Notes on Cauchy principal and Hadamard finite-part integrals
by W. T. Ang
Original version on 11 May 2009
Revised and corrected on 12 July 2014
(In the original version, a term is missing in the alternative definition for the Hadamard finite-part integrals. This is now corrected in (12) below.)

Cauchy principal integrals

Assume that $f(t)$ is well-defined over $a \leq t \leq b$. Cauchy principal integrals defined by

$$\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt = \lim_{\epsilon\to 0^+}(\int_a^{x-\epsilon} \frac{f(t)}{t-x}dt + \int_{x+\epsilon}^b \frac{f(t)}{t-x}dt) \text{ for } a < x < b \quad (1)$$

arise in the formulation of many problems in engineering science.

For convenience in our discussion, let us assume that the function $f(t)$ $a \leq t \leq b$ can be expanded as a Taylor series about $t = x$, that is, it can be written as

$$\begin{aligned} f(t) &= f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \cdots \\ &= f(x) + \sum_{m=1}^{\infty} \frac{f^{(m)}(x)}{m!}(t-x)^m \text{ for } a \leq t \leq b. \end{aligned} \quad (2)$$

Substituting (2) into the right hand side of (1), we find that

$$\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt = f(x)\ln|\frac{b-x}{a-x}| + \sum_{m=1}^{\infty} \frac{f^{(m)}(x)}{m!m}\{(b-x)^m - (a-x)^m\}. \quad (3)$$

Consider now the limit

$$\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)f(t)}{(t-x)^2+\epsilon^2}dt.$$

Using (2), we find that

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)f(t)}{(t-x)^2+\epsilon^2}dt \\
= &\lim_{\epsilon\to 0^+}\{f(x)\int_a^b \frac{(t-x)}{(t-x)^2+\epsilon^2}dt + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b \frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt\} \\
= &\lim_{\epsilon\to 0^+}\{\frac{1}{2}f(x)[\ln|(t-x)^2+\epsilon^2|]_{t=a}^{t=b} + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b \frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt\} \\
= &f(x)\ln|\frac{b-x}{a-x}| + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b \lim_{\epsilon\to 0^+}\frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt \\
= &f(x)\ln|\frac{b-x}{a-x}| + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!m}\{(b-x)^m-(a-x)^m\} \\
= &\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt.
\end{aligned}$$

Note that we can write

$$\lim_{\epsilon\to 0^+}\sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b \frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt = \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b \lim_{\epsilon\to 0^+}\frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt$$

because the integrand is well-defined over $a \le t \le b$ as $\epsilon \to 0^+$.

It is also possible to work out the integral

$$\int_a^b \frac{(t-x)^{m+1}}{(t-x)^2+\epsilon^2}dt$$

first before letting $\epsilon \to 0^+$ to obtain the same final result.

Thus, we obtain an alternative but equivalent definition for the Cauchy principal integrals, that is,

$$\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt = \lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)f(t)}{(t-x)^2+\epsilon^2}dt. \tag{4}$$

Hadamard finite-part integrals

Let us write

$$F(x) = \mathcal{C}\int_a^b \frac{f(t)}{t-x}dt.$$

If we use (3), we find that

$$\begin{aligned} F'(x) &= f(x)[-\frac{1}{b-x} + \frac{1}{a-x}] \\ &\quad + f'(x) \ln |\frac{b-x}{a-x}| \\ &\quad + \sum_{m=2}^{\infty} \frac{f^{(m)}(x)}{m!(m-1)} [(b-x)^{m-1} - (a-x)^{m-1}]. \end{aligned} \tag{5}$$

An interesting question is, "What can we make of the terms on the right hand side of (5)?"

To answer the question, let us consider

$$\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2} dt + \int_{x+\epsilon}^{b} \frac{f(t)}{(t-x)^2} dt. \tag{6}$$

On using (2), we obtain

$$\begin{aligned} & \int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2} dt + \int_{x+\epsilon}^{b} \frac{f(t)}{(t-x)^2} dt \\ = \; & \frac{2}{\epsilon} f(x) + f(x)[-\frac{1}{b-x} + \frac{1}{a-x}] \\ & + f'(x) \ln |\frac{b-x}{a-x}| \\ & + \sum_{m=2}^{\infty} \frac{f^{(m)}(x)}{m!(m-1)} [(b-x)^{m-1} - (a-x)^{m-1}]. \end{aligned} \tag{7}$$

