Chapter 1
Assets, Portfolios, and Arbitrage

We consider a simplified single step financial model with only two time instants $t = 0$ and $t = 1$. In this simple setting we introduce the notions of portfolio, arbitrage, market completeness, pricing and hedging using the notation of Föllmer and Schied (2004). A binary asset price model is considered as an example in Section 1.7.

1.1 Definitions and Notation

We will use the following notation. An element $\bar{x}$ of $\mathbb{R}^{d+1}$ is a vector $\bar{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(d)})$ made of $d + 1$ components. The scalar product $\bar{x} \cdot \bar{y}$ of two vectors $\bar{x}, \bar{y} \in \mathbb{R}^{d+1}$ is defined by

$$\bar{x} \cdot \bar{y} = x^{(0)}y^{(0)} + x^{(1)}y^{(1)} + \cdots + x^{(d)}y^{(d)}.$$ 

The vector $S_0 = (S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)})$ denotes the prices at time $t = 0$ of $d + 1$ assets, namely $S_0^{(i)} > 0$ is the price at time $t = 0$ of asset $\nu^o$ $i = 0, 1, \ldots, d$. 

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The asset values \( S_1^{(i)} > 0 \) of assets \( n = 0, 1, \ldots, d \) at time \( t = 1 \) are represented by the vector

\[
S_1 = (S_1^{(0)}, S_1^{(1)}, \ldots, S_1^{(d)}),
\]
whose components \( (S_1^{(1)}, \ldots, S_1^{(d)}) \) are random variables defined on a probability space \( (\Omega, \mathcal{F}, P) \).

In addition we will assume that asset \( n = 0 \) is a risk-free asset (of savings account type) that yields an interest rate \( r > 0 \), i.e. we have

\[
S_1^{(0)} = (1 + r)S_0^{(0)}.
\]

1.2 Portfolio Allocation and Short Selling

A portfolio based on the assets \( 0, 1, \ldots, d \) is viewed as a vector

\[
\xi = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}) \in \mathbb{R}^{d+1},
\]
in which \( \xi^{(i)} \) represents the (possibly fractional) quantity of asset \( n = i \) owned by an investor, \( i = 0, 1, \ldots, d \). The price of such a portfolio is given by

\[
\xi \cdot S_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)} = \xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} + \cdots + \xi^{(d)} S_0^{(d)}
\]
at time \( t = 0 \).

At time \( t = 1 \) the value of the portfolio has evolved into

\[
\xi \cdot S_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)}.
\]

If \( \xi^{(0)} > 0 \), the investor puts the amount \( \xi^{(0)} S_0^{(0)} > 0 \) on a savings account with interest rate \( r \), while if \( \xi^{(0)} < 0 \) he borrows the amount \( -\xi^{(0)} S_0^{(0)} > 0 \) with the same interest rate \( r \).

For \( i = 1, 2, \ldots, d \), if \( \xi^{(i)} > 0 \) then the investor buys a (possibly fractional) quantity \( \xi^{(i)} > 0 \) of the asset \( n = i \), while if \( \xi^{(i)} < 0 \) he borrows a quantity \( -\xi^{(i)} > 0 \) of asset \( i \) and sells it to obtain the amount \( -\xi^{(i)} S_0^{(i)} > 0 \). In the latter case one says that the investor short sells a quantity \( -\xi^{(i)} > 0 \) of the asset \( n = i \).
Definition 1.1. The short selling ratio, or percentage of daily turnover activity related to short selling, is defined as the ratio of the number of daily short sold shares divided by daily volume.

Profits are usually made by first buying at a low price and then selling at a high price. Short sellers apply the same rule but in the reverse time order: first sell high, and then buy low if possible, by applying the following procedure.

1. Borrow the asset \( n^o \).

2. At time \( t = 0 \), sell the asset \( n^o \) on the market at the price \( S_0^{(i)} \) and invest the amount \( S_0^{(i)} \) at the interest rate \( r > 0 \).

3. Buy back the asset \( n^o \) at time \( t = 1 \) at the price \( S_1^{(i)} \), with hopefully \( S_1^{(i)} < (1 + r)S_0^{(i)} \).

4. Return the asset to its owner, with possibly a (small) fee \( p > 0 \). *

At the end of the operation the profit made on share \( n^o \) equals

\[
(1 + r)S_0^{(i)} - S_1^{(i)} - p > 0,
\]

which is positive provided that \( S_1^{(i)} < (1 + r)S_0^{(i)} \) and \( p > 0 \) is sufficiently small.

1.3 Arbitrage

Arbitrage can be described as:

“the purchase of currencies, securities, or commodities in one market for immediate resale in others in order to profit from unequal prices” †

In other words, an arbitrage opportunity is the possibility to make a strictly positive amount of money starting from zero, or even from a negative amount. In a sense, the existence of an arbitrage opportunity can be seen as a way to “beat” the market.

For example, triangular arbitrage is a way to realize arbitrage opportunities based on discrepancies in the cross exchange rates of foreign currencies, as seen in Figure 1.1. ‡

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* The cost \( p \) of short selling will not be taken into account in later calculations.
† https://www.collinsdictionary.com/dictionary/english/arbitrage
‡ https://en.wikipedia.org/wiki/Triangular_arbitrage
As an attempt to realize triangular arbitrage based on the data of Figure 1.1b, one could:

1. Change US$1.00 into €0.89347,
2. Change €0.89347 into £0.89347 × 0.86167 = £0.769876295,
3. Change back £0.769876295 into US$0.769876295 × 1.2981 = US$0.999376418,
which would actually result into a small loss. Alternatively, one could:

1. Change US$1.00 into £0.76988,
2. Change £0.76988 into €1.16054 × 0.76988 = €0.893476535,
3. Change back €0.893476535 into US$0.893476535 × 1.11923 = US$1.000005742,
which would result into a small gain, assuming the absence of transaction costs.

### Realizing arbitrage

In the example below we realize arbitrage by buying and holding an asset.

1. Borrow the amount $-\xi^{(0)} S^{(0)}_0 > 0$ on the risk-free asset $n^o 0$.
2. Use the amount $-\xi^{(0)} S^{(0)}_0 > 0$ to purchase a quantity $\xi^{(i)} = -\xi^{(0)} S^{(0)}_0 / S^{(i)}_0$, of the risky asset $n^o i \geq 1$ at time $t = 0$ and price $S^{(i)}_0$ so that the initial portfolio cost is

   $$\xi^{(0)} S^{(0)}_0 + \xi^{(i)} S^{(i)}_0 = 0.$$

3. At time $t = 1$, sell the risky asset $n^o i$ at the price $S^{(i)}_1$, with hopefully $S^{(i)}_1 > (1 + r) S^{(i)}_0$.
4. Refund the amount $-(1 + r)\xi^{(0)} S^{(0)}_0 > 0$ with interest rate $r > 0$. 

