Chapter 10
Asian Options

Asian options are particular cases of options on average, and they were first traded in Tokyo in 1987. Given an underlying asset $S_t$ with exercise date $T$ and strike price $K$, the payoff of the Asian call option is given by

$$C := \left( \frac{1}{T} \int_0^T S_t \, dt - K \right)^+$$

whereas the payoff of the Asian put option is

$$C := \left( K - \frac{1}{T} \int_0^T S_t \, dt \right)^+.$$

This chapter covers several probabilistic and PDE techniques for the pricing and hedging of Asian options. Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal value of the underlying asset.

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10.1 Options on Averages

In this case we can take

$$C = \phi \left( \frac{1}{T} \int_0^T S_t \, dt \right)$$
where
\[
\frac{1}{T} \int_0^T S_t dt
\]
represents the average of \((S_t)_{t \in \mathbb{R}^+}\) over the time interval \([0, T]\) and \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) is a payoff function.

Figure 10.1: Brownian motion \(B_t\) and its moving average \(X_t\).

Figure 10.1 shows a graph of Brownian motion and its moving average process
\[
X_t := \frac{1}{t} \int_0^t B_s ds, \quad t > 0.
\]

**Arithmetic Asian options**

The payoff of the Asian call option on the underlying asset \(S_t\) with exercise date \(T\) and strike price \(K\) is given by
\[
C = \left( \frac{1}{T} \int_0^T S_t dt - K \right)^+.
\]

Similarly, the payoff of the Asian put option on the underlying asset \(S_t\) with exercise date \(T\) and strike price \(K\) is
\[
C = \left( K - \frac{1}{T} \int_0^T S_t dt \right)^+.
\]

Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal

* The animation works in Acrobat Reader on the entire pdf file.
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value of the underlying asset. Asian options have become particularly popular in commodities trading.

Other types of exotic options include called Asian-American options, or Hawaiian options, that combine the Asian option payoff with American style exercise, and can be priced by variational PDEs, cf. §8.6.3.2 of Crépey (2013).

An option on average is an option whose payoff has the form

\[ C = \phi(\Lambda_T, S_T), \]

where

\[ \Lambda_T = S_0 \int_0^T e^{\sigma B_u + r u - \sigma^2 u/2} du = \int_0^T S_u du, \quad T \in \mathbb{R}_+. \]

- For example when \( \phi(y, x) = (y/T - K)^+ \) this yields the Asian call option with payoff

\[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ = \left( \frac{\Lambda_T}{T} - K \right)^+, \quad (10.1) \]

which is a path-dependent option whose price at time \( t \in [0, T] \) is given by

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]. \quad (10.2) \]

- As another example, when \( \phi(y, x) := e^{-y} \) this yields the price

\[ P(0, T) = \mathbb{E}^* \left[ e^{-\int_0^T S_u du} \right] = \mathbb{E}^* \left[ e^{-\Lambda_T} \right] \]

at time 0 of a bond with underlying short term rate process \( S_t \).

The option with payoff \( C = \phi(\Lambda_T, S_T) \) can be priced as

\[ e^{-(T-t)r} \mathbb{E}^*[\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*[\phi(\Lambda_t + \int_t^T S_u du, S_T) \mid \mathcal{F}_t] \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_t^T \frac{S_u}{S_t} du, x \frac{S_T}{S_t} \right) \right]_{y=\Lambda_t, x=S_t} \]

\[ = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right]_{y=\Lambda_t, x=S_t}. \quad (10.3) \]

Using the Markov property of the process \((S_t, \Lambda_t)_{t \in \mathbb{R}_+}\), we can write down the option price as a function

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^*[\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*[\phi(\Lambda_T, S_T) \mid S_t, \Lambda_t], \]

\( \diamond \)
of \((t, S_t, \Lambda_t)\), where the function \(f(t, x, y)\) is given by
\[
 f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right].
\]

As we will see below there exists no easily tractable closed-form solution for the price of an arithmetically averaged Asian option.

**Geometric Asian options**

On the other hand, replacing the arithmetic average
\[
\frac{1}{T} \sum_{k=1}^{n} S_{t_k}(t_k - t_{k-1}) \simeq \frac{1}{T} \int_0^{T} S_u du
\]
with the geometric average
\[
\prod_{k=1}^{n} S_{t_k}^{(t_k - t_{k-1})/T} = \exp \left( \log \prod_{k=1}^{n} S_{t_k}^{(t_k - t_{k-1})/T} \right)
\]
\[
= \exp \left( \frac{1}{T} \sum_{k=1}^{n} \log S_{t_k}^{(t_k - t_{k-1})} \right)
\]
\[
= \exp \left( \frac{1}{T} \sum_{k=1}^{n} (t_k - t_{k-1}) \log S_{t_k} \right)
\]
\[
\simeq \exp \left( \frac{1}{T} \int_0^{T} \log S_u du \right)
\]
leads to closed-form solutions using the Black Scholes formula, cf. Exercise 10.3.

**Pricing using probability density functions**

We note that the prices of option on averages can be estimated numerically using the joint probability density function \(\psi_{\Lambda_{T-t}, B_{T-t}}\) of \((\Lambda_{T-t}, B_{T-t})\), as follows:
\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right] e^{-(T-t)r} \int_0^{\infty} \int_{-\infty}^{\infty} \phi \left( y + xz, xe^{\sigma u+(T-t)r} - (T-t)\sigma^2/2 \right) \psi_{\Lambda_{T-t}, B_{T-t}}(z, u) dz du.
\]

In Yor (1992), Proposition 2, the joint probability density function of
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\[(A_t, B_t) = \left( \int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds, B_t - p\sigma t/2 \right), \quad t > 0,\]

has been computed in the case \(\sigma = 2\), cf. also Matsumoto and Yor (2005). In the next proposition we restate this result for an arbitrary variance parameter \(\sigma\) after rescaling. Let \(\theta(v, \tau)\) denote the function defined as

\[
\theta(v, \tau) = \frac{v e^{x^2/(2\tau)}}{\sqrt{2\pi^3 \tau}} \int_0^\infty e^{-x^2/(2\tau)} e^{-v \cosh x \sinh (\pi x/\tau)} d\xi, \quad v, \tau > 0.
\]

(10.4)

**Proposition 10.1.** For all \(t > 0\) we have

\[
P\left( \int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds \in dy, B_t - p\sigma t/2 \in dz \right)
= \frac{\sigma}{2} e^{-p\sigma z/2 - p^2\sigma^2 t/8} \exp \left( -2 \frac{1 + e^{\sigma z}}{\sigma^2 y} \right) \theta \left( \frac{4e^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 t}{4} \right) dy \frac{dy}{dz},
\]

\(y > 0, z \in \mathbb{R}.

