

A numerical method based on integro-differential formulation for solving a one-dimensional Stefan problem

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Abstract

A numerical method based on an integro-differential formulation is proposed for solving a one-dimensional moving boundary Stefan problem involving heat conduction in a solid with phase change. Some specific test problems are solved using the proposed method. The numerical results obtained indicate that it can give accurate solutions and may offer an interesting and viable alternative to existing numerical methods for solving the Stefan problem.

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1 Introduction

A moving boundary Stefan problem of interest requires solving the differential equations

$$\frac{\partial^2 T(x, t)}{\partial x^2} + xR(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} = R^2(t) \frac{\partial T(x, t)}{\partial t} \quad \text{for } x \in [0, 1] \text{ and } t \geq 0, \quad (1)$$

and

$$R(t) \frac{dR(t)}{dt} = -\text{Ste} \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=1} \quad \text{for } t \geq 0, \quad (2)$$

subject to the initial conditions

$$R(0) = 0 \text{ and } T(x, 0) = 0 \text{ for } x \in [0, 1], \quad (3)$$

and the boundary conditions

$$\alpha T(0, t) + \beta \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} = f(t) \text{ and } T(1, t) = 0 \text{ for } t > 0, \quad (4)$$

where $T(x, t)$ is the temperature, $R(t) \geq 0$ gives the position of the moving boundary, Ste denotes a constant known as the Stefan number and the constants α and β (not both zero) and the function $f(t)$ are assumed to be suitably prescribed such that $R(t)$ is an increasing function of t . Note that (2) is known as the Stefan condition.

Equations (1)-(4) arise in the formulation of the one-dimensional moving boundary problem for the liquid region of a melting solid at the phase change temperature. They are derived from the original formulation of the Stefan problem to immobilize the solution domain from the physical region given by $[0, R(t)]$ to the non-dimensionalized interval $[0, 1]$ (Rizwan-uddin [13]). A semi-analytical technique known as the nodal integral method for solving the

problem for the special case $\alpha = 1$ and $\beta = 0$ was proposed by Rizwan-uddin [14]. Caldwell, Savović and Kwan [6] and Savović and Caldwell [15] presented a finite-difference method for determining $T(x, t)$ and $R(t)$ numerically, also for $\alpha = 1$ and $\beta = 0$. The more general boundary condition at $x = 0$ as in (4) (with $\alpha^2 + \beta^2 \neq 0$) allows for the heat flux to be specified at $x = 0$, as may occur when heat transfer takes place through convective process on the boundary. Reviews on numerical methods for solving various one-dimensional Stefan problems were recently given by Caldwell and Kwan [5] and Javierre, Vuik, Vermolen and van der Zwaag [9]. Other earlier related references which may be of interest here include Asaithambi [3], [4], Crank [8], Kutluay, Bahadir and Özdeş [10] and Lesaint and Touzani [11].

In the present paper, an alternative numerical method based on an integro-differential equation of (1) is proposed for solving the Stefan problem defined by (1)-(4). Together with the boundary conditions in (4), the integro-differential equation is reduced to a system of algebraic-differential equations by approximating $T(x, t)$ through the use of local interpolating spatial functions. The system contains functions of t giving the unknown temperature at selected nodal points, the boundary heat flux functions (at $x = 0$ and $x = 1$) and the unknown position $R(t)$ of the moving boundary. The first order time derivatives of the nodal temperature functions are approximated using quadratic functions of t in order to further reduce algebraic-differential equations to purely algebraic equations. A predictor-corrector approach is used to solve the non-linear algebraic equations and (2). The numerical procedure here does not require the boundary heat flux to be approximated using a finite-difference formula for the first order spatial derivative of the temperature. Instead, the heat flux function at either $x = 0$ or $x = 1$ is to be determined directly as a function of t , if it is not known. To test its

validity, the proposed numerical method is applied to solve some specific test problems.

The integro-differential approach offers an interesting and viable alternative to more conventional numerical techniques like the finite-difference method for solving initial-boundary value problems in engineering and physical science. The main advantage in using the approach for the numerical solution of (1)-(4) is that the formulation does not contain any spatial derivative of the unknown function in the interior of the solution domain. Thus, it is not necessary to approximate any spatial derivative using finite-difference formulae. For some examples of problems solved using integro-differential formulations, one may refer to Ang [1], [2] and Chen and You [7].

