4  On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price $S$ of the MTR Corporation with maturity $T = 23/12/2008$, strike price $K = \text{HK}$ 36.08 and entitlement ratio=10. Recall that in the Black-Scholes model, the price at time $t$ of a European claim on the underlying asset $S_t$, with strike price $K$, maturity $T$, interest rate $r$ and volatility $\sigma > 0$ is given by the Black-Scholes formula as

$$f(t, S_t) = S_t \Phi(d_+(T-t)) - K e^{-(T-t)r} \Phi(d_-(T-t)),$$

where

$$\begin{cases}
  d_-(T-t) = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}, \\
  d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t} = \frac{(r + \sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}.
\end{cases}$$

Recall that by Proposition 5.14 we have

$$\frac{\partial f}{\partial x}(t, S_t) = \Phi(d_+(T-t)), \quad 0 \leq t \leq T.$$

(a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time $t = \text{November 07, 2008}$ with $S_t = \text{HK}$ 17.20, assuming a volatility $\sigma = 90\% = 0.90$ and an annual risk-free interest rate $r = 4.377\% = 0.04377$,

(b) Still using the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time $t = \text{November 07, 2008}$ in order to hedge one such option at maturity $T = 23/12/2008$.

(c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

![Fig. 5.14: Option price as a function of the volatility $\sigma > 0$.](http://www.ntu.edu.sg/home/nprivault/index.html)
Knowing that the closing price of the warrant on November 07, 2008 was HK$ 0.023, which value can you infer for the implied volatility $\sigma$ at this date?

Exercise 5.5 Forward contracts. Recall that the price $\pi_t(C)$ of a claim $C = h(S_T)$ of maturity $T$ can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ satisfies the Black-Scholes PDE

\[
\begin{cases}
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x), \\
g(T, x) = h(x),
\end{cases}
\]

with terminal condition $g(T, x) = h(x)$, $x > 0$.

a) Assume that $C$ is a forward contract with payoff

$$C = S_T - K,$$

at time $T$. Find the function $h(x)$ in (1).

b) Find the solution $g(t, x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0, T]$.

*Hint*: search for a solution of the form $g(t, x) = x - \alpha(t)$ where $\alpha(t)$ is a function of $t$ to be determined.

c) Compute the quantity

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t)$$

of risky assets in a self-financing portfolio hedging $C$.

d) Repeat the above questions with the terminal condition $g(T, x) = x$.

Exercise 5.6

a) Solve the Black-Scholes PDE

\[
rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx \frac{\partial g}{\partial x}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(t, x)
\]

with terminal condition $g(T, x) = 1$, $x > 0$.

*Hint*: Try a solution of the form $g(t, x) = f(t)$ and find $f(t)$.

b) Find the respective quantities $\xi_t$ and $\eta_t$ of the risky asset $S_t$ and risk-free asset $A_t = e^{rt}$ in the portfolio with value

* Download the corresponding R code or the IPython notebook that can be run here.
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\[ V_t = g(t, S_t) = \xi_t S_t + \eta_t A_t \]

hedging the contract with payoff $1$ at maturity.

Exercise 5.7 Log-contracts, see also Exercise 7.5.

a) Solve the PDE

\[ 0 = \frac{\partial g}{\partial t}(x, t) + r x \frac{\partial g}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 g}{\partial x^2}(x, t) \]

with the terminal condition \( g(x, T) := \log x, \, x > 0 \).

\textbf{Hint:} Try a solution of the form \( g(x, t) = f(t) + \log x \), and find \( f(t) \).

b) Solve the Black-Scholes PDE

\[ rh(x, t) = \frac{\partial h}{\partial t}(x, t) + r x \frac{\partial h}{\partial x}(x, t) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 h}{\partial x^2}(x, t) \]  \hspace{1cm} (5.39)

with the terminal condition \( h(x, T) := \log x, \, x > 0 \).

\textbf{Hint:} Try a solution of the form \( h(x, t) = u(t)g(x, t) \), and find \( u(t) \).

c) Find the respective quantities \( \xi_t \) and \( \eta_t \) of the risky asset \( S_t \) and risk-free asset \( A_t = e^{rt} \) in the portfolio with value

\[ V_t = g(S_t, t) = \xi_t S_t + \eta_t A_t \]

hedging a log-contract with payoff \( \log S_T \) at maturity.