The expression of this probability density function can then be used for the pricing of options on average such as (10.3), as

\[
f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \phi \left( y + x \int_0^{T-t} S_u d\nu, x \frac{S_{T-t}}{S_0} \right) \right]
= e^{-(T-t)r} \times \int_0^\infty \phi \left( y + xz, x e^{\sigma u + (T-t)r - (T-t)\sigma^2/2} \right) \mathbb{P} \left( \int_0^{T-t} S_u \frac{d\nu}{S_0} \in dz, B_{T-t} \in du \right)
= \frac{\sigma}{2} e^{-(T-t)r + (T-t)p^2\sigma^2/8} \int_0^\infty \phi \left( y + xz, x e^{\sigma u + (T-t)r - (T-t)\sigma^2(1+p)/2} \right)
\times \exp \left( -2 \frac{1 + e^{\sigma u - (T-t)p\sigma^2/2}}{\sigma^2 z} - \frac{p\sigma u}{2} \right) \theta \left( \frac{4e^{\sigma u - (T-t)p\sigma^2/4}}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4} \right) \frac{dz}{z}
\times \frac{1}{v} \exp \left( -2z^2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4vz^2}{\sigma^2}, \frac{(T-t)\sigma^2}{4} \right) \frac{dz}{z},
\]

which actually stands as a triple integral due to the definition (10.4) of \(\theta(v, \tau)\). Note that here the order of integration between \(du\) and \(dz\) cannot be exchanged without particular precautions, at the risk of wrong computations.
10.2 The Asian Call Option

By repeating the argument of (10.3) for $\varphi(x, y) := (x - K)^+$, the Asian call option can be priced as

$$e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( \Lambda_t + \int_t^T S_u du \right) - K \right)^+ \bigg| \mathcal{F}_t \right]$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \int_0^T S_u du \right) - K \right)^+ \bigg| \mathcal{F}_t \right]$$

$$= e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \int_0^{T-t} S_u \frac{S}{S_0} du \right) - K \right)^+ \bigg|_{x=S_t, y=\Lambda_t} \right]$$

Hence the Asian option can be priced as

$$f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right],$$

where the function $f(t, x, y)$ is given by

$$f(t, x, y) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \left( y + x \int_0^{T-t} S_u \frac{S}{S_0} du \right) - K \right)^+ \bigg|_{x=S_t, y=\Lambda_t} \right], \quad x, y > 0. \quad (10.5)$$

### Bounds on Asian option prices

We note (see Lemma 1 of Kemna and Vorst (1990) and Exercise 10.5 below for the discrete-time version of that result), that the Asian call option price can be upper bounded by the corresponding European call price using convexity arguments.

**Proposition 10.2.** Assume that $r \geq 0$. We have the bound

$$e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] \leq e^{-rT} \mathbb{E}^* \left[ (S_T - K)^+ \right].$$

**Proof.** By Jensen’s inequality for the uniform measure with density $(1/T) \mathbb{I}_{[0,T]}$ on $[0,T]$ and for the probability measure $\mathbb{P}^*$, we have
As an Asian Options contract, the expected payoff is given by:

\[ e^{-rT} E^* \left[ \left( \int_0^T S_u \frac{du}{T} - K \right)^+ \right] = e^{-rT} E^* \left[ \left( \frac{1}{T} \int_0^T (S_u - K) \frac{du}{T} \right)^+ \right] \]

\[ \leq e^{-rT} E^* \left[ \int_0^T (S_u - K)^+ \frac{du}{T} \right] \]

\[ = e^{-rT} E^* \left[ \int_0^T e^{-(T-u)r} E^*[S_T | \mathcal{F}_u] - K)^+ \frac{du}{T} \right] \]

\[ \leq e^{-rT} \int_0^T E^* \left[ (e^{-(T-u)r} S_T - K)^+ \mathcal{F}_u \right] \frac{du}{T} \quad (10.6) \]

\[ = e^{-rT} \int_0^T E^* \left[ (S_T - K)^+ \right] \frac{du}{T} \quad (10.7) \]

\[ = e^{-rT} E^* \left[ (S_T - K)^+ \right], \]

where from (10.6) to (10.7) we used the fact that \( r \geq 0 \).

More generally, given that

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} E^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right], \]

where, from Proposition 10.2,

\[ f(t, x, y) = e^{-(T-t)r} E^* \left[ \left( \frac{1}{T} \left( y + x \int_0^{T-t} S_u \frac{du}{S_0} \right) - K \right)^+ \right] \]

\[ = e^{-(T-t)r} E^* \left[ \left( \frac{1}{T} \left( y + \frac{x}{S_0} \Lambda_{T-t} \right) - K \right)^+ \right] \]

\[ = e^{-(T-t)r} E^* \left[ \left( \frac{y}{T} - K + \frac{x}{TS_0} \Lambda_{T-t} \right)^+ \right] \]

\[ = \frac{(T-t)x}{TS_0} e^{-(T-t)r} E^* \left[ \left( \frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + \frac{\Lambda_{T-t}}{T-t} \right)^+ \right] \]

\[ \leq \frac{(T-t)x}{TS_0} e^{-(T-t)r} E^* \left[ \left( \frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + S_{T-t} \right)^+ \right] \]

we find the bound

\[ \leq e^{-(T-t)r} E^* \left[ \left( \frac{y}{T} - K + \frac{(T-t)xS_{T-t}}{TS_0} \right)^+ \right], \quad x, y > 0, \]
at time $t \in [0, T]$. See also Proposition 3.2-(ii) of Geman and Yor (1993) for lower bounds when $r$ takes negative values. We also have the following bound which yields the behavior of Asian option prices in large time.

**Proposition 10.3.** The Asian option price satisfies the bound

$$
eq (T-t)^r \mathbb{E}^{*}\left[\left(\frac{1}{T} \int_0^T S_u du - K\right)^+ \mid \mathcal{F}_t\right] \leq \frac{e^{- (T-t)^r}}{T} \int_0^t S_u du + S_t \frac{1 - e^{- (T-t)^r}}{rT},$$

$t \in [0, T]$, and tends to zero (almost surely) as time to maturity $T$ tends to infinity:

$$
\lim_{T \to \infty} \left( e^{- (T-t)^r} \mathbb{E}^{*}\left[\left(\frac{1}{T} \int_0^T S_u du - K\right)^+ \mid \mathcal{F}_t\right]\right) = 0, \quad t \in \mathbb{R}_+.
$$

**Proof.** We have the bound

$$0 \leq e^{- (T-t)^r} \mathbb{E}^{*}\left[\left(\frac{1}{T} \int_0^T S_u du - K\right)^+ \mid \mathcal{F}_t\right]$$

$$\leq e^{- (T-t)^r} \mathbb{E}^{*}\left[\frac{1}{T} \int_0^T S_u du \mid \mathcal{F}_t\right]$$

$$= e^{- (T-t)^r} \mathbb{E}^{*}\left[\frac{1}{T} \int_t^T S_u du \mid \mathcal{F}_t\right] + e^{- (T-t)^r} \mathbb{E}^{*}\left[\frac{1}{T} \int_t^T S_u du \mid \mathcal{F}_t\right]$$

$$= e^{- (T-t)^r} \frac{1}{T} \int_0^t \mathbb{E}^{*}[S_u \mid \mathcal{F}_t] du + \frac{1}{T} e^{- (T-t)^r} \int_t^T \mathbb{E}^{*}[S_u \mid \mathcal{F}_t] du$$

$$= e^{- (T-t)^r} \frac{1}{T} \int_0^t S_u du + \frac{1}{T} e^{- (T-t)^r} \int_t^T e^{(u-t)^r} S_t du$$

$$= \frac{1}{T} e^{- (T-t)^r} \int_0^t S_u du + \frac{S_t}{T} e^{- (T-t)^r} \int_0^T e^{- (T-u)^r} du$$

$$= \frac{1}{T} e^{- (T-t)^r} \int_0^t S_u du + S_t \frac{1 - e^{- (T-t)^r}}{rT}.$$
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Pricing by the Hartman-Watson distribution