The expression in (6) contains two parts – one whose magnitude blows up to infinity and the other that remains finite in magnitude as $\epsilon \to 0^+$. The part that remains finite in magnitude is given by the right hand side of (5) (that is, by $F'(x)$). Thus, we conclude

$$\frac{d}{dx}[\mathcal{C} \int_a^b \frac{f(t)}{t-x} dt] = \text{finite part of } \lim_{\epsilon \to 0^+} [\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2} dt + \int_{x+\epsilon}^{b} \frac{f(t)}{(t-x)^2} dt].$$

If we define

$$\mathcal{H} \int_a^b \frac{f(t)}{(t-x)^2} dt = \text{finite part of } \lim_{\epsilon \to 0^+} [\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2} dt + \int_{x+\epsilon}^{b} \frac{f(t)}{(t-x)^2} dt], \tag{8}$$

then we may write

$$\frac{d}{dx}[\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt] = \mathcal{H}\int_a^b \frac{f(t)}{(t-x)^2}dt, \tag{9}$$

or

$$\frac{d}{dx}[\mathcal{C}\int_a^b \frac{f(t)}{t-x}dt] = \mathcal{H}\int_a^b f(t)\frac{\partial}{\partial x}[\frac{1}{(t-x)}]dt. \tag{10}$$

Perhaps (8) may be better rewritten as

$$\begin{aligned}\mathcal{H}\int_a^b \frac{f(t)}{(t-x)^2}dt &= \lim_{\epsilon\to 0^+}[\int_a^{x-\epsilon} \frac{f(t)}{(t-x)^2}dt + \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^2}dt - \frac{2}{\epsilon}f(x)] \\ &\text{for } a < x < b. \end{aligned} \tag{11}$$

An alternative definition for the Hadamard finite-part integrals is given by[1]

$$\begin{aligned}\mathcal{H}\int_a^b \frac{f(t)dt}{(t-x)^2} &= \lim_{\epsilon\to 0^+}[\int_a^b \frac{(t-x)^2 f(t)dt}{[(t-x)^2+\epsilon^2]^2} - \frac{\pi}{2\epsilon}f(x)] \\ &-\frac{f(x)}{2}[\frac{1}{b-x} - \frac{1}{a-x}] \text{ for } a < x < b. \end{aligned} \tag{12}$$

The alternative definition (12) can be easily verified as follows. If we use (2) and

$$\begin{aligned}\int \frac{(t-x)^2dt}{((t-x)^2+\epsilon^2)^2} &= -\frac{t-x}{2\left((t-x)^2+\epsilon^2\right)} + \frac{1}{2\epsilon}\arctan(\frac{1}{2}\frac{2t-2x}{\epsilon}), \\ \int \frac{(t-x)^3dt}{((t-x)^2+\epsilon^2)^2} &= \frac{\epsilon^2}{2\left((t-x)^2+\epsilon^2\right)} + \frac{1}{2}\ln\left((t-x)^2+\epsilon^2\right), \end{aligned} \tag{13}$$

---

[1] This alternative definition is suggested in *Engineering Analysis with Boundary Elements* **23** (1999) 713-720 by WT Ang and DL Clements, but the term $-\frac{f(x)}{2}[\frac{1}{b-x} - \frac{1}{a-x}]$ on the second line of (12) is missing in the paper.

we find that

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^2 f(t)dt}{[(t-x)^2+\epsilon^2]^2} &= f(x)\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^2 dt}{[(t-x)^2+\epsilon^2]^2} \\
&\quad + f'(x)\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^3 dt}{[(t-x)^2+\epsilon^2]^2} \\
&\quad + \sum_{m=1}^{\infty}\frac{f^{(m+1)}(x)}{(m+1)!}\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^{3+m} dt}{[(t-x)^2+\epsilon^2]^2} \\
&= \lim_{\epsilon\to 0^+}\frac{\pi}{2\epsilon}f(x) - \frac{f(x)}{2(b-x)} + \frac{f(x)}{2(a-x)} \qquad (14) \\
&\quad + f'(x)\{\ln|b-x| - \ln|a-x|\} \\
&\quad + \sum_{m=1}^{\infty}\frac{f^{(m+1)}(x)}{(m+1)!}\int_a^b (t-x)^{m-1}dt.
\end{aligned}$$