Exercise 5.8 Binary options. Consider a price process \( (S_t)_{t \in \mathbb{R}_+} \) given by

\[ \frac{dS_t}{S_t} = r dt + \sigma dB_t, \quad S_0 = 1, \]

under the risk-neutral probability measure \( \mathbb{P}^* \). A binary (or digital) call option is a contract with maturity \( T \), strike price \( K \), and payoff

\[ C_d := \mathbb{1}_{[K, \infty)}(S_T) = \begin{cases} \$1 & \text{if } S_T \geq K, \\ 0 & \text{if } S_T < K. \end{cases} \]

a) Derive the Black-Scholes PDE satisfied by the pricing function \( C_d(t, S_t) \) of the binary call option, together with its terminal condition.
b) Show that the solution $C_d(t, x)$ of the Black-Scholes PDE of Question (a) is given by

$$C_d(t, x) = e^{-(T-t)r} \Phi \left( \frac{(r - \sigma^2 / 2)(T - t) + \log(x / K)}{\sigma \sqrt{T - t}} \right)$$

$$= e^{-(T-t)r} \Phi(d_-(T - t)),$$

where

$$d_-(T - t) := \frac{(r - \sigma^2 / 2)(T - t) + \log(S_t / K)}{\sigma \sqrt{T - t}}, \quad 0 \leq t < T.$$  

Exercise 5.9

a) Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t$$

in terms of $\alpha, \sigma \in \mathbb{R}$, and the initial condition $S_0$.

b) Write down the Black-Scholes PDE satisfied by the function $C(t, x)$, where $C(t, S_t)$ is the price at time $t \in [0, T]$ of the contingent claim with payoff $\phi(S_T) = \exp(S_T)$, and identify the process Delta $(\xi_t)_{t \in [0, T]}$ that hedges this claim.

c) Solve the Black-Scholes PDE of Question (b) with the terminal condition $\phi(x) = e^x, \quad x \in \mathbb{R}$.

Hint: Search for a solution of the form

$$C(t, x) = \exp \left( -(T-t)r + x h(t) + \frac{\sigma^2}{4r} (h^2(t) - 1) \right),$$

where $h(t)$ is a function to be determined, with $h(T) = 1$.

d) Compute the strategy $(\xi_t, \eta_t)_{t \in [0, T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

Exercise 5.10

a) Show that for every fixed value of $S$, the function

$$d \mapsto h(S, d) := S \Phi(d + |\sigma| \sqrt{T}) - K e^{-rT} \Phi(d),$$

reaches its maximum at $d_*(S) := \frac{\log(S/K) + (r - \sigma^2 / 2)T}{|\sigma| \sqrt{T}}$.
b) By the differentiation rule
\[
\frac{d}{dS} h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d'_*(S) \frac{\partial h}{\partial d}(S, d_*(S)),
\]
recover the value of the Black-Scholes Delta.

Exercise 5.11 Consider the backward induction relation (3.13), i.e.
\[
\tilde{v}(t, x) = (1 - p^*_N)\tilde{v}(t + 1, x(1 + a_N)) + p^*_N \tilde{v}(t + 1, x(1 + b_N)),
\]
using the renormalizations \( r_N := rT/N \) and
\[
a_N := (1 + r_N)e^{-|\sigma|\sqrt{T/N}} - 1, \quad b_N := (1 + r_N)e^{\sigma|\sqrt{T/N}} - 1, \quad N \geq 1,
\]
of Section 3.6, with
\[
p^*_N = \frac{r_N - a_N}{b_N - a_N} \quad \text{and} \quad p^*_N = \frac{b_N - r_N}{b_N - a_N}.
\]

a) Show that the Black-Scholes PDE (5.13) of Proposition 5.12 can be recovered from the induction relation (3.13) when the number \( N \) of time steps tends to infinity.

b) Show that the expression of the Delta \( \xi_t = \frac{\partial g_c}{\partial x}(t, S_t) \) can be similarly recovered from the finite difference relation (3.19), i.e.
\[
\xi_t^{(1)}(S_{t-1}) = \frac{v(t, (1 + b_N)S_{t-1}) - v(t, (1 + a_N)S_{t-1})}{S_{t-1}(b_N - a_N)}
\]
as \( N \) tends to infinity.

Problem 5.12 (Leung and Sircar (2015)) ProShares Ultra S&P500 and ProShares UltraShort S&P500 are leveraged investment funds that seek daily investment results, before fees and expenses, that correspond to \( \beta \) times \( (\beta x) \) the daily performance of the S&P500, with respectively \( \beta = 2 \) for ProShares Ultra and \( \beta = -2 \) for ProShares UltraShort. Here, leveraging with a factor \( \beta : 1 \) aims at multiplying the potential return of an investment by a factor \( \beta \). The following 10 questions are interdependent and should be treated in sequence.