First we note that the numerical computation of Asian option prices can be done using the probability density function of

$$\Lambda_T = \int_0^T S_t dt.$$ 

From Proposition 10.1, we deduce the marginal probability density function of

$$\Lambda_T := \int_0^T e^{\sigma B_t - p \sigma^2 t/2} dt,$$

also called the Hartman-Watson distribution Barrieu et al. (2004), as follows:

$$\mathbb{P}\left( \int_0^T e^{\sigma B_t - p \sigma^2 t/2} dt \in du \right) = \frac{\sigma}{2u} e^{p^2 \sigma^2 T/8} \int_{-\infty}^{\infty} \exp\left(-2 \frac{1 + e^{\sigma v - p \sigma^2 T/2}}{\sigma^2 u} - \frac{p}{2} \sigma v\right) \theta\left(\frac{4 e^{\sigma v/2 - p \sigma^2 T/4 \sigma^2 T/4}}{\sigma^2 u}, \frac{\sigma^2 T}{4}\right) dv du$$

$$= e^{-p^2 \sigma^2 T/8} \int_0^\infty v^{-1-p} \exp\left(-2 \frac{1 + v^2}{\sigma^2 u}\right) \theta\left(\frac{4v}{\sigma^2 u}, \frac{\sigma^2 T}{4}\right) dv du,$$

where $S_t = S_0 e^{\sigma B_t - p \sigma^2 t/2}$ and $p = 1 - 2r / \sigma^2$. By (10.5), this probability density function can then be used for the pricing of Asian options, as

$$f(t, x, y) = e^{-(T-t)r} \mathbb{E}^*\left[\left(\frac{1}{T}\left(\frac{y + x S_0 \Lambda_{T-t}}{T} - K\right)\right)^+\right]$$

$$= e^{-(T-t)r} \int_0^\infty \left(\frac{y + x z}{T} - K\right)^+ \mathbb{P}(\Lambda_{T-t}/S_0 \in dz)$$

$$= e^{-(T-t)r} \frac{\sigma}{2} e^{-(T-t)p^2 \sigma^2 / 8} \int_0^\infty \int_0^\infty \left(\frac{y + x z}{T} - K\right)^+ \times v^{-1-p} \exp\left(-2 \frac{1 + v^2}{\sigma^2 z}\right) \theta\left(\frac{4v}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4}\right) dv dz$$

$$= \frac{1}{T} e^{-(T-t)r} \frac{-(T-t)p^2 \sigma^2 / 8}{(KT-y)/x} \int_0^{(KT-y)/x} \int_0^\infty (xz + y - KT)$$
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\[ \times \exp \left( -2 \frac{1 + v^2}{\sigma^2 z} \right) \theta \left( \frac{4v}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{dz}{z} \]

\[ = \frac{4x}{\sigma^2 T} e^{-(T-t)r-(T-t)p^2\sigma^2/8} \int_0^\infty \int_0^\infty \left( \frac{1}{z} - \frac{\sigma^2(KT-y)}{4x} \right)^+ \]

\[ \times v^{-1-p} \exp \left( -z \frac{1 + v^2}{2} \right) \theta \left( vz, \frac{(T-t)\sigma^2}{4} \right) dv \frac{dz}{z}, \]

cf. Theorem in § 5 of Carr and Schröder (2004), which is actually a triple integral due to the definition (10.4) of \( \theta(v, t) \). Note that since the integrals are not absolutely convergent, here the order of integration between \( dv \) and \( dz \) cannot be exchanged without particular precautions, at the risk of wrong computations.

**Time Laplace transform**

The time Laplace transform of the rescaled price

\[ C(t) := \mathbb{E}^* \left[ \left( \frac{1}{t} \int_0^t S_u du - \kappa \right)^+ \right], \quad t \in \mathbb{R}_+, \]

as

\[ \int_0^\infty e^{-\lambda t} C(t) dt = \frac{\int_0^{K/2} e^{-x} e^{-x-2+(p+\sqrt{2\lambda+p^2})/2} (1 - 2Kx)^2 + (\sqrt{2\lambda+p^2}-p)/2} \lambda (\lambda - 2 + 2p) \Gamma \left( -1 + (p + \sqrt{2\lambda + p^2})/2 \right) dx, \]

with here \( \sigma := 2 \), and \( \Gamma(z) \) denotes the gamma function, see Relation (3.10) in Geman and Yor (1993). This expression can be used for pricing by numerical inversion of the Laplace transform using e.g. the Widder method, the Gaver-Stehfest method, the Durbin-Crump method, or the Papoulis method. The following Figure 10.2 represents Asian option prices computed by the Geman-Yor Geman and Yor (1993) method.

Fig. 10.2: Graph of Asian option prices with \( \sigma = 1 \), \( r = 0.1 \) and \( K = 90 \).
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We refer to *e.g.* Carr and Schröder (2004), Dufresne (2000), and references therein for more results on Asian option pricing using the probability density function of the averaged geometric Brownian motion.

Figure 7.6 presents a graph of implied volatility surface for Asian options on light sweet crude oil futures.

### 10.3 Moment Matching Approximations

**Lognormal approximation**

Other numerical approaches to the pricing of Asian options include Levy (1992), Turnbull and Wakeman (1992) which rely on approximations of the average price distribution based on the Lognormal distribution. The lognormal distribution has the probability density function

\[
g(x) = \frac{1}{\eta \sqrt{2\pi}} e^{-(\mu - \log x)^2 / (2\eta^2)} \frac{dx}{x}, \quad x > 0,
\]

where \(\mu \in \mathbb{R}, \eta > 0\), with moments

\[
\mathbb{E}[X] = e^{\mu + \eta^2/2} \quad \text{and} \quad \mathbb{E}[X^2] = e^{2\mu + 2\eta^2}.
\]

The approximation is implemented by matching the above first two moments to those of time integral

\[
\Lambda_T := \int_0^T S_t \, dt
\]

of geometric Brownian motion

\[
S_t = S_0 e^{\sigma B_t + (r - \sigma^2/2)t}, \quad t \in [0, T],
\]

as computed in the next proposition, cf. also (7) and (8) page 480 of Levy (1992).