2 Integro-differential formulation

Integrating (1) partially with respect to x over the interval $0 \leq x \leq \eta$ (with $0 < \eta < 1$) yields

$$\begin{aligned} & \left. \frac{\partial T(\eta, t)}{\partial \eta} - \frac{\partial T(x, t)}{\partial x} \right|_{x=0} \\ &= \int_0^{\eta} \left[R^2(t) \frac{\partial T(x, t)}{\partial t} - xR(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} \right] dx. \end{aligned} \quad (5)$$

Equation (5) is now partially integrated with respect to η over the interval $0 \leq \eta \leq \xi$ (with $0 < \xi < 1$) to obtain

$$\begin{aligned} & T(\xi, t) - T(0, t) - \xi \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} \\ &= \int_0^{\xi} \int_0^{\eta} \left[R^2(t) \frac{\partial T(x, t)}{\partial t} - xR(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} \right] dx d\eta. \end{aligned} \quad (6)$$

Interchanging the order of integration in the double integral on the right hand side of (6) leads to

$$\begin{aligned} & T(\xi, t) - T(0, t) - \xi \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} \\ &= \int_0^\xi |x - \xi| \left[R^2(t) \frac{\partial T(x, t)}{\partial t} - xR(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} \right] dx. \end{aligned} \quad (7)$$

If the exercise above is repeated using the intervals $\eta \leq x \leq 1$ and $\xi \leq \eta \leq 1$ in place of $0 \leq x \leq \eta$ and $0 \leq \eta \leq \xi$ respectively, one obtains

$$\begin{aligned} & T(1, t) - T(\xi, t) + (\xi - 1) \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=1} \\ &= - \int_\xi^1 |x - \xi| \left[R^2(t) \frac{\partial T(x, t)}{\partial t} - xR(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} \right] dx. \end{aligned} \quad (8)$$

Taking the difference between (7) and (8) and using the boundary condition at $x = 1$ in (4) give

$$\begin{aligned} & 2T(\xi, t) - T(0, t) - \xi \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} - (\xi - 1) \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=1} \\ &= R^2(t) \int_0^1 |x - \xi| \frac{\partial T(x, t)}{\partial t} dx - \int_0^1 x|x - \xi| R(t) \frac{dR(t)}{dt} \frac{\partial T(x, t)}{\partial x} dx. \end{aligned} \quad (9)$$

Performing an integration by part on the second integral in (9) (to remove the first order spatial derivative of T), one may derive the integro-differential equation

$$\begin{aligned} 2T(\xi, t) &= T(0, t) + \xi\theta_0(t) + (\xi - 1)\theta_1(t) + R^2(t) \int_0^1 |x - \xi| \frac{\partial T(x, t)}{\partial t} dx \\ &\quad + R(t) \frac{dR(t)}{dt} \left[\int_0^\xi (\xi - 2x)T(x, t) dx + \int_\xi^1 (2x - \xi)T(x, t) dx \right], \end{aligned} \quad (10)$$

where $\theta_0(t)$ and $\theta_1(t)$ are the flux functions defined by

$$\theta_0(t) = \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} \quad \text{and} \quad \theta_1(t) = \left. \frac{\partial T(x, t)}{\partial x} \right|_{x=1}. \quad (11)$$

Similar integro-differential equations were used to obtain numerical methods for solving one-dimensional heat and wave equations in Ang [1], [2].

The one-dimensional Stefan problem stated in Section 1 may now be reformulated as one which requires solving for $T(x, t)$, $\theta_0(t)$, $\theta_1(t)$ and $R(t)$ from (10) together with (2) subject to the initial conditions (3) and the boundary condition at $x = 0$ in (4). In view of (11), one may rewrite the Stefan condition (2) as

$$R(t) \frac{dR(t)}{dt} = -\text{Ste} \theta_1(t) \quad \text{for } t \geq 0. \quad (12)$$

3 Approximation of $T(x, t)$

As in Ang [1], [2], the temperature $T(x, t)$ is approximated using

$$T(x, t) \simeq \sum_{m=1}^N T_m(t) \sum_{n=1}^N c_{nm} \sigma_n(x), \quad (13)$$

where $T_m(t) = T(\xi_m, t)$, $\xi_1, \xi_2, \dots, \xi_{N-1}$ and ξ_N are N distinct well-spaced nodes selected from the interval $[0, 1]$ with $\xi_1 = 0$ and $\xi_N = 1$, $\sigma_n(x) = 1 + |x - \xi_n|^{3/2}$ is the local interpolating function centred about ξ_n and c_{nm} are constant coefficients defined by

$$\sum_{k=1}^N \sigma_n(\xi_k) c_{pk} = \begin{cases} 1 & \text{if } n = p, \\ 0 & \text{if } n \neq p. \end{cases} \quad (14)$$

Equation (14) implies that $[c_{pk}]$ is the inverse matrix of $[a_{ij}]$, where $a_{ij} = \sigma_j(\xi_i)$.