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At the end of the operation the profit made is
\[
\xi^{(i)} S_1^{(i)} - (1 + r)\xi^{(0)} S_0^{(0)} = \xi^{(i)} S_1^{(i)} + (1 + r)\xi^{(0)} S_0^{(0)}
\]
\[
= -\xi^{(0)} S_0^{(0)} S_1^{(i)} + (1 + r)\xi^{(0)} S_0^{(0)}
\]
\[
= -\xi^{(0)} S_0^{(0)} (S_1^{(i)} - (1 + r)S_0^{(i)})
\]
\[
= \xi^{(i)} (S_1^{(i)} - (1 + r)S_0^{(i)})
\]
\[
> 0,
\]
or \( S_1^{(i)} - (1 + r)S_0^{(i)} \) per unit of stock invested, which is positive provided that \( S_1^{(i)} > S_0^{(i)} \) and \( r \) is sufficiently small.

Arbitrage opportunities can be similarly realized using the short selling procedure described in Section 1.2.

<table>
<thead>
<tr>
<th>City</th>
<th>Currency</th>
<th>US$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokyo</td>
<td>38,800 yen</td>
<td>$346</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>HK$2,956.67</td>
<td>$381</td>
</tr>
<tr>
<td>Seoul</td>
<td>378,533 won</td>
<td>$400</td>
</tr>
<tr>
<td>Taipei</td>
<td>NT$12,980</td>
<td>$404</td>
</tr>
<tr>
<td>New York</td>
<td></td>
<td>$433</td>
</tr>
<tr>
<td>Sydney</td>
<td>A$633.28</td>
<td>$483</td>
</tr>
<tr>
<td>Frankfurt</td>
<td>€399</td>
<td>$513</td>
</tr>
<tr>
<td>Paris</td>
<td>€399</td>
<td>$513</td>
</tr>
<tr>
<td>Rome</td>
<td>€399</td>
<td>$513</td>
</tr>
<tr>
<td>Brussels</td>
<td>€399.66</td>
<td>$514</td>
</tr>
<tr>
<td>London</td>
<td>£279.99</td>
<td>$527</td>
</tr>
<tr>
<td>Manila</td>
<td>29,500 pesos</td>
<td>$563</td>
</tr>
<tr>
<td>Jakarta</td>
<td>5,754,1676 rupiah</td>
<td>$627</td>
</tr>
</tbody>
</table>

Fig. 1.2: Arbitrage: Retail prices around the world for the Xbox 360 in 2006.

Next, we state a mathematical formulation of the concept of arbitrage.

**Definition 1.2.** A portfolio allocation \( \bar{\xi} \in \mathbb{R}^{d+1} \) constitutes an arbitrage opportunity if the three following conditions are satisfied:

i) \( \bar{\xi} \cdot S_0 \leq 0 \) at time \( t = 0 \), \( \) [start from a zero-cost portfolio or in debt]

ii) \( \bar{\xi} \cdot \bar{S}_1 \geq 0 \) at time \( t = 1 \), \( \) [finish with a nonnegative amount]

iii) \( \mathbb{P}(\bar{\xi} \cdot \bar{S}_1 > 0) > 0 \) at time \( t = 1 \). [profit made with non-zero probability]
Note that there exist multiple ways to break the assumptions of Definition 1.2 in order to achieve absence of arbitrage. For example, under absence of arbitrage, satisfying Condition (i) means that either $\bar{\xi} \cdot \bar{S}_1$ cannot be a.s. non-negative (i.e., potential losses cannot be avoided), or $\mathbb{P}(\bar{\xi} \cdot \bar{S}_1 > 0) = 0$, (i.e., no strictly positive profit can be made).

The are many real-life examples of situations where arbitrage opportunities can occur, such as:

- assets with different returns (finance),
- servers with different speeds (queueing, networking, computing),
- highway lanes with different speeds (driving).

In the latter two examples, the absence of arbitrage is consequence of the fact that switching to a faster lane or server may result into congestion, thus annihilating the potential benefit of the shift.

### Table 1.1: Absence of arbitrage - the Mark Six “Investment Table”.

<table>
<thead>
<tr>
<th>模式</th>
<th>一注形式</th>
<th>两注形式</th>
<th>三注形式</th>
<th>四注形式</th>
<th>五注形式</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of Selections</td>
<td>HK$</td>
<td>No. of Legs</td>
<td>HK$</td>
<td>No. of Legs</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td>6</td>
<td>30</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>140</td>
<td>7</td>
<td>105</td>
<td>6</td>
<td>75</td>
</tr>
<tr>
<td>9</td>
<td>420</td>
<td>8</td>
<td>280</td>
<td>7</td>
<td>175</td>
</tr>
<tr>
<td>10</td>
<td>1,050</td>
<td>9</td>
<td>630</td>
<td>8</td>
<td>350</td>
</tr>
<tr>
<td>11</td>
<td>2,310</td>
<td>10</td>
<td>1,260</td>
<td>9</td>
<td>630</td>
</tr>
<tr>
<td>12</td>
<td>4,620</td>
<td>11</td>
<td>2,510</td>
<td>10</td>
<td>1,050</td>
</tr>
<tr>
<td>13</td>
<td>8,580</td>
<td>12</td>
<td>3,960</td>
<td>11</td>
<td>1,650</td>
</tr>
<tr>
<td>14</td>
<td>15,015</td>
<td>13</td>
<td>6,435</td>
<td>12</td>
<td>2,475</td>
</tr>
<tr>
<td>15</td>
<td>25,025</td>
<td>14</td>
<td>10,010</td>
<td>13</td>
<td>3,575</td>
</tr>
<tr>
<td>49</td>
<td>69,919,080</td>
<td>48</td>
<td>8,561,520</td>
<td>47</td>
<td>891,825</td>
</tr>
</tbody>
</table>

In the table of Figure 1.1 the absence of arbitrage opportunities is materialized by the fact that the price of each combination is found to be proportional to its probability, thus making the game fair and disallowing any opportunity or arbitrage that would result of betting on a more profitable combination.

In the sequel we will work under the assumption that arbitrage opportunities do not occur and we will rely on this hypothesis for the pricing of financial instruments.
Example: share rights

Let us give a market example of pricing by absence of arbitrage.

From March 24 to 31, 2009, HSBC issued rights to buy shares at the price of $28. This right behaves similarly to an option in the sense that it gives the right (with no obligation) to buy the stock at $K = 28$. On March 24, the HSBC stock price closed at $41.70.

The question is: how to value the price $R$ of the right to buy one share? This question can be answered by looking for arbitrage opportunities. Indeed, there are two ways to purchase the stock:

1. Directly buy the stock on the market at the price of $41.70. Cost: $41.70,

or:

2. First, purchase the right at price $R$, and then the stock at price $28$. Total cost: $R + 28$.

a) In case $R + 28 < 41.70$, (1.1)

arbitrage would be possible for an investor who owns no stock and no rights, by:

i) Buying the right at a price $R$, and then
ii) Buying the stock at price $28$, and
iii) Reselling the stock at the market price of $41.70.