**Proposition 10.4.** We have

\[
\mathbb{E}^* [\Lambda_T] = S_0 \frac{e^{rT} - 1}{r},
\]

and

\[
\mathbb{E}^* [(\Lambda_T)^2] = 2S_0^2 r \frac{e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r) e^{rT} + (\sigma^2 + r)}{(\sigma^2 + r)(\sigma^2 + 2r)r}.
\]

**Proof.** The computation of the first moment is straightforward: we have

\[
\mathbb{E}^* [\Lambda_T] = \mathbb{E}^* \left[ \int_0^T S_u \, du \right] = \int_0^T \mathbb{E}^* [S_u] \, du = S_0 \int_0^T e^{ru} \, du = S_0 \frac{e^{rT} - 1}{r}.
\]
For the second moment we have, letting $p := 1 - 2r / \sigma^2$,
\[
\mathbb{E}^* \left[ (\Lambda_T)^2 \right] = S_0^2 \int_0^T \int_0^T e^{-p\sigma^2 a / 2 - p\sigma^2 b / 2} \mathbb{E}^* \left[ e^{\sigma B_a} e^{\sigma B_b} \right] dbda
\]
\[= 2S_0^2 \int_0^T \int_0^a e^{-p\sigma^2 a / 2 - p\sigma^2 b / 2} e^{\sigma^2(a+b) / 2} e^{p\sigma^2 dbda}
\]
\[= 2S_0^2 \int_0^T e^{-(p-1)\sigma^2 a / 2} \int_0^a e^{-(p-3)\sigma^2 b / 2} dbda
\]
\[= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a / 2} \left( 1 - e^{-(p-3)\sigma^2 a / 2} \right) da
\]
\[= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} \left( 1 - e^{-(p-1)\sigma^2 T / 2} \right) - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(2p-4)\sigma^2 a / 2} da
\]
\[= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} \left( 1 - e^{-(p-1)\sigma^2 T / 2} \right) - \frac{4S_0^2}{(p-3)(p-2)\sigma^4} \left( 1 - e^{-(p-2)\sigma^2 T} \right)
\]
\[= 2S_0^2 \mu e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r) e^{rT} + (\sigma^2 + r)
\]
\[= (\sigma^2 + r)(\sigma^2 + 2r) \frac{e^T}{r^2}
\]
since $r - \sigma^2 / 2 = -p\sigma^2 / 2$.

By matching (10.4) with the moments of Proposition 10.4 we estimate $\hat{\mu}, \hat{\eta}$
as
\[
\hat{\eta}^2 = \frac{1}{T} \log \left( \frac{\mathbb{E}[\Lambda_T^2]}{(\mathbb{E}^*[\Lambda_T])^2} \right) \quad \text{and} \quad \hat{\mu} = \frac{1}{T} \log \mathbb{E}^*[\Lambda_T] - \frac{1}{2} \hat{\eta}^2.
\]

Fig. 10.3: Lognormal approximation for the probability density function of $\Lambda_T$.

As a consequence, from Lemma 6.5 we find the approximation
\[
e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] \approx \frac{1}{T} e^{(\mu + \eta^2 / 2)T} \Phi(d_1) - K \Phi(d_2), \quad (10.11)
\]
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where

\[
d_1 = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\eta \sqrt{T}} + \frac{\sqrt{T}}{2} = \frac{\tilde{\mu}T + \tilde{\eta}^2T - \log(KT)}{\eta \sqrt{T}}
\]

and

\[
d_2 = d_1 - \frac{\eta \sqrt{T}}{2} = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\eta \sqrt{T}} - \frac{\eta \sqrt{T}}{2}.
\]

The next Figure 10.4 compares the lognormal approximation to a Monte Carlo estimate of Asian option prices with \(\sigma = 0.5\), \(r = 0.05\) and \(K/S_t = 1.1\)

![Figure 10.4: Lognormal approximation to the Asian option price.](image)

Figure 10.4 also includes the stratified approximation

\[
e^{-rT} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] \approx \frac{e^{-rT}}{T} \int_0^\infty \left( e^{-p(z/x)\sigma^2(z/x)T/2 + \sigma^2(z/x)T/2} \Phi(d_+(K, z, x)) - KT \Phi(d_-(K, z, x)) \right) \times d\mathbb{P}(S_T \leq z \mid S_0 = x),
\]

of Privault and Yu (2016), where

\[
d_\pm(K, z, x) := \frac{1}{2\sigma(z/x)\sqrt{T}} \log \left( \frac{2x(b_T(z/x) - (1 + z/x)a_T(z/x))}{\sigma^2K^2T^2} \right) \pm \frac{\sigma(z/x)\sqrt{T}}{2}
\]

and
Pricing Asian options by conditioning on the geometric mean price

Asian options on the arithmetic average

\[ \frac{1}{T} \int_0^T S_t dt \]

have been priced by conditioning on the geometric mean price

\[ G := \exp \left( \frac{1}{T} \int_0^T \log S_t dt \right) = \frac{1}{T} \int_0^T S_t dt \]

in Curran (1994), as

\[
e^{-rT} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right]
\]

\[
= e^{-rT} \int_0^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \mid G = x \right] d\mathbb{P}(G \leq x)
\]

\[
= e^{-rT} \int_0^K \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \mid G = x \right] d\mathbb{P}(G \leq x)
\]

\[
+ e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \mid G = x \right] d\mathbb{P}(G \leq x)
\]

\[ = C_1 + C_2, \]

where

\[ C_1 := e^{-rT} \int_0^K \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \mid G = x \right] d\mathbb{P}(G \leq x), \]

\[ C_2 := e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \mid G = x \right] d\mathbb{P}(G \leq x). \]
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and

\[ C_2 := e^{-rT} \int_K^\infty \mathbb{E}^* \left[ \frac{1}{T} \int_0^T S_u du - K \right]^+ \bigg| G = x \bigg] d\mathbb{P}(G \leq x) \]

\[ = e^{-rT} \int_K^\infty \mathbb{E}^* \left\{ \int_0^T S_u du \bigg| G = x \right\} d\mathbb{P}(G \leq x) \]

\[ = \frac{e^{-rT}}{T} \int_K^\infty \mathbb{E}^* \left\{ \int_0^T S_u du \bigg| G = x \right\} d\mathbb{P}(G \leq x) - K e^{-rT} \int_K^\infty d\mathbb{P}(G \leq x) \]

\[ = \frac{e^{-rT}}{T} \mathbb{E}^* \left[ \int_0^T S_u du \mathbb{1}_{\{G \geq K\}} \right] - K e^{-rT} \mathbb{P}(G \geq K). \]

The term \( C_1 \) can be estimated by a lognormal approximation given that \( G = x \). As for \( C_2 \) we note that

\[ G = \exp \left( \frac{1}{T} \int_0^T \log S_t dt \right) \]

\[ = \exp \left( \frac{1}{T} \int_0^T \left( \mu t + \sigma B_t - \frac{\sigma^2 t}{2} \right) dt \right) \]

\[ = \exp \left( T \left( \frac{\mu - \sigma^2}{2} \right) + \frac{\sigma}{T} \int_0^T B_t dt \right), \]

hence

\[ \log G = \frac{T}{2} (\mu - \sigma^2 / 2) + \frac{\sigma}{T} \int_0^T B_t dt \]

is Gaussian \( N \left( T(\mu - \sigma^2 / 2) / 2, \sigma^2 T / 3 \right) \) with mean \( T(\mu - \sigma^2 / 2) / 2 \), and variance

\[ \mathbb{E} \left[ \left( \int_0^T B_t dt \right)^2 \right] = \mathbb{E} \left[ \int_0^T \int_0^T B_s B_t dsdt \right] \]

\[ = \int_0^T \int_0^T \mathbb{E} [B_s B_t] dsdt \]

\[ = 2 \int_0^T \int_0^t sdsdt \]

\[ = \int_0^T t^2 dt \]

\[ = \frac{T^3}{3}. \]

Hence we have

\[ \mathbb{P}(G \geq K) = \mathbb{P}(\log G \geq \log K) \]

\[ = \mathbb{P} \left( \frac{T}{2} \left( \mu - \sigma^2 \right) + \frac{\sigma}{T} \int_0^T B_t dt \geq \log K \right) \]
= \mathbb{P} \left( \int_0^T B_t \, dt \geq \frac{T}{\sigma} \left( -\frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) + \log K \right) \right)
= \Phi \left( \frac{\sqrt{3}}{\sigma \sqrt{T}} \left( \frac{T}{2} \left( \mu - \frac{\sigma^2}{2} \right) - \log K \right) \right).