Note that the choice of the local interpolating function $\sigma_n(x)$ in (13) is not unique. The function $\sigma_n(x) = 1 + |x - \xi_n|^{3/2}$ may be regarded as a one-dimensional analogue of the local interpolating function proposed in Zhang and Zhu [16] for use in the dual-reciprocity boundary element method.

From (4), $T_N(t)$ is known, that is, $T_N(t) = 0$. In general, the functions $T_1(t), T_2(t), \dots, T_{N-2}(t)$ and $T_{N-1}(t)$ may be regarded as unknowns yet to be determined.

4 An initial-value problem

If one substitutes (13) into (10) and lets $\xi = \xi_r$ for $r = 1, 2, \dots, N$, one obtains

$$\begin{aligned} 2T_r(t) &= T_1(t) + \xi_r \theta_0(t) + (\xi_r - 1)\theta_1(t) \\ &\quad + R^2(t) \sum_{m=1}^{N-1} F_{rm} \frac{dT_m(t)}{dt} + R(t) \frac{dR(t)}{dt} \sum_{m=1}^{N-1} G_{rm} T_m(t) \end{aligned} \quad \text{for } r = 1, 2, \dots, N, \quad (15)$$

where

$$\begin{aligned} F_{rm} &= \sum_{n=1}^N c_{nm} \left(\frac{1}{2} [(1 - \xi_r)^2 + \xi_r^2] + \frac{2}{5} [(1 - \xi_r)(1 - \xi_n)^{5/2} + \xi_r \xi_n^{5/2}] \right. \\ &\quad \left. - \frac{4}{35} [(1 - \xi_n)^{7/2} + \xi_n^{7/2}] + \frac{8}{35} |\xi_r - \xi_n|^{7/2} \right), \end{aligned} \quad (16)$$

$$\begin{aligned} G_{rm} &= \sum_{n=1}^N c_{nm} \left(1 - \xi_r + \frac{4}{7} [(1 - \xi_n)^{7/2} - 2 |\xi_r - \xi_n|^{7/2}] \right. \\ &\quad \left. + \left(\frac{4}{5} \xi_n - \frac{2}{5} \xi_r \right) [(1 - \xi_n)^{5/2} - 2 \operatorname{sgn}(\xi_r - \xi_n) |\xi_r - \xi_n|^{5/2}] \right. \\ &\quad \left. - \frac{8}{35} \xi_n^{7/2} + \frac{2}{5} \xi_n^{5/2} \xi_r \right). \end{aligned} \quad (17)$$

Note that if $R(t)$ is known then (15) constitutes a system of N linear algebraic-differential equations containing $(N + 1)$ unknown functions of t . The unknown functions are $\theta_0(t)$, $\theta_1(t)$, $T_1(t)$, $T_2(t)$, \dots , $T_{N-2}(t)$ and $T_{N-1}(t)$. To obtain another equation, the boundary condition at $x = 0$ in (4) is written as

$$\alpha T_1(t) + \beta \theta_0(t) = f(t). \quad (18)$$

Thus, the Stefan problem under consideration is now approximately reduced to an initial-value problem which requires solving (12), (15) and (18) subject to

$$R(0) = 0 \text{ and } T_r(0) = 0 \text{ for } r = 1, 2, \dots, N - 1. \quad (19)$$

Note that (19) is obtained from (3). Mathematically, initial values of $\theta_0(t)$ and $\theta_1(t)$ are not required, as (15) does not contain any time derivative of these functions. Nevertheless, if needed, they may be deduced from the initial condition $T(x, 0) = 0$ in (3) to be given by $\theta_0(0) = 0$ and $\theta_1(0) = 0$.