Examples

$$\begin{aligned}
\mathcal{C}\int_{-1}^{1}\frac{1}{t}dt &= \lim_{\epsilon\to 0^+}(\int_{-1}^{-\epsilon}\frac{1}{t}dt + \int_{+\epsilon}^{1}\frac{1}{t}dt) \\
&= \lim_{\epsilon\to 0^+}(\ln|-\epsilon| - \ln|-1| + \ln|1| - \ln|\epsilon|) = 0
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}\int_0^1 \frac{t}{t-\frac{1}{4}}dt &= \int_0^1 dt + \frac{1}{4}\mathcal{C}\int_0^1 \frac{1}{t-\frac{1}{4}}dt \\
&= 1 + \frac{1}{4}\lim_{\epsilon\to 0^+}(\int_0^{1/4-\epsilon}\frac{1}{t-\frac{1}{4}}dt + \int_{1/4+\epsilon}^{1}\frac{1}{t-\frac{1}{4}}dt) = 1 + \frac{1}{4}\ln(3)
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}\int_{-1}^{1}\frac{1}{t^2}dt &= \text{finite part of } \lim_{\epsilon\to 0^+}(\int_{-1}^{-\epsilon}\frac{1}{t^2}dt + \int_{+\epsilon}^{1}\frac{1}{t^2}dt) \\
&= \text{finite part of } \lim_{\epsilon\to 0^+}(\frac{2}{\epsilon} - 2) = -2.
\end{aligned}$$

Hypersingular integral equations for a simple crack problem

We will now show how elastic crack problems may be formulated in terms of equations containing Hadamard finite-part integrals. For clarity, let us consider a mode III crack problem that requires us to solve the two-dimensional

Laplace's equation for $\phi(x,y)$ on the whole of the Oxy plane containing a finite cut (a crack) in the region $-a < x < a$, $y = 0$. The solution $\phi(x,y)$ is required to satisfy the following conditions:

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0^{\pm}} = q(x) \text{ for } -a < x < a, \tag{15}$$

$$\phi \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty, \tag{16}$$

where $q(x)$ is a given function.

If $\phi(x,y)$ satisfies the 2D Laplace's equation in the region R bounded by a simple closed curve C, the boundary integral equation for the 2D Laplace's equation is

$$\begin{aligned} \phi(\xi,\eta) &= \int\limits_C [\phi(x,y)\Lambda(x,y;\xi,\eta) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ &\quad \text{for } (\xi,\eta) \in R, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \frac{\partial}{\partial n}(\phi(x,y)) &= n_1(x,y)\frac{\partial \phi}{\partial x} + n_2(x,y)\frac{\partial \phi}{\partial y}, \\ \Phi(x,y;\xi,\eta) &= \frac{1}{4\pi}\ln([x-\xi]^2 + [y-\eta]^2), \\ \Lambda(x,y;\xi,\eta) &= \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ &= \frac{[x-\xi]n_1(x,y) + [y-\eta]n_2(x,y)}{2\pi([x-\xi]^2 + [y-\eta]^2)}, \end{aligned} \tag{18}$$

where $[n_1(x,y), n_2(x,y)]$ is the outward unit normal vector to C at the point (x,y).

Here we take the boundary C to comprise two parts: the boundary at infinity denoted by C_∞ and the crack L. The crack L consists of two opposite faces and the function $\phi(x,y)$ may jump across opposite crack faces. We may write (17) as

$$
\begin{aligned}
\phi(\xi,\eta) &= \int\limits_{C_\infty}[\phi(x,y)\Lambda(x,y;\xi,\eta)-\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\
&\quad +\int_{-a}^{a}[\phi(x,0^+)\Lambda(x,0^+;\xi,\eta)+\phi(x,0^-)\Lambda(x,0^-;\xi,\eta) \\
&\quad +\Phi(x,0^+;\xi,\eta)\left.\frac{\partial}{\partial y}(\phi(x,y))\right|_{y=0^+} \\
&\quad -\Phi(x,0^-;\xi,\eta)\left.\frac{\partial}{\partial y}(\phi(x,y))\right|_{y=0^-}]dx \qquad (19)
\end{aligned}
$$

for (ξ,η) lying in the interior of the Oxy plane with a finite cut at $-a<x<a$, $y=0$.