The profit made by this investor would equal

$41.70 - (R + 28) > 0$.

b) On the other hand, in case

$R + 28 > 41.70$, (1.2)

arbitrage would be possible for an investor who owns the rights, by:

i) Buying the stock on the market at $41.70,
ii) Selling the right by contract at the price $R$, and then
iii) Selling the stock at $28$ to that other investor.

In this case, the profit made would equal

$R + 28 - 41.70 > 0$. 

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In the absence of arbitrage opportunities, the combination of (1.1) and (1.2) implies that $R$ should satisfy

$$R + 28 - 41.70 = 0,$$

\textit{i.e.} the \textit{arbitrage} price of the right is given by the equation

$$R = 41.70 - 28 = 13.70. \quad (1.3)$$

Interestingly, the \textit{market} price of the right was $13.20 at the close of the session on March 24. The difference of $0.50 can be explained by the presence of various market factors such as transaction costs, the time value of money, or simply by the fact that asset prices are constantly fluctuating over time. It may also represent a small arbitrage opportunity, which cannot be at all excluded. Nevertheless, the absence of arbitrage argument (1.3) prices the right at $13.70, which is quite close to its market value. Thus the absence of arbitrage hypothesis appears as an accurate tool for pricing.

Exercise: A company issues share rights, so that ten rights allow one to purchase three shares at the price of €6.35. Knowing that the stock is currently valued at €8, estimate the price of the right by absence of arbitrage.

\textit{Answer}: Letting $R$ denote the price of one right, it will require $10R/3$ to purchase one stock at €6.35, hence absence of arbitrage tells us that

$$\frac{10}{3} R + 6.35 = 8,$$

from which it follows that

$$R = \frac{3}{10} (8 - 6.35) = €0.495.$$

Note that the actual share right was quoted at €0.465 according to market data.

1.4 Risk-Neutral Probability Measures

In order to use absence of arbitrage in the general context of pricing financial derivatives, we will need the notion of \textit{risk-neutral probability measure}.

The next definition says that under a risk-neutral probability measure, the risky assets $n^o 1, 2, \ldots, d$ have same \textit{average} rate of return as the risk-free asset $n^o 0$.

\textbf{Definition 1.3.} A probability measure $\mathbb{P}^*$ on $\Omega$ is called a risk-neutral measure if

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\[ \mathbb{E}^* \left[ S_1^{(i)} \right] = (1 + r)S_0^{(i)}, \quad i = 1, 2, \ldots, d. \]  

Here, \( \mathbb{E}^* \) denotes the expectation under the probability measure \( \mathbb{P}^* \). Note that for \( i = 0 \), the condition \( \mathbb{E}^* \left[ S_1^{(0)} \right] = (1 + r)S_0^{(0)} \) is always satisfied by definition.

In other words, \( \mathbb{P}^* \) is called “risk neutral” because taking risks under \( \mathbb{P}^* \) by buying a stock \( S_1^{(i)} \) has a neutral effect: on average the expected yield of the risky asset equals the risk-free rate obtained by investing on the savings account with interest rate \( r \).

On the other hand, under a “risk premium” probability measure \( \mathbb{P}^\# \), the expected return of the risky asset \( S_1^{(i)} \) would be higher than \( r \), i.e. we would have

\[ \mathbb{E}^\# \left[ S_1^{(i)} \right] > (1 + r)S_0^{(i)}, \quad i = 1, 2, \ldots, d, \]

whereas under a “negative premium” measure \( \mathbb{P}^b \), the expected return of the risky asset \( S_1^{(i)} \) would be lower than \( r \), i.e. we would have

\[ \mathbb{E}^b \left[ S_1^{(i)} \right] < (1 + r)S_0^{(i)}, \quad i = 1, 2, \ldots, d. \]

In the sequel we will only consider probability measures \( \mathbb{P}^* \) that are equivalent to \( \mathbb{P} \) in the sense that they have the same events of zero probability.

**Definition 1.4.** A probability measure \( \mathbb{P}^* \) on \( (\Omega, \mathcal{F}) \) is said to be equivalent to another probability measure \( \mathbb{P} \) when

\[ \mathbb{P}^*(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathcal{F}. \]  

The following Theorem 1.5 can be used to check for the existence of arbitrage opportunities, and is known as the first fundamental theorem of asset pricing.

**Theorem 1.5.** A market is without arbitrage opportunity if and only if it admits at least one (equivalent) risk-neutral probability measure \( \mathbb{P}^* \).

**Proof.** (i) Sufficiency. Assume that there exists a risk-neutral probability measure \( \mathbb{P}^* \) equivalent to \( \mathbb{P} \). Since \( \mathbb{P}^* \) is risk neutral we have

\[ \xi \cdot S_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)} = \frac{1}{1 + r} \sum_{i=0}^{d} \xi^{(i)} \mathbb{E}^* \left[ S_1^{(i)} \right] = \frac{1}{1 + r} \mathbb{E}^* \left[ \xi \cdot S_1 \right] \geq 0, \]  

by Definition 1.2-(ii). We proceed by contradiction, and suppose that the market admits an arbitrage opportunity. In this case, Definition 1.2-(iii) implies \( \mathbb{P}(\xi \cdot S_1 > 0) > 0 \), hence \( \mathbb{P}^*(\xi \cdot S_1 > 0) > 0 \) because \( \mathbb{P} \) is equivalent to \( \mathbb{P}^* \). Since

\[ \mathbb{P}^*(\xi \cdot S_1 > 0) > 0, \]  

the market is not arbitrage free.
N. Privault

\[ 0 < \mathbb{P}^* (\xi \cdot S_1 > 0) \]
\[ = \mathbb{P}^* \left( \bigcup_{n \geq 1} \{ \xi \cdot S_1 > 1/n \} \right) \]
\[ = \lim_{n \to \infty} \mathbb{P}^* (\xi \cdot S_1 > 1/n) \]
\[ = \lim_{\varepsilon \searrow 0} \mathbb{P}^* (\xi \cdot S_1 > \varepsilon), \]

there exists \( \varepsilon > 0 \) such that \( \mathbb{P}^* (\xi \cdot S_1 \geq \varepsilon) > 0 \), hence

\[ \mathbb{E}^* [\xi \cdot S_1] \geq \mathbb{E}^* [\xi \cdot S_1 1_{\{\xi \cdot S_1 \geq \varepsilon\}}] \]
\[ \geq \varepsilon \mathbb{E}^* [1_{\{\xi \cdot S_1 \geq \varepsilon\}}] \]
\[ = \varepsilon \mathbb{P}^* (\xi \cdot S_1 \geq \varepsilon) \]
\[ > 0, \]

and by (1.6) we conclude that

\[ \xi \cdot S_0 = \frac{1}{1 + r} \mathbb{E}^* [\xi \cdot S_1] > 0, \]

which results into a contradiction by Definition 1.2-(i). We conclude that the market is without arbitrage opportunities.