5 Numerical method

The unknown functions $T_n(t)$ ($n = 1, 2, \dots, N - 1$) are approximated as cubic functions of time t over the interval $[\tau, \tau + 3\Delta t]$, that is (as in Ang [1]),

$$\begin{aligned} T_n(t) \simeq & \frac{1}{(\Delta t)^3} \left[-\frac{1}{6}(t - \tau - \Delta t)(t - \tau - 2\Delta t)(t - \tau - 3\Delta t)T_n(\tau) \right. \\ & + \frac{1}{2}(t - \tau)(t - \tau - 2\Delta t)(t - \tau - 3\Delta t)T_n(\tau + \Delta t) \\ & - \frac{1}{2}(t - \tau)(t - \tau - \Delta t)(t - \tau - 3\Delta t)T_n(\tau + 2\Delta t) \\ & \left. + \frac{1}{6}(t - \tau)(t - \tau - \Delta t)(t - \tau - 2\Delta t)T_n(\tau + 3\Delta t) \right] \\ & \text{for } t \in [\tau, \tau + 3\Delta t], \end{aligned} \quad (20)$$

Differentiation of (20) with respect to t gives

$$\begin{aligned}
\frac{dT_n(t)}{dt} &\simeq \frac{1}{(\Delta t)^3} \left[-\left(\frac{1}{2}[t-\tau]^2 - 2[t-\tau]\Delta t + \frac{11}{6}[\Delta t]^2\right)T_n(\tau) \right. \\
&\quad + \left(\frac{3}{2}[t-\tau]^2 - 5[t-\tau]\Delta t + 3[\Delta t]^2\right)T_n(\tau + \Delta t) \\
&\quad - \left(\frac{3}{2}[t-\tau]^2 - 4[t-\tau]\Delta t + \frac{3}{2}[\Delta t]^2\right)T_n(\tau + 2\Delta t) \\
&\quad \left. + \left(\frac{1}{2}[t-\tau]^2 - [t-\tau]\Delta t + \frac{1}{3}[\Delta t]^2\right)T_n(\tau + 3\Delta t) \right] \\
&\quad \text{for } t \in [\tau, \tau + 3\Delta t]. \tag{21}
\end{aligned}$$

If one lets $t = \tau + j\Delta t$ (for $j = 1, 2, 3$) in (15), after using (21), one obtains

$$\begin{aligned}
&2T_r(\tau + j\Delta t) - T_1(\tau + j\Delta t) \\
&= \xi_r\theta_0(\tau + j\Delta t) + (\xi_r - 1)\theta_1(\tau + j\Delta t) \\
&+ R(\tau + j\Delta t) \frac{dR}{dt} \Big|_{t=\tau+j\Delta t} \sum_{m=1}^{N-1} G_{rm}T_m(\tau + j\Delta t) \\
&+ \frac{R^2(\tau + j\Delta t)}{\Delta t} \sum_{m=1}^{N-1} F_{rm} \left[-\left(\frac{1}{2}j^2 - 2j + \frac{11}{6}\right)T_m(\tau) \right. \\
&\quad + \left(\frac{3}{2}j^2 - 5j + 3\right)T_m(\tau + \Delta t) - \left(\frac{3}{2}j^2 - 4j + \frac{3}{2}\right)T_m(\tau + 2\Delta t) \\
&\quad \left. + \left(\frac{1}{2}j^2 - j + \frac{1}{3}\right)T_m(\tau + 3\Delta t) \right] \\
&\quad \text{for } r = 1, 2, \dots, N \text{ and } j = 1, 2, 3. \tag{22}
\end{aligned}$$

In a similar manner, (18) gives

$$\alpha T_1(\tau + j\Delta t) + \beta \theta_0(\tau + j\Delta t) = f(\tau + j\Delta t) \text{ for } j = 1, 2, 3. \tag{23}$$

Integrating (12) with respect to t over the interval $[\tau, \tau + j\Delta t]$ (for $j = 1, 2, 3$) gives

$$R^2(\tau + j\Delta t) - R^2(\tau) = -2 \text{Ste} \int_{\tau}^{\tau+j\Delta t} \theta_1(t)dt \text{ for } j = 1, 2, 3. \tag{24}$$

