LIMITS ON JUMP INVERSION FOR STRONG REDUCIBILITIES

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Abstract. We show that Sacks’ and Shoenfield’s analogs of jump inversion fail for both tt- and wtt-reducibilities in a strong way. In particular we show that there is a \( \Delta^0_2 \) set \( B >_{tt} \emptyset' \) such that there is no c.e. set \( A \) with \( A' \equiv_{wtt} B \). We also show that there is a \( \Sigma^0_2 \) set \( C >_{tt} \emptyset' \) such that there is no \( \Delta^0_2 \) set \( D \) with \( D' \equiv_{wtt} C \).

§1. Introduction. The concern of this paper is the interaction of two basic notions from computability theory. These are the jump operator and reducibilities stronger than Turing reducibility which are of the tabular type. We answer a question of Anderson [And08] by showing that there are no analogs of Sacks Jump Inversion Theorem [Sac63] and Shoenfield’s Jump Inversion Theorem [Sho59] for these strong reducibilities.

The study of strong reducibilities has been part of computability since the dawn of the subject, as witnessed by Post’s paper [Pos44]. \( A \) is Turing reducible to \( B \), \( A \leq_T B \), means that \( A \) can be computed by \( B \) via any oracle access mechanism. It is clearly natural to ask what happens when we restrict the access mechanism in the reduction from \( A \) to \( B \). Tabular reducibilities such as weak truth table (wtt-) and truth table (tt-) reducibilities do not allow the reducibility to be adaptive. Thus, as is well known, \( A \leq_{tt} B \), is defined as \( x \in A \) iff \( B \models \sigma_f(x) \) where \( f \) is a computable function and \( \sigma_f(x) \) is the \( f(x)^{th} \) truth table. As is also well known a truth table reduction is simply a Turing reduction \( \Phi \) which is total for all oracles. Weak truth table reducibility simply has the truth table being partial, or \( \Phi^B = A \) where the use of the computation \( \varphi(x) \) is a computable function. Thus in either case we are not allowed to adapt the size of the reduction as the oracle \( B \) varies. \( A \leq_{tt} B \) implies \( A \leq_{wtt} B \) but it is easy to construct examples where the converse fails.

These reducibilities also arise very naturally when we consider reducibilities coming from reductions in mathematical structures. For example, the reduction of the word problem to the conjugacy problem in combinatorial group theory is a tt-reduction and the degrees of bases of c.e. vector spaces are naturally represented by weak truth table degrees (Downey and Remmel [DR89]).
In recent times, truth table reducibility has become a central area of interest as it has been shown to be a natural reducibility to study in algorithmic randomness, a fact first realized by Demuth [Dem88]. The point here is that if $A \leq_{tt} B$, via $\Phi^B = A$, with $\Phi$ total on all oracles, then we can use $\Phi$ to translate between measures effectively. For instance if $B$ is random with respect to uniform measure, and $A$ is noncomputable, $A$ will be random with respect to the measure generated by the inverse of $\Phi$. Thus, for instance, truth table degrees are absolutely central to the deep investigations of Reimann and Slaman [RS08a, RS08b] on sets never continuously random. They are also deeply connected with things like the Cantor-Bendixson rank of sets for a similar reason.

All of this recent work has highlighted our lack of understanding as to how the finer structure of the (w)tt-degrees relates to the jump operator. The halting problem is a fundamental object of computability theory, and the jump $A' = \{e : \Phi^e_A(e) \downarrow\}$ is the relativized form of the halting problem.

For Turing reducibility, we know a lot about how the jump operator behaves. The most basic theorem is Friedberg’s Jump Inversion Theorem [Fri58], that if $X \geq_T \emptyset'$ then there is a set $A$ with $A' \equiv_T X \equiv_T \emptyset \oplus A$. Early on, Mohrherr [Moh84] proved that if $X \geq_{tt} \emptyset'$ then there is a set $A$ with $A' \equiv_{tt} X$. Mohrherr’s proof came from an analysis of Friedberg’s Theorem, and resulted in a 1-generic set $A$. It was only much later that Anderson [And08] proved that indeed the full analog of Friedberg’s Theorem held: if $X \geq_{tt} \emptyset'$ then there is a set $A$ with $A' \equiv_{tt} X \equiv_{tt} A \oplus \emptyset'$. Anderson’s theorem was more difficult than Mohrherr’s, and the method employed by Friedberg (which will give generics sets) provably fails, so that arguments akin to those from information theory were necessary.

All of this led to the present paper. The most important sets in computability theory are the c.e. sets as well as those computable from the halting problem, the $\Delta^0_2$ sets. Shoenfield [Sho59] proved a jump theorem for such sets. Namely for any $\Sigma^0_2$ set $X \geq_T \emptyset'$ there is a $\Delta^0_2$ set $A$ with $A' \equiv_T X$. Famously, Sacks used the infinite injury method to show that the same result held with $A$ a computably enumerable set, and after that many other intricate jump theorems were found culminating in Robinson’s Jump Interpolation Theorem [Rob71]. (See Soare [Soa87] for more details.)