(ii) The proof of necessity relies on the theorem of separation of convex sets by hyperplanes Proposition 1.6 below, see Theorem 1.6 in Föllmer and Schied (2004). It can be briefly sketched as follows. Given two financial assets with net discounted gains \( X, Y \) and a portfolio made of one unit of \( X \) and \( c \) unit(s) of \( Y \), the absence of arbitrage opportunities can be reformulated by saying that for any portfolio choice determined by \( c \in \mathbb{R} \), we have

\[ X + cY \geq 0 \implies X + cY = 0, \quad \mathbb{P} - a.s., \quad (1.7) \]

\( i.e. \) a risk-free (no loss) portfolio cannot entail a strictly positive gain. In other words, if one wishes to make a strictly positive gain on the market, one has to accept the possibility of a loss.

To show that this implies the existence of a risk-neutral probability measure \( \mathbb{P}^* \) under which the risky investments have zero discounted return, \( i.e. \)

\[ \mathbb{E}_{\mathbb{P}^*} [X] = \mathbb{E}_{\mathbb{P}^*} [Y] = 0, \quad (1.8) \]

the convex separation Proposition 1.6 below is applied to the convex set

\[ \mathcal{C} = \{ (\mathbb{E}_Q[X], \mathbb{E}_Q[Y]) : Q \in \mathcal{P} \} \]

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in $\mathbb{R}^2$, where $\mathcal{P}$ is the family of probability measures $Q$ on $\Omega$ equivalent to $\mathbb{P}$. If (1.8) does not hold under any $\mathbb{P}^* \in \mathcal{P}$ then $(0,0) \notin C$ and the convex separation Proposition 1.6 below applied to the convex sets $C$ and $\{(0,0)\}$ shows the existence of $c \in \mathbb{R}$ such that

$$\mathbb{E}_Q[X] + c \mathbb{E}_Q[Y] \geq 0 \text{ for all } Q \in \mathcal{P},$$

and

$$\mathbb{E}_{\mathbb{P}^*}[X] + c \mathbb{E}_{\mathbb{P}^*}[Y] > 0 \text{ for some } \mathbb{P}^* \in \mathcal{P}. \quad (1.10)$$

Condition (1.9) shows that $X + cY \geq 0$ a.s. while Condition (1.10) implies $\mathbb{P}^*(X + cY > 0) > 0$, which contradicts absence of arbitrage by Definition 1.2-(iii).

Next is a version of the separation theorem for convex sets, which relies on the more general Theorem 1.7 below.

**Proposition 1.6.** Let $C$ be a convex set in $\mathbb{R}^2$ such that $(0,0) \notin C$. Then there exists $c \in \mathbb{R}$ such that

$$x + cy \geq 0,$$

for all $(x,y) \in C$, and

$$x + cy > 0,$$

for some $(x,y) \in C$.

**Proof.** Theorem 1.7 below applied to $C_1 := (0,0)$ and $C_2 := C$ shows that

$$0 \leq a \leq x + cy$$

for all $(x,y) \in C$. On the other hand, if $x + cy = 0$ for all $(x,y) \in C$ then the convex $C$ is contained in a line for which there clearly exist $\tilde{c} \in \mathbb{R}$ such that $x + \tilde{c}y \geq 0$ for all $(x,y) \in C$ and $x + \tilde{c}y > 0$ for some $(x,y) \in C$, because $(0,0) \notin C$.

The proof of Proposition 1.6 relies on the following result, see e.g. Theorem 4.14 of Hiriart-Urruty and Lemaréchal (2001).

**Theorem 1.7.** Let $C_1$ and $C_2$ be two disjoint convex sets in $\mathbb{R}^2$. Then there exists $a,c \in \mathbb{R}$ such that

$$x^{(1)} + cy^{(1)} \leq a \quad \text{and} \quad a \leq x^{(2)} + cy^{(2)},$$

for all $(x^{(1)},y^{(1)}) \in C_1$ and $(x^{(2)},y^{(2)}) \in C_2$ (up to exchange of $C_1$ and $C_2$).

### 1.5 Hedging Contingent Claims

In this section we consider the notion of contingent claim. “Contingent” is an adjective that means:
1. subject to chance.

2. occurring or existing only if (certain circumstances) are the case; dependent on.

More generally, we will work according to the following broad definition.

**Definition 1.8.** A contingent claim is any nonnegative random variable $C : \Omega \rightarrow [0, \infty)$.

In practice, the random variable $C$ will represent the payoff of an (option) contract at time $t = 1$.

Referring to Definition 0.2, a European call option with maturity $t = 1$ on the asset $n^o i$ is a contingent claim whose the payoff $C$ is given by

$$ C = (S_1^{(i)} - K)^+ := \begin{cases} S_1^{(i)} - K & \text{if } S_1^{(i)} \geq K, \\ 0 & \text{if } S_1^{(i)} < K, \end{cases} $$

where $K$ is called the strike price. The claim $C$ is called “contingent” because its value may depend on various market conditions, such as $S_1^{(i)} > K$. A contingent claim is also called a financial “derivative” for the same reason.

Similarly, referring to Definition 0.1, a European put option with maturity $t = 1$ on the asset $n^o i$ is a contingent claim with payoff

$$ C = (K - S_1^{(i)})^+ := \begin{cases} K - S_1^{(i)} & \text{if } S_1^{(i)} \leq K, \\ 0 & \text{if } S_1^{(i)} > K, \end{cases} $$

**Definition 1.9.** A contingent claim with payoff $C$ is said to be attainable if there exists a portfolio allocation $\tilde{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)})$ such that...
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\[ C = \bar{\xi} \cdot S_1 = \sum_{i=0}^{d} \xi^{(i)} S^{(i)}_1 = \xi^{(0)} S^{(0)}_1 + \xi^{(1)} S^{(1)}_1 + \cdots + \xi^{(d)} S^{(d)}_1, \]

with \( P \)-probability one.

When a contingent claim \( C \) is attainable, a trader will be able to:

1. at time \( t = 0 \), build a portfolio allocation \( \bar{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}) \in \mathbb{R}^{d+1} \),

2. invest the amount
   \[ \bar{\xi} \cdot \bar{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S^{(i)}_0 \]
   in this portfolio at time \( t = 0 \),

3. at time \( t = 1 \), pay the claim amount \( C \) using the value \( \bar{\xi} \cdot S_1 = \sum_{i=0}^{d} \xi^{(i)} S^{(i)}_1 \)
   of the portfolio.