If $\theta_1(t)$ is approximately given by

$$\begin{aligned}
\theta_1(t) \simeq & \frac{1}{(\Delta t)^3} \left[-\frac{1}{6}(t - \tau - \Delta t)(t - \tau - 2\Delta t)(t - \tau - 3\Delta t)\theta_1(\tau) \right. \\
& + \frac{1}{2}(t - \tau)(t - \tau - 2\Delta t)(t - \tau - 3\Delta t)\theta_1(\tau + \Delta t) \\
& - \frac{1}{2}(t - \tau)(t - \tau - \Delta t)(t - \tau - 3\Delta t)\theta_1(\tau + 2\Delta t) \\
& \left. + \frac{1}{6}(t - \tau)(t - \tau - \Delta t)(t - \tau - 2\Delta t)\theta_1(\tau + 3\Delta t) \right] \\
& \text{for } t \in [\tau, \tau + 3\Delta t], \tag{25}
\end{aligned}$$

then

$$\begin{aligned}
R^2(\tau + j\Delta t) - R^2(\tau) \simeq & -2 \text{Ste}(\Delta t) \left[\left(-\frac{1}{24}j^4 + \frac{1}{3}j^3 - \frac{11}{12}j^2 + j \right) \theta_1(\tau) \right. \\
& + \left(\frac{1}{8}j^4 - \frac{5}{6}j^3 + \frac{3}{2}j^2 \right) \theta_1(\tau + \Delta t) \\
& + \left(-\frac{1}{8}j^4 + \frac{2}{3}j^3 - \frac{3}{4}j^2 \right) \theta_1(\tau + 2\Delta t) \\
& \left. + \left(\frac{1}{24}j^4 - \frac{1}{6}j^3 + \frac{1}{6}j^2 \right) \theta_1(\tau + 3\Delta t) \right] \\
& \text{for } j = 1, 2, 3. \tag{26}
\end{aligned}$$

Letting $t = \tau + j\Delta t$ (for $j = 1, 2, 3$) in (12) gives

$$R(\tau + j\Delta t) \left. \frac{dR(t)}{dt} \right|_{t=\tau+j\Delta t} = -\text{Ste} \theta_1(\tau + j\Delta t) \quad \text{for } j = 1, 2, 3. \tag{27}$$

Assuming that $R(\tau)$, $\theta_1(\tau)$, $T_m(\tau)$ ($m = 1, 2, 3, \dots, N - 1$) are known, one may solve for the unknowns $R(\tau + j\Delta t)$, $\theta_i(\tau + j\Delta t)$ and $T_m(\tau + j\Delta t)$ ($j = 1, 2, 3$; $i = 0, 1$; $m = 1, 2, \dots, N - 1$) by using a predictor-corrector procedure which iterates between (22)-(23) and (26)-(27).

More specifically, the procedure starts off with an initial guess of $R^2(j\Delta t)$ and $R(j\Delta t)R'(j\Delta t)$ ($j = 1, 2, 3$). With this initial guess and $\tau = 0$, (22)-(23) may be solved as a system of $3(N + 1)$ linear algebraic equations for the

$3(N + 1)$ unknowns given by $\theta_i(j\Delta t)$ and $T_m(j\Delta t)$ ($j = 1, 2, 3$; $i = 0, 1$; $m = 1, 2, \dots, N - 1$). From (3), one may use $R(0) = 0$ and $T_m(0) = 0$ ($m = 1, 2, \dots, N - 1$) in (22)-(23) and (26)-(27) when $\tau = 0$. Once these unknowns are determined, $R(j\Delta t)R'(j\Delta t)$ and $R^2(j\Delta t)$ ($j = 1, 2, 3$) are calculated from (26)-(27) respectively, with $\tau = 0$, using the values of $\theta_1(j\Delta t)$ ($j = 1, 2, 3$) just obtained. The newly updated values of $R(j\Delta t)R'(j\Delta t)$ and $R^2(j\Delta t)$ ($j = 1, 2, 3$) are checked for convergence against those values from the initial guess. If the two sets of values do not agree to within a prescribed level, one returns to (22)-(23) (still with $\tau = 0$), solve again for $\theta_i(j\Delta t)$ and $T_m(j\Delta t)$ ($j = 1, 2, 3$; $i = 0, 1$; $m = 1, 2, \dots, N - 1$), applies (26)-(27) with the latest values of $\theta_1(j\Delta t)$ ($j = 1, 2, 3$) to recompute $R(j\Delta t)R'(j\Delta t)$ and $R^2(j\Delta t)$ ($j = 1, 2, 3$) respectively, and checks again for convergence in the values of $R(j\Delta t)R'(j\Delta t)$ and $R^2(j\Delta t)$ ($j = 1, 2, 3$). The iteration between (22)-(23) and (26)-(27) for $\tau = 0$ continues until the values of $R(j\Delta t)R'(j\Delta t)$ and $R^2(j\Delta t)$ ($j = 1, 2, 3$) converge to within the prescribed level.