Basket options

Basket options on the portfolio

\[ A_T := \sum_{k=1}^{N} \alpha_k S_T^{(k)} \]

have also been priced in Milevsky (1998) by approximating \( A_T \) by a lognormal or a reciprocal gamma random variable, see also Deelstra et al. (2004) for additional conditioning on the geometric average of asset prices.

Asian basket options

Moment matching techniques combined with conditioning have been applied to Asian basket options in Deelstra et al. (2010). See also Dahl and Benth (2002) for the pricing of Asian basket options using quasi Monte Carlo simulation.

10.4 PDE Method - Two Variables

The price at time \( t \) of the Asian option with payoff (10.1) can be written as

\[ f(t, S_t, \Lambda_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u \, du - K \right)^{+} \mid \mathcal{F}_t \right], \quad t \in [0, T]. \]

Next, we derive the Black-Scholes partial differential equation (PDE) for the price of a self-financing portfolio. Until the end of this chapter we model the asset price \( (S_t)_{t \in [0, T]} \) as

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \quad t \in \mathbb{R}_+, \]

where \( (B_t)_{t \in \mathbb{R}_+} \) is a standard Brownian motion under the historical probability measure \( \mathbb{P} \).

**Proposition 10.5.** Let \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) be a portfolio strategy such that

(i) \( (\eta_t, \xi_t)_{t \in \mathbb{R}_+} \) is self-financing.

(ii) the value \( V_t := \eta_t A_t + \xi_t S_t, \ t \in \mathbb{R}_+, \) takes the form
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\[ V_t = f(t, S_t, \Lambda_t), \quad t \in \mathbb{R}_+, \]

for some function \( f \in C^{1,2,1}( (0, \infty)^3 ) \).

Then the function \( f(t, x, y) \) in (10.13) satisfies the PDE

\[
rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + x \frac{\partial f}{\partial y}(t, x, y) + rx \frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x, y),
\]

\( t, x > 0 \), under the boundary conditions

\[
\begin{align*}
  f(t, 0^+, y) &= \lim_{x \to 0} f(t, x, y) = e^{-(T-t)r} \left( \frac{y}{T} - K \right)^+, \\
  f(t, x, 0^+) &= \lim_{y \to 0} f(t, x, y) = 0, \\
  f(T, x, y) &= \left( \frac{y}{T} - K \right)^+,
\end{align*}
\]

(10.14a) \( (10.14b) \) \( (10.14c) \)

and \( \xi_t \) is given by

\[ \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t), \quad t \in \mathbb{R}_+. \] (10.15)

**Proof.** We note that the self-financing condition (5.8) implies

\[
dV_t = \eta_t dA_t + \xi_t dS_t
\]

\[= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \]

(10.16)

Since \( d\Lambda_t = S_t dt \), an application of Itô’s formula to \( f(t, x, y) \) leads to

\[
dV_t = f(t, S_t, \Lambda_t) = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) dt + \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) d\Lambda_t
\]

\[+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dt + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dB_t
\]

\[= \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) dt + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) dt
\]

\[+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dt + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) dB_t.
\]

(10.17)

By respective identification of the terms in \( dB_t \) and \( dt \) in (10.16) and (10.17) we get
\[
\begin{aligned}
\sum_{k=1}^{n} a_{kk} x_k + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j &= c,
\end{aligned}
\]
discretization steps and $d$ is the dimension of the problem (curse of dimensionality).

### 10.5 PDE Method - One Variable

#### (1) Time-independent coefficients

Following Lamberton and Lapeyre (1996), page 91, we define the auxiliary process

$$Z_t = \frac{1}{S_t} \left( \frac{1}{T} \int_0^t S_u du - K \right) = \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right), \quad t \in [0, T].$$

With this notation, the price of the Asian option at time $t$ becomes

$$e^{- (T-t)r} \mathbb{E}^*[\left( \frac{1}{T} \int_0^T S_u du - K \right)^+ | \mathcal{F}_t] = e^{- (T-t)r} \mathbb{E}^*[S_T(Z_T)^+ | \mathcal{F}_t].$$

**Lemma 10.6.** The price (10.2) at time $t$ of the Asian option with payoff (10.1) can be written as

$$f(t, S_t, \Lambda_t) = S_t g(t, Z_t) = e^{- (T-t)r} \mathbb{E}^*[\left( \frac{1}{T} \int_0^T S_u du - K \right)^+ | \mathcal{F}_t],$$

$t \in [0, T]$, with the relation

$$f(t, x, y) = x g \left( t, \frac{1}{x} \left( \frac{y}{T} - K \right) \right), \quad x, y \in \mathbb{R}_+, \quad 0 \leq t \leq T,$$

where

$$g(t, z) = e^{- (T-t)r} \mathbb{E}^*[\left( z + \frac{1}{T} \int_0^{T-t} \frac{S_u}{S_0} du \right)^+]$$

$$= e^{- (T-t)r} \mathbb{E}^*[\left( z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+],$$

and with the boundary condition

$$g(T, z) = z^+, \quad z \in \mathbb{R}.$$

**Proof.** For $0 \leq s \leq t \leq T$, we have

$$d(S_t Z_t) = \frac{1}{T} d \left( \int_0^t S_u du - K \right) = \frac{S_t}{T} dt,$$
hence
\[ \frac{S_T Z_t}{S_s} = Z_s + \frac{1}{T} \int_s^t \frac{S_u}{S_s} du, \quad t \geq s. \]

Since for any \( t \in [0, T] \), \( S_t \) is positive and \( \mathcal{F}_t \)-measurable, and \( S_u / S_t \) is independent of \( \mathcal{F}_t \), \( u \geq t \), we have:

\[ e^{-(T-t) r} \mathbb{E}^* \left[ S_T (Z_T)^+ | \mathcal{F}_t \right] = e^{-(T-t) r} S_t \mathbb{E}^* \left[ \left( \frac{S_T Z_T}{S_t} \right)^+ | \mathcal{F}_t \right] \]

\[ = e^{-(T-t) r} S_t \mathbb{E}^* \left[ \left( Z_t + \frac{1}{T} \int_t^T \frac{S_u}{S_t} du \right)^+ | \mathcal{F}_t \right] \]

\[ = e^{-(T-t) r} S_t \mathbb{E}^* \left[ \left( z + \frac{1}{T} \int_t^T \frac{S_u}{S_t} du \right)^+ \right] \bigg|_{z=Z_t} \]

\[ = e^{-(T-t) r} S_t \mathbb{E}^* \left[ \left( z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+ \right] \bigg|_{z=Z_t} \]

\[ = S_t g(t, Z_t), \]

which proves (10.21). \( \square \)

When \( \Lambda_t / T \geq K \) we have \( Z_t \geq 0 \), hence by (10.18) and (10.20) we find

\[ g(t, Z_t) = e^{-(T-t) r} Z_t + \frac{1 - e^{-(T-t) r}}{r T}, \quad t \in [0, T]. \] (10.22)