The above shows that in order to attain the claim, an initial investment \( \bar{\xi} \cdot \bar{S}_0 \)
is needed at time \( t = 0 \). This amount, to be paid by the buyer to the issuer of
the option (the option writer), is also called the \textit{arbitrage price}, or premium,
of the contingent claim \( C \), and is denoted by
\[
\pi_0(C) := \bar{\xi} \cdot \bar{S}_0. \tag{1.11}
\]

The action of allocating a portfolio allocation \( \bar{\xi} \) such that
\[
C = \bar{\xi} \cdot S_1 \tag{1.12}
\]
is called \textit{hedging}, or \textit{replication}, of the contingent claim \( C \).

In case the value \( \bar{\xi} \cdot \bar{S}_1 \) exceeds the amount of the claim, \textit{i.e.} if
\[
\bar{\xi} \cdot \bar{S}_1 \geq C,
\]
we talk about \textit{super-hedging}. In this book we only focus on hedging, \textit{i.e.} on
\textit{replication} of the contingent claim \( C \), and we will not consider super-hedging.

As a simplified illustration of the principle of hedging, one may an buy
oil-related asset in order to hedge oneself against a potential price rise of
gasoline. In this case, any increase in the price of gasoline that would result
in a higher value of the financial derivative \( C \) would be correlated to an
increase in the underlying stock value, so that the equality (1.12) would be maintained.

1.6 Market Completeness

Market completeness is a strong property saying that any contingent claim can be perfectly hedged.

**Definition 1.10.** A market model is said to be complete if every contingent claim \( C \) is attainable.

The next result is the second fundamental theorem of asset pricing.

**Theorem 1.11.** A market model without arbitrage opportunities is complete if and only if it admits only one (equivalent) risk-neutral probability measure \( \mathbb{P}^* \).

*Proof.* See the proof of Theorem 1.40 in Föllmer and Schied (2004). □

Theorem 1.11 will give us a concrete way to verify market completeness by searching for a unique solution \( \mathbb{P}^* \) to Equation (1.4).

1.7 Example: Binary Market

In this section we work out a simple example that allows us to apply Theorem 1.5 and Theorem 1.11. We take \( d = 1 \), i.e. the portfolio is made of

- a risk-free asset with interest rate \( r \) and priced \((1 + r)S_0^{(0)}\) at time \( t = 1 \),
- and a risky asset priced \( S_1^{(1)} \) at time \( t = 1 \).

We use the probability space

\[ \Omega = \{ \omega^-, \omega^+ \}, \]

which is the simplest possible non-trivial choice of probability space, made of only two possible outcomes with

\[ \mathbb{P}(\{\omega^-\}) > 0 \quad \text{and} \quad \mathbb{P}(\{\omega^+\}) > 0, \]

in order for the setting to be non-trivial. In other words the behavior of the market is subject to only two possible outcomes, for example, one is expecting an important binary decision of “yes/no” type, which can lead to two distinct scenarios called \( \omega^- \) and \( \omega^+ \).

In this context, the asset price \( S_1^{(1)} \) is given by a random variable

\[ S_1^{(1)} : \Omega \rightarrow \mathbb{R} \]
whose value depends whether the scenario $\omega^-$, resp. $\omega^+$, occurs.

Precisely, we set

$$S_1^{(1)}(\omega^-) = a, \quad \text{and} \quad S_1^{(1)}(\omega^+) = b,$$

*i.e.* the value of $S_1^{(1)}$ becomes equal $a$ under the scenario $\omega^-$, and equal to $b$ under the scenario $\omega^+$, where $0 < a < b$.

### Arbitrage

The first natural question is:

- Arbitrage: Does the market allow for arbitrage opportunities?

We will answer this question using Theorem 1.5, by searching for a risk-neutral probability measure $P^*$ satisfying the relation

$$\mathbb{E}^*[S_1^{(1)}] = (1 + r)S_0^{(1)}, \quad (1.13)$$

where $r > 0$ denotes the risk-free interest rate, cf. Definition 1.3.

In this simple framework, any measure $P^*$ on $\Omega = \{\omega^-, \omega^+\}$ is characterized by the data of two numbers $P^*(\{\omega^-\}) \in [0, 1]$ and $P^*(\{\omega^+\}) \in [0, 1]$, such that

$$P^*(\Omega) = P^*(\{\omega^-\}) + P^*(\{\omega^+\}) = 1. \quad (1.14)$$

Here, saying that $P^*$ is *equivalent* to $P$ simply means that

$$P^*(\{\omega^-\}) > 0 \quad \text{and} \quad P^*(\{\omega^+\}) > 0.$$  

Although we should solve (1.13) for $P^*$, at this stage it is not yet clear how $P^*$ is involved in the equation. In order to make (1.13) more explicit we write the expected value as

$$\mathbb{E}^*[S_1^{(1)}] = aP^*(\{\omega^-\}) + bP^*(\{\omega^+\}) = (1 + r)S_0^{(1)},$$

hence Condition (1.13) for the existence of a risk-neutral probability measure $P^*$ reads

$$aP^*(\{\omega^-\}) + bP^*(\{\omega^+\}) = (1 + r)S_0^{(1)}. \quad (1.15)$$

Using the Condition (1.14) we obtain the system of two equations

$$\begin{cases}
aP^*(\{\omega^-\}) + bP^*(\{\omega^+\}) = (1 + r)S_0^{(1)} \\
P^*(\{\omega^-\}) + P^*(\{\omega^+\}) = 1,
\end{cases}$$

* The case $a = b$ leads to a trivial, constant market.
with unique risk-neutral solution

\[ p^* := \mathbb{P}^*(\{\omega^+\}) = \frac{(1 + r)S_0^{(1)} - a}{b - a} \quad \text{and} \quad q^* := \mathbb{P}^*(\{\omega^-\}) = \frac{b - (1 + r)S_0^{(1)}}{b - a}. \] (1.16)

In order for a solution \( \mathbb{P}^* \) to exist as a probability measure, the numbers \( \mathbb{P}^*(\{\omega^-\}) \) and \( \mathbb{P}^*(\{\omega^+\}) \) must be nonnegative. In addition, for \( \mathbb{P}^* \) to be equivalent to \( \mathbb{P} \) they should be strictly positive from (1.5).

We deduce that if \( a, b \) and \( r \) satisfy the condition

\[ a < (1 + r)S_0^{(1)} < b, \] (1.17)

then there exists a risk-neutral (equivalent) probability measure \( \mathbb{P}^* \) which is unique, hence by Theorems 1.5 and 1.11 the market is without arbitrage and complete.