The iterative process above may be repeated by letting $\tau = 3\Delta t$ and using $R^2(j\Delta t)$ and $R(j\Delta t)R'(j\Delta t)$ (as computed with $\tau = 0$) as starting values for $R^2((3 + j)\Delta t)$ and $R((j + 3)\Delta t)R'((j + 3)\Delta t)$ ($j = 1, 2, 3$) respectively, in order to solve for $R((j + 3)\Delta t)$, $\theta_i((j + 3)\Delta t)$ and $T_m((j + 3)\Delta t)$ ($j = 1, 2, 3$; $i = 0, 1$; $m = 1, 2, \dots, N - 1$). Once convergence is achieved for $\tau = 3\Delta t$, the process is repeated with $\tau = 6\Delta t, 9\Delta t, 12\Delta t, \dots$ (consecutively) to solve for the unknowns at higher and higher time levels.

6 Test problems

The numerical method proposed above is applied here to solve two specific test problems. In both problems, the predictor-corrector procedure which iterates between (22)-(23) and (26)-(27) is stopped once $R^2(t)$ and $R(t)R'(t)$

at the relevant time levels achieve a convergence of 9 significant figures.

Problem 1. In (4), take $\alpha = 1$, $\beta = -1$, $f(t) = -1 + \exp(t) + t \exp(t)$ and $\text{Ste} = 1$.

With $\text{Ste} = 1$, one may verify that the analytic solution to this problem is given by

$$T(x, t) = -1 + \exp(t[1 - x]) \text{ and } R(t) = t. \quad (28)$$

Table 1. A comparison of numerical values of $T(x, 0.90)$ with the exact solution at selected points (Problem 1).

x	$N = 11$ $\Delta t = 0.10$	$N = 51$ $\Delta t = 0.01$	Exact
0	1.459215	1.459591	1.459603
0.10	1.247512	1.247896	1.247908
0.20	1.054071	1.054422	1.054433
0.30	0.877280	0.877601	0.877611
0.40	0.715709	0.715998	0.716007
0.50	0.568046	0.568304	0.568312
0.60	0.433094	0.433322	0.433329
0.70	0.309759	0.309958	0.309965
0.80	0.197043	0.197212	0.197217
0.90	0.094025	0.094169	0.094174

In Table 1, the numerical values of the temperature $T(x, t)$ at selected points and at time $t = 0.90$, obtained using $N = 11$ and $\Delta t = 0.10$, are found to be in good agreement with the exact solution in (28). Convergence of the numerical values to the exact ones is obviously observed when the calculation is refined using $N = 51$ and $\Delta t = 0.01$, that is, significantly more accurate numerical values of the temperature are obtained when the

number of collocation points is increased by more than 4 times and the time-step reduced by ten times. To obtain the numerical values using $N = 11$ and $\Delta t = 0.10$, the predictor-corrector procedure requires no more than 19 iterations. For the refined calculation using $N = 51$ and $\Delta t = 0.01$, less than 11 iterations are needed. (Note that the criterion for convergence used here is rather stringent. At any particular time level, the functions $R^2(t)$ and $R(t)R'(t)$ are required to converge to at least 9 significant figures. If a less stringent criterion is used instead, much fewer iterations are required in the numerical calculation.)