Note that as in (10.9), \( g(t, z) \) can be computed from the probability density function (10.8) of \( \Lambda_{T-t} \), as

\[ g(t, z) = \mathbb{E}^* \left[ \left( z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+ \right] \]

\[ = \int_0^\infty \left( z + \frac{u}{T} \right)^+ d\mathbb{P} \left( \frac{\Lambda_t}{S_0} \leq u \right) \]

\[ = e^{-p^2 \sigma^2 t / 8} \]

\[ \times \int_0^\infty \left( z + \frac{u}{T} \right)^+ \int_0^\infty v^{-1-p} \exp \left( -2 \frac{1+v^2}{\sigma^2} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{du}{u} \]

\[ = e^{-p^2 \sigma^2 t / 8} \]

\[ \times \int_{(z T) \vee 0}^{\infty} \left( z + \frac{u}{T} \right) \int_0^\infty v^{-1-p} \exp \left( -2 \frac{1+v^2}{\sigma^2} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{du}{u} \]
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\[ z e^{-p^2 \sigma^2 t/8} \int_{(-zT)\cap[0,\infty)} \int_{0}^{\infty} v^{-1-p} \exp \left( -2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{du}{u} \]

\[ + \frac{1}{T} e^{-p^2 \sigma^2 t/8} \int_{(-zT)\cap[0,\infty)} \int_{0}^{\infty} v^{-1-p} \exp \left( -2 \frac{1 + v^2}{\sigma^2} \right) \theta \left( \frac{4v}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{du}{u} \]


**Proposition 10.7.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) be a portfolio strategy such that

1. \((\eta_t, \xi_t)_{t \in \mathbb{R}^+}\) is self-financing,

2. the value \(V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}^+\), takes the form

\[ V_t = S_t g(t, Z_t), \quad t \in \mathbb{R}^+, \]

for some function \(g \in C^{1,2}((0, \infty)^2)\).

Then the function \(g(t, z)\) satisfies the PDE

\[
\frac{\partial g}{\partial t}(t, z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0, \quad (10.23)
\]

under the terminal condition

\[ g(T, z) = z^+, \quad z \in \mathbb{R}, \quad (10.24) \]

and the corresponding replicating portfolio Delta is given by

\[ \xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t), \quad t \in [0, T]. \quad (10.25) \]

**Proof.** By the Itô formula applied to \(1/S_t\) we have

\[ d \left( \frac{1}{S_t} \right) = \frac{1}{S_t} \left( (-\mu + \sigma^2) dt - \sigma dB_t \right), \]

hence

\[ dZ_t = d \left( \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) \]

\[ = d \left( \frac{\Lambda_t}{TS_t} - \frac{K}{S_t} \right) \]

\[ = \frac{1}{T} d \left( \frac{\Lambda_t}{S_t} \right) - K d \left( \frac{1}{S_t} \right) \]

\[ = \frac{1}{T} d \Lambda_t + \left( \frac{\Lambda_t}{T} - K \right) d \left( \frac{1}{S_t} \right) \]
\[
\frac{dt}{T} + S_t Z_t \left( \frac{1}{S_t} \right) = \frac{dt}{T} + Z_t (-\mu + \sigma^2) dt - Z_t \sigma dB_t.
\]

By the self-financing condition (5.8) we have
\[
dV_t = \eta_t dA_t + \xi_t dS_t
= r\eta_t dA_t + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t,
\]
(10.26)
t \in \mathbb{R}_+. An application of Itô’s formula to \(f(t,x,y)\) leads to
\[
d(S_t g(t, Z_t)) = g(t, Z_t) dS_t + S_t dg(t, Z_t) + dS_t \cdot dg(t, Z_t)
= g(t, Z_t) dS_t + S_t \frac{\partial g}{\partial t}(t, Z_t) dt + S_t \frac{\partial g}{\partial z}(t, Z_t) dZ_t
+ \frac{1}{2} S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t) (dZ_t)^2 + dS_t \cdot dg(t, Z_t)
= \mu S_t g(t, Z_t) dt + \sigma S_t g(t, Z_t) dB_t + S_t \frac{\partial g}{\partial t}(t, Z_t) dt
+ S_t Z_t (-\mu + \sigma^2) \frac{\partial g}{\partial z}(t, Z_t) dt + \frac{1}{T} S_t \frac{\partial g}{\partial z}(t, Z_t) dt - \sigma S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dB_t
+ \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dt
= \mu S_t g(t, Z_t) dt + S_t \frac{\partial g}{\partial t}(t, Z_t) dt + S_t Z_t (-\mu + \sigma^2) \frac{\partial g}{\partial z}(t, Z_t) dt + \frac{1}{T} S_t \frac{\partial g}{\partial z}(t, Z_t) dt
+ \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dt
+ \sigma S_t g(t, Z_t) dB_t - \sigma S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dB_t.
\]

By respective identification of the terms in \(dB_t\) and \(dt\) in (10.26) and (10.17) we get
\[
\begin{align*}
    r\eta_t A_t + \mu \xi_t S_t &= \mu S_t g(t, Z_t) + S_t \frac{\partial g}{\partial t}(t, Z_t) - \mu S_t Z_t \frac{\partial g}{\partial z}(t, Z_t)
    + \frac{1}{T} S_t \frac{\partial g}{\partial z}(t, Z_t) + \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t),
    \\
    \xi_t S_t \sigma &= \sigma S_t g(t, Z_t) - \sigma S_t Z_t \frac{\partial g}{\partial z}(t, Z_t),
\end{align*}
\]

hence

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\[
\begin{align*}
V_t - r \xi_t S_t &= S_t \frac{\partial g}{\partial t}(t, Z_t) + \frac{1}{T} S_t \frac{\partial g}{\partial z}(t, Z_t) + \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t), \\
\xi_t &= g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t),
\end{align*}
\]

i.e.

\[
\begin{align*}
\frac{\partial g}{\partial t}(t, z) + \left( \frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z \frac{\partial^2 g}{\partial z^2}(t, z) &= 0, \\
\xi_t &= g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t),
\end{align*}
\]

under the terminal condition \( g(T, z) = z^+ \), \( z \in \mathbb{R} \), which follows from (10.21).

When \( \Lambda_t / T \geq K \) we have \( Z_t \geq 0 \) and (10.22) and (10.25) show that

\[
\xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t) = e^{-(T-t)r} Z_t + \frac{1 - e^{-(T-t)r}}{rT} - e^{-(T-t)r} Z_t
\]

\[
= \frac{1 - e^{-(T-t)r}}{rT}, \quad t \in [0, T],
\]

which recovers (10.19). Similarly, from (10.24) we recover

\[
\xi_T = g(T, Z_T) - Z_T \frac{\partial g}{\partial z}(T, Z_T) = Z_T \mathbb{1}_{\{Z_T \geq 0\}} - Z_T \mathbb{1}_{\{Z_T > 0\}} = 0
\]

at maturity.