**Remark 1.12.**

i) If \( a = (1 + r)S_0^{(1)} \), resp. \( b = (1 + r)S_0^{(1)} \), then \( \mathbb{P}^*(\{\omega^+\}) = 0 \), resp. \( \mathbb{P}^*(\{\omega^-\}) = 0 \), and \( \mathbb{P}^* \) is not equivalent to \( \mathbb{P} \).

ii) If \( a = b = (1 + r)S_0^{(1)} \) then (1.4) admits an infinity of solutions, hence the market is without arbitrage but it is not complete. More precisely, in this case both the risk-free and risky assets yield a deterministic return rate \( r \) and the portfolio value becomes

\[ \xi \cdot S_1 = (1 + r)\xi \cdot S_0, \]

at time \( t = 1 \), hence the terminal value \( \xi \cdot S_1 \) is deterministic and this single value can not always match the value of a random contingent claim \( C \) that would be allowed to take two distinct values \( C(\omega^-) \) and \( C(\omega^+) \). Therefore, market completeness does not hold when \( a = b = (1 + r)S_0^{(1)} \).

iii) On the other hand, we check from (1.16) that under the conditions

\[ a < b \leq (1 + r)S_0^{(1)} \quad \text{or} \quad (1 + r)S_0^{(1)} \leq a < b, \] (1.18)

no (equivalent) risk-neutral probability measure exists, and as a consequence there exist arbitrage opportunities in the market.

Let us give a financial interpretation of Condition (1.18).

1. If \( (1 + r)S_0^{(1)} \leq a < b \), let \( \xi^{(1)} := 1 \) and choose \( \xi^{(0)} \) such that

\[ \xi^{(0)}S_0^{(0)} + \xi^{(1)}S_0^{(1)} = 0. \]
According to Definition 1.2-(i), i.e.
\[
\xi^{(0)} = -\xi^{(1)} S_0^{(1)}/S_0^{(0)} < 0.
\]

In particular, Condition (i) of Definition 1.2 is satisfied, and the investor borrows the amount \(-\xi^{(0)} S_0^{(0)} > 0\) on the risk-free asset and uses it to buy one unit \(\xi^{(1)} = 1\) of the risky asset. At time \(t = 1\) she sells the risky asset \(S_1^{(1)}\) at a price at least equal to \(a\) and refunds the amount \(-(1 + r)\xi^{(0)} S_0^{(0)} > 0\) she borrowed, with interests. Her profit is
\[
\xi \cdot S_1 = (1 + r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_1^{(1)} \\
\geq (1 + r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} a \\
= -(1 + r)\xi^{(1)} S_0^{(1)} + \xi^{(1)} a \\
= \xi^{(1)} (- (1 + r) S_0^{(1)} + a) \\
\geq 0,
\]
which satisfies Condition (ii) of Definition 1.2. In addition, Condition (iii) of Definition 1.2 is also satisfied because
\[
P(\xi \cdot S_1 > 0) = P(S_1^{(1)} = b) = P(\{\omega^+\}) > 0.
\]

2. If \(a < b \leq (1 + r)S_0^{(1)}\), let \(\xi^{(0)} > 0\) and choose \(\xi^{(1)}\) such that
\[
\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} = 0,
\]
according to Definition 1.2-(i), i.e.
\[
\xi^{(1)} = -\xi^{(0)} S_0^{(0)}/S_0^{(1)} < 0.
\]
This means that the investor borrows a (possibly fractional) quantity $-\xi^{(1)} > 0$ of the risky asset, sells it for the amount $-\xi^{(1)}S^{(1)}_0$, and invests this money on the risk-free account for the amount $\xi^{(0)}S^{(0)}_0 > 0$. As mentioned in Section 1.2, in this case one says that the investor short sells the risky asset. At time $t = 1$ she obtains $(1 + r)\xi^{(0)}S^{(0)}_0 > 0$ from the risk-free asset and she spends at most $b$ to buy the risky asset and return it to its original owner. Her profit is

$$\bar{\xi} \cdot \bar{S}_1 = (1 + r)\xi^{(0)}S^{(0)}_0 + \xi^{(1)}S^{(1)}_1$$

$$\geq (1 + r)\xi^{(0)}S^{(0)}_0 + \xi^{(1)}b$$

$$= -(1 + r)\xi^{(1)}S^{(1)}_0 + \xi^{(1)}b$$

$$= \xi^{(1)}(- (1 + r)S^{(1)}_0 + b)$$

$$\geq 0,$$

since $\xi^{(1)} < 0$. Note that here, $a \leq S^{(1)}_1 \leq b$ became

$$\xi^{(1)}b \leq \xi^{(1)}S^{(1)}_1 \leq \xi^{(1)}a$$

because $\xi^{(1)} < 0$. We can check as in Part 1 above that Conditions (i)-(iii) of Definition 1.2 are satisfied.

3. Finally if $a = b \neq (1 + r)S^{(1)}_0$ then (1.4) admits no solution as a probability measure $\mathbb{P}^*$ hence arbitrage opportunities exist and can be constructed by the same method as above.

Under Condition (1.17) there is absence of arbitrage and Theorem 1.5 shows that no portfolio strategy can yield both $\bar{\xi} \cdot \bar{S}_1 \geq 0$ and $\mathbb{P}(\bar{\xi} \cdot \bar{S}_1 > 0) > 0$ starting from $\xi^{(0)}S^{(0)}_0 + \xi^{(1)}S^{(1)}_0 \leq 0$, however this is less simple to show directly.

**Market completeness**

The second natural question is:

- Completeness: Is the market complete, *i.e.*, are all contingent claims attainable?

In the sequel we work under the condition

$$a < (1 + r)S^{(1)}_0 < b,$$

under which Theorems 1.5 and 1.11 show that the market is without arbitrage and complete since the risk-neutral probability measure $\mathbb{P}^*$ exists and
is unique.

Let us recover this fact by elementary calculations. For any contingent claim $C$ we need to show that there exists a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)})$ such that $C = \bar{\xi} \cdot S_1$, i.e.

\[
\begin{align*}
\xi^{(0)} (1 + r) S_0^{(0)} + \xi^{(1)} b &= C(\omega^+) \\
\xi^{(0)} (1 + r) S_0^{(0)} + \xi^{(1)} a &= C(\omega^-).
\end{align*}
\] (1.19)

These equations can be solved as

\[
\xi^{(0)} = \frac{b C(\omega^-) - a C(\omega^+)}{S_0^{(0)} (1 + r) (b - a)} \quad \text{and} \quad \xi^{(1)} = \frac{C(\omega^+) - C(\omega^-)}{b - a}.
\] (1.20)

In this case we say that the portfolio allocation $(\xi^{(0)}, \xi^{(1)})$ *hedges* the contingent claim $C$. In other words, any contingent claim $C$ is attainable and the market is indeed complete. Here, the quantity

\[
\xi^{(0)} S_0^{(0)} = \frac{b C(\omega^-) - a C(\omega^+)}{(1 + r) (b - a)}
\]

represents the amount invested on the risk-free asset.