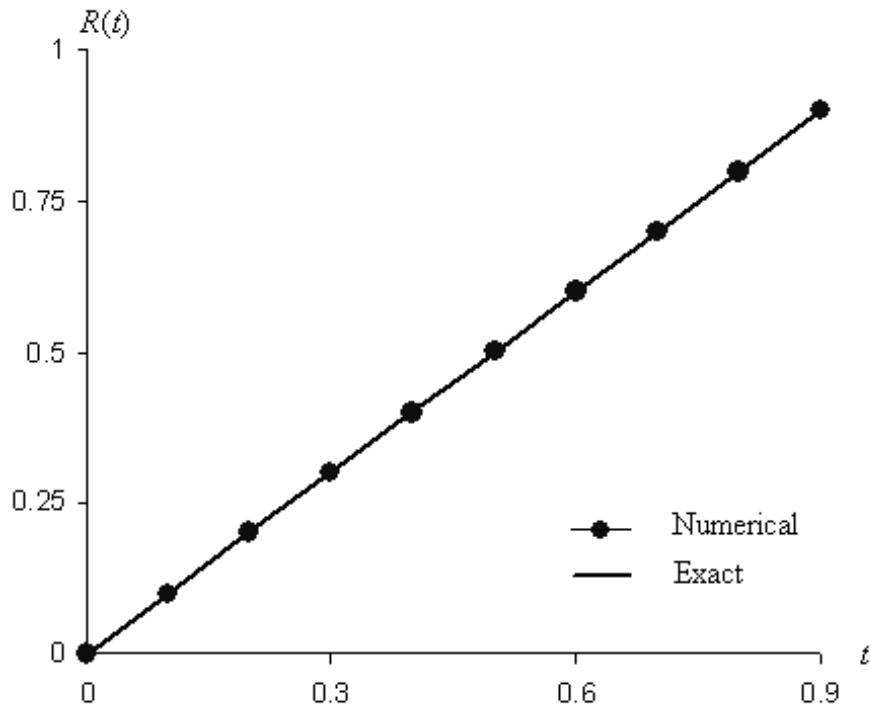


Figure 1. A graphical comparison between the numerical and the exact of the function $R(t)$ (which describes the moving front) over the time interval $0 \leq t \leq 0.90$ (Problem 1).

A graphical comparison between the numerical and the exact $R(t)$ over the interval $0 \leq t \leq 0.90$ is made in Figure 1. The numerical $R(t)$ is calculated using $N = 21$ and $\Delta t = 0.10$. Since the numerical and the exact values agree to at least 4 significant figures, the two graphs in Figure 1 are visually indistinguishable.

Problem 2. In (4), take $\alpha = 1$, $\beta = 0$ and $f(t) = 1 - \exp(-t)$.

For this particular problem, no analytic solution is apparently available, but it may be shown that (Özişik [12])

$$T(x, t) \simeq 1 - \frac{\operatorname{erf}(\lambda x)}{\operatorname{erf}(\lambda)} \text{ and } R(t) \simeq \sqrt{4\lambda^2(t - t_{\text{large}}) + R^2(t_{\text{large}})} \text{ for } t \geq t_{\text{large}}, \quad (29)$$

where $\operatorname{erf}(x)$ is the error function, t_{large} is sufficiently large positive number and the value of λ is obtained from

$$\sqrt{\pi}\lambda \exp(\lambda^2) \operatorname{erf}(\lambda) = \text{Ste}. \quad (30)$$

For the purpose of carrying out numerical calculation, the Stefan number Ste is taken to be 1. (With Ste = 1, the constant λ in (29), obtained from solving (30) numerically, is given by 0.620063.) The calculation is carried out using $N = 21$ and $\Delta t = 0.05$. Less than 14 iterations are required to satisfy the criterion for convergence in the predictor-corrector procedure. A plot of the numerical $T(0.50, t)$ against t over the interval $0 \leq t \leq 6.0$ is given in Figure 2. As pointed out earlier on, the problem does not have any known exact solution. One may view the first formula in (29) as a time-independent asymptotic solution which $T(x, t)$ should approach as time t increases. According to (29), the numerical value of $T(0.50, t)$ should tend to the asymptotic value 0.452845 as $t \rightarrow \infty$. This is observed in Figure 2.