We also check that

\[
\xi_t = e^{-(T-t)r} \sigma S_t \frac{\partial f}{\partial x}(t, S_t, Z_t) - \sigma Z_t \frac{\partial f}{\partial z}(t, S_t, Z_t)
\]

\[
= e^{-(T-t)r} \left( -Z_t \frac{\partial g}{\partial z}(t, Z_t) + g(t, Z_t) \right)
\]

\[
= e^{-(T-t)r} \left( S_t \frac{\partial g}{\partial z}(t, \frac{1}{T} \int_0^t S_u du - K) \right) |_{x=S_t} + g(t, Z_t)
\]

\[
= \frac{\partial}{\partial x} \left( x e^{-(T-t)r} g \left( t, \frac{1}{T} \int_0^t S_u du - K \right) \right) |_{x=S_t}, \quad t \in [0, T].
\]

We also find that the amount invested on the risk-free asset is given by

\[
\eta_t A_t = Z_t S_t \frac{\partial g}{\partial z}(t, Z_t).
\]
Next we note that a PDE with no first order derivative term can be obtained using time-dependent coefficients.

(2) - Time-dependent coefficients

Define now the auxiliary process

\[ U_t := \frac{1 - e^{-(T-t)r}}{rT} + e^{-(T-t)r} \frac{1}{S_t} \left( \frac{1}{T} \int_0^t S_u du - K \right) \]

\[ = \frac{1}{rT} (1 - e^{-(T-t)r}) + e^{-(T-t)r} Z_t, \quad t \in [0, T], \]

i.e.

\[ Z_t = e^{(T-t)r} U_t + \frac{e^{(T-t)r} - 1}{rT}, \quad t \in [0, T]. \]

We have

\[ dU_t = -\frac{1}{T} e^{-(T-t)r} dt + r e^{-(T-t)r} Z_t dt + e^{-(T-t)r} dZ_t \]

\[ = e^{-(T-t)r} \sigma^2 Z_t dt - e^{-(T-t)r} \sigma Z_t dB_t - (\mu - r) e^{-(T-t)r} Z_t dt \]

\[ = -e^{-(T-t)r} \sigma Z_t d\tilde{B}_t, \quad t \in \mathbb{R}_+, \]

where

\[ d\tilde{B}_t = dB_t - \sigma dt + \frac{\mu - r}{\sigma} dt = d\tilde{B}_t - \sigma dt \]

is a standard Brownian motion under

\[ d\hat{P} = e^{\sigma B_T - \sigma^2 t/2} d\mathbb{P}^* = e^{-rT S_T / S_0} d\mathbb{P}^*. \]

**Lemma 10.8.** The Asian option price can be written as

\[ S_t h(t, U_t) = e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right], \]

where the function \( h(t, y) \) is given by

\[ h(t, y) = \mathbb{E}^*[(U_T)^+ \big| U_t = y], \quad 0 \leq t \leq T. \]

**Proof.** We have

\[ U_T = \frac{1}{S_T} \left( \frac{1}{T} \int_0^T S_u du - K \right) = Z_T, \]

and
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\[
\frac{d\hat{P}_t}{d\hat{P}^*_{t}} = e^{r(B_T - B_t) - (T-t)\sigma^2/2} = \frac{e^{-rT}S_T}{e^{-rt}S_t},
\]
hence the price of the Asian option is

\[
e^{-(T-t)r} \mathbb{E}^*[S_T(Z_T)^+ | \mathcal{F}_t] = e^{-(T-t)r} \mathbb{E}^*[S_T(U_T)^+ | \mathcal{F}_t] \]

\[
= S_t \mathbb{E}^* \left[ \frac{e^{-rT}S_T}{e^{-rt}S_t} (U_T)^+ | \mathcal{F}_t \right] \]

\[
= S_t \mathbb{E}^* \left[ \frac{d\hat{P}_t}{d\hat{P}^*_t} (U_T)^+ | \mathcal{F}_t \right] \]

\[
= S_t \hat{E}[(U_T)^+ | \mathcal{F}_t].
\]

□

The next proposition gives a replicating hedging strategy for Asian options. See § 7.5.3 of Shreve (2004) and references therein for a different derivation of the PDE (10.27).

**Proposition 10.9.** Let \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) be a portfolio strategy such that

(i) \((\eta_t, \xi_t)_{t \in \mathbb{R}_+}\) is self-financing,

(ii) the value \(V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}_+\), takes the form

\[V_t = S_t h(t, U_t), \quad t \in \mathbb{R}_+,
\]

for some function \(h \in C^{1,2}((0, \infty)^2)\).

Then the function \(h(t, z)\) satisfies the PDE

\[
\frac{\partial h}{\partial t}(t, y) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-(T-t)r}}{rT} - y \right)^2 \frac{\partial^2 h}{\partial y^2}(t, y) = 0, \tag{10.27}
\]

under the terminal condition

\[h(T, z) = z^+,
\]

and the corresponding replicating portfolio is given by

\[\xi_t = h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t), \quad t \in [0, T].\]

**Proof.** By the self-financing condition (10.16) we have

\[dV_t = rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \tag{10.28}
\]

\[\hat{\nabla}
\]
\( t \in \mathbb{R}_+ \). By Itô’s formula we get

\[
\begin{align*}
\frac{d(S_t h(t, U_t))}{dt} &= h(t, U_t) dS_t + S_t dh(t, U_t) + dS_t \cdot dh(t, U_t) \\
&= \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t \\
&\quad + S_t \left( \frac{\partial h}{\partial t}(t, U_t) dt + \frac{\partial h}{\partial y}(t, U_t) dU_t + \frac{1}{2} \frac{\partial^2 h}{\partial y^2}(t, U_t)(dU_t)^2 \right) \\
&\quad + \frac{\partial h}{\partial y}(t, U_t) dS_t \cdot dU_t \\
&= \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t - S_t (\mu - r) \frac{\partial h}{\partial y}(t, U_t) Z_t dt \\
&\quad + S_t \left( \frac{\partial h}{\partial t}(t, U_t) dt - \sigma \frac{\partial h}{\partial y}(t, U_t) Z_t dB_t + \frac{\sigma^2}{2} Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t) dt \right) \\
&\quad - \sigma^2 S_t \frac{\partial h}{\partial y}(t, U_t) Z_t dt \\
&= \mu S_t h(t, U_t) dt + \sigma S_t h(t, U_t) dB_t - S_t (\mu - r) \frac{\partial h}{\partial y}(t, U_t) Z_t dt \\
&\quad + S_t \left( \frac{\partial h}{\partial t}(t, U_t) dt - \sigma \frac{\partial h}{\partial y}(t, U_t) Z_t dB_t - \sigma dt \right) + \frac{\sigma^2}{2} Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t) dt \\
&\quad - \sigma^2 S_t \frac{\partial h}{\partial y}(t, U_t) Z_t dt.
\end{align*}
\]

By respective identification of the terms in \( dB_t \) and \( dt \) in (10.28) and (10.17) we get

\[
\begin{cases}
\begin{align*}
\eta A_t + \xi_t S_t &= \mu S_t h(t, U_t) - (\mu - r) S_t Z_t \frac{\partial h}{\partial y}(t, U_t) dt + S_t \frac{\partial h}{\partial t}(t, U_t) \\
&\quad + \frac{\sigma^2}{2} S_t Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t), \\
\xi_t &= h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t),
\end{align*}
\end{cases}
\]

hence

\[
\begin{cases}
\end{cases}
\]

\[
\begin{cases}
\begin{align*}
\eta A_t &= -r S_t (\xi_t - h(t, U_t)) + S_t \frac{\partial h}{\partial t}(t, U_t) + \frac{\sigma^2}{2} S_t Z_t^2 \frac{\partial^2 h}{\partial y^2}(t, U_t), \\
\xi_t &= h(t, U_t) - Z_t \frac{\partial h}{\partial y}(t, U_t),
\end{align*}
\end{cases}
\]

and
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\[
\begin{align*}
\frac{\partial h}{\partial t}(t,y) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-(T-t)r}}{rT} - y \right)^2 \frac{\partial^2 h}{\partial y^2}(t,y) &= 0, \\
\xi_t &= h(t, U_t) + \left( \frac{1 - e^{-(T-t)r}}{rT} - U_t \right) \frac{\partial h}{\partial y}(t, U_t),
\end{align*}
\]

under the terminal condition

\[h(T, z) = z^+.\]