In view of the far field condition in (16), if we assume that $\phi(x,y)$ behaves as $O([x^2+y^2]^{-\alpha}$ $(\alpha>0)$ for large x^2+y^2, then we can show that

$$
\int\limits_{C_\infty}[\phi(x,y)\Lambda(x,y;\xi,\eta)-\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)=0. \qquad (20)
$$

From (18), $\Phi(x,0^+;\xi,\eta)=\Phi(x,0^-;\xi,\eta)$ and $\Lambda(x,0^+;\xi,\eta)=-\Lambda(x,0^-;\xi,\eta)$. Noting (15) and (20), we may now reduce (19) to

$$
\phi(\xi,\eta)=\int_{-a}^{a}\frac{\eta\Delta\phi(x)dx}{2\pi([x-\xi]^2+\eta^2)}, \qquad (21)
$$

where $\Delta\phi(x)=\phi(x,0^+)-\phi(x,0^-)$ for $-a<x<a$.

If we expand $\Delta\phi(x)$ as a Taylor series about $x=\xi$, we find that, for $-a<\xi<a$,

$$
\begin{aligned}
\phi(\xi,0^+) &= \lim_{\eta\to0^+}\int_{-a}^{a}\frac{\eta\Delta\phi(x)dx}{2\pi([x-\xi]^2+\eta^2)}=\frac{1}{2}\Delta\phi(\xi), \\
\phi(\xi,0^-) &= \lim_{\eta\to0^-}\int_{-a}^{a}\frac{\eta\Delta\phi(x)dx}{2\pi([x-\xi]^2+\eta^2)}=-\frac{1}{2}\Delta\phi(\xi), \qquad (22)
\end{aligned}
$$

which leads to $\Delta\phi(\xi)=\phi(\xi,0^+)-\phi(\xi,0^-)$. For $\xi\notin[-a,a]$, note that the integrands of the integrals in (22) are not singular and hence $\phi(\xi,0^+)=\phi(\xi,0^-)$, that is, $\phi(x,y)$ is continuous on the uncracked part of the plane $y=0$.

From (21), we obtain

$$\frac{\partial}{\partial\eta}[\phi(\xi,\eta)] = \int_{-a}^{a} \frac{[x-\xi]^2\Delta\phi(x)dx}{2\pi([x-\xi]^2+\eta^2)^2} - \eta^2\int_{-a}^{a} \frac{\Delta\phi(x)dx}{2\pi([x-\xi]^2+\eta^2)^2}. \quad (23)$$

If we expand the function $\Delta\phi(x)$ in the second integral on the right hand side of (23) as a Taylor series about $x = \xi$, we find that

$$\begin{aligned} \left.\frac{\partial}{\partial\eta}[\phi(\xi,\eta)]\right|_{\eta=0^+} &= \frac{1}{2\pi}\lim_{\eta\to 0^+}\{\int_{-a}^{a}\frac{[x-\xi]^2\Delta\phi(x)dx}{([x-\xi]^2+\eta^2)^2} - \frac{\pi\Delta\phi(\xi)}{2\eta} \\ &\quad -\frac{\Delta\phi(\xi)}{2}[\frac{1}{b-\xi} - \frac{1}{a-\xi}] + O(\eta)\} \\ &\quad \text{for } -a<\xi<a, \end{aligned} \quad (24)$$

which may be rewritten as

$$\left.\frac{\partial}{\partial\eta}[\phi(\xi,\eta)]\right|_{\eta=0^+} = \frac{1}{2\pi}\mathcal{H}\int_{-a}^{a}\frac{\Delta\phi(x)dx}{[x-\xi]^2} \text{ for } -a<\xi<a, \quad (25)$$

if we take (12) into consideration.

Thus, the condition on the crack in (15) gives rise to

$$\frac{1}{2\pi}\mathcal{H}\int_{-a}^{a}\frac{\Delta\phi(x)dx}{[x-\xi]^2} = q(x) \text{ for } -a<\xi<a, \quad (26)$$

a Hadamard finite-part (hypersingular) integral equation with $\Delta\phi(x)$ (for $-a<x<a$) as an unknown function to be determined. For crack problems, the unknown function $\Delta\phi(x)$ takes the form $\Delta\phi(x) = \sqrt{a^2-x^2}\psi(x)$ (for $-a<x<a$) and for given $q(x)$, it may be possible to invert (26) to obtain analytically. Even if we do not know how to invert (26) analytically, there are numerical methods[2] for determining $\psi(x)$ from the hypersingular integral equation.

[2]See, for example, Kaya A, Erdogan F, On the solution of integral equations with strongly singular kernels, *Quarterly of Applied Mathematics* **45** (1987) 105-122.