a) Consider a risky asset priced \( S_0 := $4 \) at time \( t = 0 \) and taking two possible values \( S_1 = $5 \) and \( S_1 = $2 \) at time \( t = 1 \). Compute the two possible returns (in \%) achieved when investing $4 in one share of the asset \( S \), and the expected return under the risk-neutral probability measure, assuming that the risk-free rate is zero.
b) Leveraging. Still based on an initial $4 investment, we decide to leverage by a factor $\beta = 3$ by borrowing another $(\beta - 1) \times $4 = 2 \times $4 at rate zero to purchase a total of $\beta = 3$ shares of the asset $S$. Compute the two returns (in %) possibly achieved in this case, and the expected return under the risk-neutral probability measure, assuming that the risk-free rate is zero.

c) Denoting by $F_t$ the ProShares value at time $t$, how much should the fund invest in the underlying asset $S_t$, and how much $ should it borrow or save on the risk-free market at any time $t$ in order to leverage with a factor $\beta : 1$?

d) Find the portfolio allocation $(\xi_t, \eta_t)$ for the fund value

$$F_t = \xi_t S_t + \eta_t A_t, \quad t \in \mathbb{R}_+,$$

according to Question (c), where $A_t := A_0 e^{rt}$ is the risk-free money market account.

e) We choose to model the S&P500 index $S_t$ as the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+,$$

under the risk-neutral probability measure $\mathbb{P}^*$. Find the stochastic differential equation satisfied by $(F_t)_{t \in \mathbb{R}_+}$ under the self-financing condition $dF_t = \xi_t dS_t + \eta_t dA_t$.

f) Is the discounted fund value $(e^{-rt} F_t)_{t \in \mathbb{R}_+}$ a martingale under the risk-neutral probability measure $\mathbb{P}^*$?

g) Find the relation between the fund value $F_t$ and the index $S_t$ by solving the stochastic differential equation obtained for $F_t$ in Question (e). For simplicity we normalize $F_0 := S_0^\beta$.

h) Write the price at time $t = 0$ of the call option with payoff $(F_T - K)^+$ on the ProShares index using the Black-Scholes formula.

i) Show that when $\beta > 0$, the Delta at time $t \in [0, T)$ of the call option with payoff $(F_T - K)^+$ on ProShares Ultra is equal to the Delta of the call option with payoff $(S_T - K_\beta(t))^+$ on the S&P500, for a certain strike price $K_\beta(t)$ to be determined explicitly.

j) When $\beta < 0$, find the relation between the Delta at time $t \in [0, T)$ of the call option with payoff $(F_T - K)^+$ on ProShares UltraShort and the Delta of the put option with payoff $(K_\beta(t) - S_T)^+$ on the S&P500.

Problem 5.13 Stop-loss start-gain strategy (Lipton (2001) § 8.3.3., Exercise 4.20 continued). Let $(B_t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion started at $B_0 \in \mathbb{R}$.

a) We consider a simplified foreign exchange model in which the AUD is a risky asset and the AUD/SGD exchange rate at time $t$ is modeled by $B_t$, i.e. AU$1 equals SG$B_t at time $t$. A foreign exchange (FX) European call option gives to its holder the right (but not the obligation) to receive
AU$1 in exchange for $K = SG$1 at maturity $T$. Give the option payoff at maturity, quoted in SGD.

In the sequel, for simplicity we assume no time value of money ($r = 0$), i.e. the (risk-free) SGD account is priced $A_t = A_0 = 1$, $t \in [0, T]$.

b) Consider the following hedging strategy for the European call option of Question (a):

i) If $B_0 > 1$, charge the premium $B_0 - 1$ at time 0, borrow SG$1 and purchase AU$1.
ii) If $B_0 < 1$, issue the option for free.
iii) From time 0 to time $T$, purchase* AU$1 every time $B_t$ crosses $K = 1$ from below, and sell† AU$1 each time $B_t$ crosses $K = 1$ from above.

Show that this strategy effectively hedges the foreign exchange European call option at maturity $T$.

*Note. Note that it suffices to consider four scenarios based on $B_0 < 1$ vs $B_0 < 1$ and $B_T > 1$ vs $B_T < 1$.

c) Determine the quantities $\eta_t$ of SGD cash and $\xi_t$ of (risky) AUDs to be held in the portfolio and express the portfolio value

$$V_t = \eta_t + \xi_t B_t$$

at all times $t \in [0, T]$.

d) Compute the integral summation

$$\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s$$

of portfolio profits and losses at any time $t \in [0, T]$.

*Hint. Apply the result of Question (e).

e) Is the portfolio strategy $(\eta_t, \xi_t)_{t \in [0, T]}$ self-financing? How to interpret the answer in practice?

* We need to borrow SG$1 if this is the first AUD purchase.
† We use the SG$1 product of the sale to refund the loan.