Note that if $C(\omega^+) \geq C(\omega^-)$ then $\xi^{(1)} \geq 0$ and there is not short selling. This occurs in particular if $C$ has the form $C = h(S_1^{(1)})$ with $x \mapsto h(x)$ a nondecreasing function, since

\[
\xi^{(1)} = \frac{C(\omega^+) - C(\omega^-)}{b - a} \\
= \frac{h(S_1^{(1)}(\omega^+)) - h(S_1^{(1)}(\omega^-))}{b - a} \\
= \frac{h(b) - h(a)}{b - a} \\
\geq 0,
\]

thus there is no short selling. This applies in particular to European call options with strike price $K$, for which the function $h(x) = (x - K)^+$ is nondecreasing. Similarly we will find that $\xi^{(1)} \leq 0$, i.e. short selling always occurs when $h$ is a nonincreasing function, which is the case in particular for European put options with payoff function $h(x) = (K - x)^+$. 

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http://www.ntu.edu.sg/home/nprivault/index.html
Arbitrage price

The arbitrage price $\pi_0(C)$ of the contingent claim $C$ is defined in (1.11) as the initial value at time $t = 0$ of the portfolio hedging $C$, i.e.

$$\pi_0(C) = \xi \cdot S_0,$$

(1.21)

where $(\xi(0), \xi(1))$ are given by (1.20). Arbitrage prices can be used to ensure that financial derivatives are “marked” at their fair value (mark to market).* Note that $\pi_0(C)$ cannot be 0 since this would entail the existence of an arbitrage opportunity according to Definition 1.2.

The next proposition shows that the arbitrage price $\pi_0(C)$ of the claim can be computed as the expected value of its payoff $C$ under the risk-neutral probability measure, after discounting at the rate $1 + r$ for the time value of money.

**Proposition 1.13.** The arbitrage price $\pi_0(C)$ of the contingent claim $C$ is given by

$$\pi_0(C) = \frac{1}{1 + r} \mathbb{E}^*[C].$$

(1.22)

**Proof.** Using the expressions (1.16) of the risk-neutral probabilities $\mathbb{P}^*(\{\omega^-(\cdot)\})$, $\mathbb{P}^*(\{\omega^+(\cdot)\})$, and (1.20) of the portfolio allocation $(\xi(0), \xi(1))$, we have

$$\pi_0(C) = \xi \cdot S_0$$

$$= \xi(0)S_0(0) + \xi(1)S_0(1)$$

$$= bC(\omega^-) - aC(\omega^+) + S_0(1)C(\omega^+) - C(\omega^-)$$

$$= \frac{1}{1 + r} \left( C(\omega^-) \frac{b - S_0(1)}{b - a} + C(\omega^+) \frac{(1 + r)S_0(1) - a}{b - a} \right)$$

$$= \frac{1}{1 + r} \left( C(\omega^-) \mathbb{P}^*(S_0(1) = a) + C(\omega^+) \mathbb{P}^*(S_0(1) = b) \right)$$

$$= \frac{1}{1 + r} \mathbb{E}^*[C].$$

□

In the case of a European call options with strike price $K \in [a, b]$ we have $C = (S_1^1 - K)^+$ and

$$\pi_0((S_1^1 - K)^+) = \frac{1}{1 + r} \mathbb{E}^* [(S_1^1 - K)^+]$$

* Not to be confused with “market making”.

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In the case of a European put options we have $C = (K - S_1^{(1)})^+$ and

$$
\pi_0((K - S_1^{(1)})^+) = \frac{1}{1 + r} \mathbb{E}^*[\pi_0((K - S_1^{(1)})^+)]
= \frac{1}{1 + r} (K - a) \mathbb{P}^*(S_1^{(1)} = a)
= \frac{1}{1 + r} (K - a) \frac{b - (1 + r) S_0^{(1)}}{b - a}
= \frac{K - a}{b - a} \left( \frac{b}{1 + r} - S_0^{(1)} \right).
$$

Here, $(S_0^{(1)} - K)^+$, resp. $(K - S_0^{(1)})^+$ is called the intrinsic value at time 0 of the call, resp. put option.

The simple setting described in this chapter raises several questions and remarks.

**Remarks**

1. The fact that $\pi_0(C)$ can be obtained by two different methods, i.e. an algebraic method via (1.20) and (1.21) and a probabilistic method from (1.22) is not a simple coincidence. It is actually a simple example of the deep connection that exists between probability and analysis.

In a continuous-time setting, (1.20) will be replaced with a partial differential equation (PDE) and (1.22) will be computed via the Monte Carlo method. In practice, the quantitative analysis departments of major financial institutions can be split into a “PDE team” and a “Monte Carlo team”, often trying to determine the same option prices by two different methods.

2. What if we have three possible scenarios, i.e. $\Omega = \{\omega^-, \omega^0, \omega^+\}$ and the random asset $S_1^{(1)}$ is allowed to take more than two values, e.g. $S_1^{(1)} \in \{a, b, c\}$ according to each scenario? In this case the system (1.15) would be rewritten as
\[
\left\{
\begin{array}{l}
aP^*({\omega^-}) + bP^*({\omega'}) + cP^*({\omega^+}) = (1 + r)S_0^{(1)} \\
P^*({\omega^-}) + P^*({\omega'}) + P^*({\omega^+}) = 1,
\end{array}
\right.
\]

and this system of two equations with three unknowns does not admit a unique solution, hence the market can be without arbitrage but it cannot be complete, cf. Exercise 1.4.

Market completeness can be reached by adding a second risky asset, *i.e.* taking \(d = 2\), in which case we will get three equations and three unknowns. More generally, when \(\Omega\) has \(n \geq 2\) elements, completeness of the market can be reached provided that we consider \(d\) risky assets with \(d + 1 \geq n\). This is related to the Meta-Theorem 8.3.1 of Björk (2004) in which the number \(d\) of traded underlying risky assets is linked to the number of random sources through arbitrage and market completeness.

**Exercises**

Exercise 1.1 Consider a risky asset valued \(S_0 = 3\) at time \(t = 0\) and taking only two possible values \(S_1 \in \{1, 5\}\) at time \(t = 1\), and a financial claim given at time \(t = 1\) by

\[
C := \begin{cases} 
0 & \text{if } S_1 = 5 \\
2 & \text{if } S_1 = 1.
\end{cases}
\]

Is \(C\) the payoff of a call option or of a put option? Give the strike price of the option.

Exercise 1.2 Consider a risky asset valued \(S_0 = 4\) at time \(t = 0\), and taking only two possible values \(S_1 \in \{2, 5\}\) at time \(t = 1\). Compute the initial value \(V_0 = \alpha S_0 + \beta\) of the portfolio hedging the claim

\[
C := \begin{cases} 
0 & \text{if } S_1 = 5 \\
6 & \text{if } S_1 = 2
\end{cases}
\]

at time \(t = 1\), and find the corresponding risk-neutral probability measure \(P^*\).