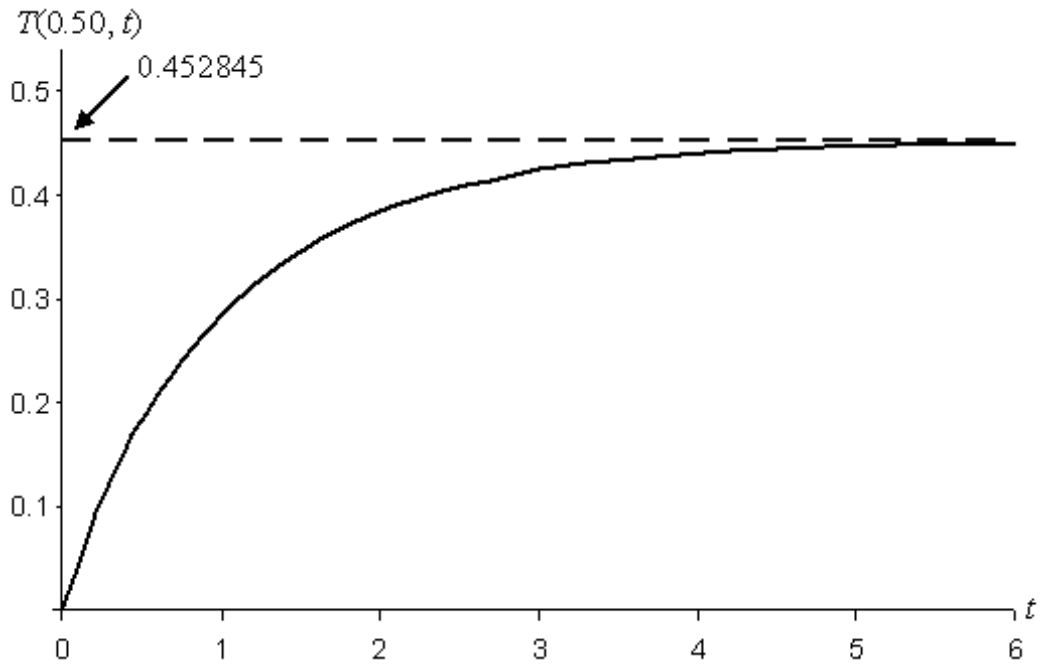


Figure 2. A plot of the numerical temperature $T(0.50, t)$ (solid line) and the large time asymptotic solution in (29) (dashed line) over the time interval $0 \leq t \leq 6$ (Problem 2).

To check the numerical $R(t)$ against the approximate formula in (29), t_{large} is selected to be 6.0. From the numerical calculation itself, $R(t_{\text{large}}) = R(6)$ is found to be 2.775477336 (approximately). A graphical comparison of the numerical $R(t)$ and the one given in (29) is given over the time interval $6.0 \leq t \leq 12.0$ in Figure 3. Over the given time interval, the two sets of approximate values of $R(t)$ agree to least 3 significant figures.

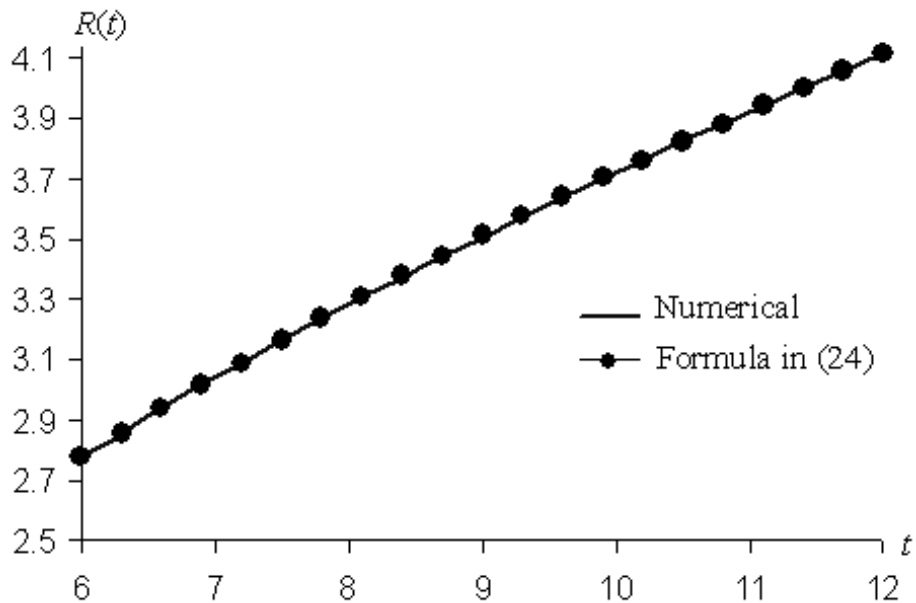


Figure 3. A graphical comparison between the numerical $R(t)$ and the asymptotic formula in (29) over the interval $6 \leq t \leq 12$ (Problem 2).

7 Final remarks

The partial differential equation (1) which governs the one-dimensional Stefan problem under consideration here is reduced to the integro-differential equation (10). The integro-differential equation is used to derive a numerical procedure for solving the Stefan problem. No approximation of the boundary heat flux is necessary. If the boundary heat flux is not known, it appears as an unknown function of time to be solved directly. Furthermore, the integro-differential formulation does not contain any spatial derivative of the temperature in the interior of the solution domain. It is not necessary to approximate any spatial derivative of the temperature through the use of finite-difference formulae.

The numerical procedure is implemented on the computer to solve some

test problems. Numerical results obtained indicate that the numerical procedure can be used to obtain accurate solution for the Stefan problem. Other numerical results like those in Table 1 and Figures 1, 2 and 3 have also been obtained for other values of the Stefan number but are not presented here. A more refined computation (with higher number N of collocation points and smaller time-step Δt) is necessary to achieve a certain level of accuracy for a larger value of the Stefan number Ste .

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