We also find the risk-free portfolio allocation

\[\eta_t A_t = e^{(T-t)r} S_t \left( U_t - \frac{1 - e^{-(T-t)r}}{rT} \right) \frac{\partial h}{\partial y}(t, U_t) = S_t Z_t \frac{\partial h}{\partial y}(t, U_t).\]

Exercises

Exercise 10.1 Consider the short rate process \(r_t = \sigma B_t\), where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion.

a) Find the probability distribution of the time integral \(\int_0^T r_s ds\).

b) Compute the price

\[e^{-rT} \mathbb{E}^* \left[ \left( \int_0^T r_u du - \kappa \right)^+ \right]\]

of a caplet on the forward rate \(\int_0^T r_s ds\).

Exercise 10.2 Asian call options with negative strike price. Consider the asset price process

\[S_t = S_0 e^{rt + \sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}^+,
\]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion. Assuming that \(\kappa \leq 0\), compute the price

\[e^{-(T-t)r} \mathbb{E}^* \left[ \left( \frac{1}{T} \int_0^T S_u du - \kappa \right)^+ \mid \mathcal{F}_t \right]\]

of the Asian option at time \(t \in [0, T]\).
Exercise 10.3  Compute the price

\[ e^{-(T-t)r} \mathbb{E}^* \left[ \left( \exp \left( \frac{1}{T} \int_t^T \log S_u du \right) - K \right)^+ \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \]

at time \( t \) of the geometric Asian option with maturity \( T \), where \( S_t = S_0 e^{r t + \sigma B_t - \sigma^2 t / 2}, \quad t \in [0, T] \).

\textit{Hint}: When \( X \sim \mathcal{N}(0, v^2) \) we have

\[ \mathbb{E}^*[(e^{m+X} - K)^+] = e^{m+v^2/2} \Phi(v + (m - \log K) / v) - K \Phi((m - \log K) / v). \]

Exercise 10.4  Consider a CIR process \((r_t)_{t \in \mathbb{R}^+}\) given by

\[ dr_t = -\lambda (r_t - m) dt + \sigma \sqrt{r_t} dB_t, \quad (10.29) \]

where \((B_t)_{t \in \mathbb{R}^+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\), and let

\[ \Lambda_t := \frac{1}{T - \tau} \int_{\tau}^t r_s ds, \quad t \in [\tau, T]. \]

Compute the price at time \( t \in [\tau, T] \) of the Asian option with payoff \((\Lambda_T - K)^+\), under the condition \(\Lambda_t \geq K\).

Exercise 10.5  Consider an asset price \((S_t)_{t \in \mathbb{R}^+}\) which is a submartingale under the risk-neutral probability measure \(\mathbb{P}^*\), in a market with risk-free interest rate \(r > 0\), and let \(\phi(x) = (x-K)^+\) be the (convex) payoff function of the European call option.

Show that, for any sequence \(0 < T_1 < \cdots < T_n\), the price of the average option with payoff

\[ \phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right) \]

can be upper bounded by the price of the European call option with maturity \(T_n\), \textit{i.e.} show that

\[ \mathbb{E}^* \left[ \phi \left( \frac{S_{T_1} + \cdots + S_{T_n}}{n} \right) \right] \leq \mathbb{E}^* [\phi(S_{T_n})]. \]

Exercise 10.6  Let \((S_t)_{t \in \mathbb{R}^+}\) denote a risky asset whose price \(S_t\) is given by

\[ dS_t = \mu S_t dt + \sigma S_t dB_t. \]
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where \((B_t)_{t \in \mathbb{R}_+}\) is a standard Brownian motion under the risk-neutral probability measure \(\mathbb{P}^*\). Compute the price at \(t \in [\tau, T]\) of the Asian option with payoff

\[
\left( \frac{1}{T - \tau} \int_\tau^T S_u du - K \right)^+,
\]

under the condition that

\[
A_t := \frac{1}{T - \tau} \int_\tau^t S_u du \geq K.
\]

Exercise 10.7 Pricing Asian options by PDEs. Show that the functions \(g(t, z)\) and \(h(t, y)\) are linked by the relation

\[
g(t, z) = h \left( t, \frac{1 - e^{-(T-t)r}}{rT} + e^{-(T-t)r}z \right), \quad t \in [0, T], \quad z > 0,
\]

and that the PDE (1.35) for \(h(t, y)\) can be derived from the PDE (1.33) for \(g(t, z)\) and the above relation.

Exercise 10.8 Hedging Asian options Yang et al. (2011).

a) Compute the Asian option price \(f(t, S_t, \Lambda_t)\) when \(\Lambda_t / T \geq K\).

b) Compute the hedging portfolio allocation \((\xi_t, \eta_t)\) when \(\Lambda_t / T \geq K\). When \(\Lambda_t / T \geq K\) we have

\[
\xi_t = \frac{1 - e^{-(T-t)r}}{rT} \quad \text{and} \quad \eta_t A_t = e^{(T-t)r} \left( \frac{\Lambda_t}{T} - K \right), \quad t \in [0, T].
\]

c) At maturity we have \(f(T, S_T, \Lambda_T) = (\Lambda_T / T - K)^+\), hence \(\xi_T = 0\) and

\[
\eta_T A_T = A_T \frac{e^{-rT}}{A_0} \left( \frac{\Lambda_T}{T} - K \right) 1_{\{\Lambda_T > KT\}} = \left( \frac{\Lambda_T}{T} - K \right)^+.
\]

d) Show that the Asian option with payoff \((\Lambda_T - K)^+\) can be hedged by the self-financing portfolio

\[
\xi_t = \frac{1}{S_t} \left( f(t, S_t, \Lambda_t) - e^{-(T-t)r} \left( \frac{\Lambda_t}{T} - K \right) h \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right) \right)
\]

in the asset \(S_t\) and

\[
\eta_t = \frac{e^{-rT}}{A_0} \left( \frac{\Lambda_t}{T} - K \right) h \left( t, \frac{1}{S_t} \left( \frac{\Lambda_t}{T} - K \right) \right), \quad t \in [0, T],
\]

in the risk-free asset \(A_t = A_0 e^{rt}\), where \(h(t, z)\) is solution to a partial differential equation to be written explicitly.
Exercise 10.9  Compute the first and second moments of the time integral
\[ \int_{\tau}^{T} S_t \, dt \] for \( \tau \in [0, T) \), where \((S_t)_{t \in \mathbb{R}_+}\) is the geometric Brownian motion \(S_t := S_0 e^{\sigma B_t + rt - \sigma^2 t / 2}, \ t \in \mathbb{R}_+\).