Exercise 1.3

a) Consider the following market model:
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\[ b \]
\[ a \]
\[ S_0^{(1)} \]
\[ (1 + r)S_0^{(1)} \]

i) Does this model allow for arbitrage? \[ \text{Yes} | \text{No} \]

ii) If this model allows for arbitrage opportunities, how can they be realized?
- By shortselling
- By borrowing in savings
- N.A.

b) Consider the following market model:

\[ b \]
\[ a \]
\[ S_0^{(1)} \]
\[ (1 + r)S_0^{(1)} \]

i) Does this model allow for arbitrage? \[ \text{Yes} | \text{No} \]

ii) If this model allows for arbitrage opportunities, how can they be realized?
- By shortselling
- By borrowing in savings
- N.A.

c) Consider the following market model:

\[ b \]
\[ a \]
\[ S_0^{(1)} \]
\[ (1 + r)S_0^{(1)} \]

i) Does this model allow for arbitrage? \[ \text{Yes} | \text{No} \]

ii) If this model allows for arbitrage opportunities, how can they be realized?
- By shortselling
- By borrowing in savings
- N.A.

Exercise 1.4 In a market model with two time instants \( t = 0 \) and \( t = 1 \), consider
- a risk-free asset with price \( S_0^{(0)} \) at time \( t = 0 \), and value \( S_1^{(0)} = (1 + r)S_0^{(0)} \) at time \( t = 1 \).

- a risky asset with price \( S_0^{(1)} \) at time \( t = 0 \), and three possible values at time \( t = 1 \), with \( a < b < c \), i.e.:

\[
S_1^{(1)} = \begin{cases} 
S_0^{(1)}(1 + a), \\
S_0^{(1)}(1 + b), \\
S_0^{(1)}(1 + c).
\end{cases}
\]

a) Show that this market is without arbitrage but not complete.

b) In general, is it possible to hedge (or replicate) a claim with three distinct payoff values \( P_a, P_b, P_c \) in this market?

Exercise 1.5 We consider a risk-free asset priced as \( S_1^{(0)} = S_0^{(0)} \), at times \( k = 0, 1 \), where the risk-free rate is \( r = 0 \), and a risky asset \( S(1) \) whose return \( R_1 := (S_1^{(1)} - S_0^{(1)})/S_0^{(1)} \) can take three values \( \{-b < 0 < b\} \) at each time step, with

\[
p^* := \mathbb{P}^*(R_1 = b) > 0, \quad \theta^* := \mathbb{P}^*(R_1 = 0) > 0, \quad q^* := \mathbb{P}^*(R_1 = -b) > 0,
\]

a) Determine all possible risk-neutral probability measures \( \mathbb{P}^* \) equivalent to \( \mathbb{P} \) in terms of the parameter \( \theta^* \in (0, 1) \) from the condition \( \mathbb{E}^*[R_1] = 0 \).

b) We assume that the variance \( \text{Var}^* \left[ \frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} \right] = \sigma^2 > 0 \) of the asset return is known to be equal to \( \sigma^2 \). Show that this condition determines a unique risk-neutral probability measure \( \mathbb{P}_\sigma^* \) under a certain condition on \( b \) and \( \sigma \).

Exercise 1.6 Consider a market model with two time instants \( t = 0 \) and \( t = 1 \) and two assets:

- a risk-free asset \( \pi \) with price \( \pi_0 \) at time \( t = 0 \) and value \( \pi_1 = \pi_0(1 + r) \) at time \( t = 1 \),
- a risky asset \( S \) with price \( S_0 \) at time \( t = 0 \) and random value \( S_1 \) at time \( t = 1 \).

We assume that \( S_1 \) can take only the values \( S_0(1 + a) \) and \( S_0(1 + b) \), where \( -1 < a < r < b \). The return of the risky asset is defined as
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\[ R = \frac{S_1 - S_0}{S_0}. \]

a) What are the possible values of \( R \)?

b) Show that under the probability measure \( \mathbb{P}^* \) defined by

\[ \mathbb{P}^*(R = a) = \frac{b - r}{b - a}, \quad \mathbb{P}^*(R = b) = \frac{r - a}{b - a}, \]

the expected return \( \mathbb{E}^*[R] \) of \( S \) is equal to the return \( r \) of the risk-free asset.

c) Does there exist arbitrage opportunities in this model? Explain why.

d) Is this market model complete? Explain why.

e) Consider a contingent claim with payoff \( C \) given by

\[ C = \begin{cases} \alpha & \text{if } R = a, \\ \beta & \text{if } R = b. \end{cases} \]

Show that the portfolio allocation \( (\eta, \xi) \) defined* by

\[ \eta = \alpha \frac{(1 + b) - \beta(1 + a)}{\pi_0(1 + r)(b - a)} \quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0(b - a)}, \]

hedges the contingent claim \( C \), i.e. show that at time \( t = 1 \) we have

\[ \eta \pi_1 + \xi S_1 = C. \]

* Hint: distinguish two cases \( R = a \) and \( R = b \).

f) Compute the arbitrage price \( \pi_0(C) \) of the contingent claim \( C \) using \( \eta, \pi_0, \xi, \) and \( S_0 \).

g) Compute \( \mathbb{E}^*[C] \) in terms of \( a, b, r, \alpha, \beta \).

h) Show that the arbitrage price \( \pi_0(C) \) of the contingent claim \( C \) satisfies

\[ \pi_0(C) = \frac{1}{1 + r} \mathbb{E}^*[C]. \] (1.23)

i) What is the interpretation of Relation (1.23) above?

j) Let \( C \) denote the payoff at time \( t = 1 \) of a put option with strike price \( K=\$11 \) on the risky asset. Give the expression of \( C \) as a function of \( S_1 \) and \( K \).

k) Letting \( \pi_0 = S_0 = 1 \), \( r = 5\% \) and \( a = 8 \), \( b = 11 \), \( \alpha = 2 \), \( \beta = 0 \), compute the portfolio allocation \( (\xi, \eta) \) hedging the contingent claim \( C \).

l) Compute the arbitrage price \( \pi_0(C) \) of the claim \( C \).

* Here, \( \eta \) is the (possibly fractional) quantity of asset \( \pi \) and \( \xi \) is the quantity held of asset \( S \).
Exercise 1.7 Consider a stock valued $S_0 = $180 at the beginning of the year. At the end of the year, its value $S_1$ can be either $152$ or $203$ and the risk-free interest rate is $r = 3\%$ per year. Given a put option with strike price $K$ on this underlying, find the value of $K$ for which the price of the option at the beginning of the year is equal to the intrinsic option payoff. This value of $K$ is called the break-even strike price. In other words, the break-even price is the value of $K$ for which immediate exercise of the option is equivalent to holding the option until maturity.

What effect would a decrease in the interest rate $r$ would have on this break-even strike price?