Anderson asked: do the analogs of any of these basic theorems hold for tt- or perhaps wtt-reducibilities? We prove that the analogs fail to hold and in fact that they fail in more or less the strongest way that they can. Our first result shows that Sacks’ Jump Inversion Theorem fails for both the tt- and wtt-reducibilities, by constructing a $\Delta^0_2$ counter-example. We will in fact prove something stronger:

**Theorem 3.3.** For any computable sequence of $\Delta^0_2$ sets $\{V_e\}_{e \in \mathbb{N}}$ (given by their $\Delta^0_2$ indices), there exists a $\Delta^0_2$ set $S \geq_{tt} \emptyset'$ such that for every $e$, $V_e' \not\equiv_{wtt} S$.

Using Theorem 3.3 we can easily show the failure of Sacks’ Jump Inversion for both tt- and wtt-reducibilities:

**Theorem 3.4.** There exists an $\omega + 1$-c.e. set $S >_{tt} \emptyset'$ such that there is no c.e. set $A$ with $A' \equiv_{wtt} S$.

We remind the reader what an $\omega + 1$-c.e. set is. $S \leq_T \emptyset'$ is said to be $\omega + 1$-c.e., if there exists a computable approximation $g : \omega^2 \mapsto \{0, 1\}$ of $S$ and a partial computable function $h : \omega \mapsto \omega$ so that for every $x$, $\lim_{s \to \infty} g(x, s) = S(x)$ and
the number of mind changes of \( g(x, -) \) is either 0, or else \( h(x) \downarrow \) and bounds the number of mind changes of \( g(x, -) \).

That is, the set \( S \) belongs to the first level of the Ershov hierarchy for which such a counter-example is not immediately ruled out by existing results. The result also gives an interesting fact about the \( \Delta^0_1 \) wtt-degrees which are realized as the jump of a low c.e. set. Clearly there are such wtt-degrees \( a > \theta'_{wtt} \), namely the wtt-degrees which are the jump of a low but not superlow c.e. set. Our result shows that not every \( \Delta^0_1 \) wtt-degree \( a > \theta'_{wtt} \) can be realized by the jump of a (low) c.e. set.

Our second result shows that the analogue of Shoenfield’s Jump Inversion Theorem fails for both the tt- and wtt-reducibilities. By Mohrherr’s result, the counter-example \( S \) has to be strictly \( \Sigma^0_2 \):

**Theorem 3.5.** There exists a \( \Sigma^0_2 \) set \( S >_R \emptyset' \) such that there is no \( \Delta^0_2 \) set \( A \) with \( A' \equiv_{wtt} S \).

**1.1. Notation.** We follow standard notation for Computability Theory, as found in Cooper [Coo04] and Soare [Soa87].

### §2. The basic module.

**2.1. The plan for the c.e. case.** Let \( (\Gamma_e, \Delta_e, \gamma_e, \delta_e)_{e \in \omega} \) run through all possible 4-tuples where \( \Gamma_e \) and \( \Delta_e \) are Turing functionals, and \( \gamma_e \) and \( \delta_e \) are partial computable functions. Let us suppose we wanted to prove Theorem 3.4 directly by constructing \( S \). We must then meet for all \( e \in \omega \) the requirements:

\[
R_e: \Gamma^{V_e}_{e} \neq S \lor \Delta^{S \oplus \emptyset'}_{e} \neq V_e.
\]

where \( V_e \) is the \( e^{th} \) c.e. set, and \( \gamma_e \) and \( \delta_e \) bound the uses of the computations of \( \Gamma_e \) and \( \Delta_e \), respectively. Then \( S \oplus \emptyset' \) will be the desired set. Note that the requirements automatically ensure that \( S \oplus \emptyset' \not\equiv_{wtt} \emptyset' \).

Suppose we wanted to satisfy \( R_e \). We can first try making \( \Delta^{S \oplus \emptyset'} \neq V_e \) (for the purpose of the discussion we drop subscript \( e \)). In particular we assume that the recursion theorem gives us infinitely many indices \( x_1, x_2, \ldots \) for which we can control \( V_e(x_i) \). The obvious plan is to keep \( V_e(x_i) = 0 \) until \( \Delta^{S \oplus \emptyset'}_{e}(x_i) \downarrow = 0 \). We then make \( V_e(x_i) = 1 \) by enumerating an axiom with some use \( V_{e} \mid u \). The only way in which \( \Delta^{S \oplus \emptyset'}_{e}(x_i) \) can later change to be 1. is for some number \( < \delta(x_i) \) to enter \( \emptyset' \). Our next step would be then to extract \( x_1 \) out from \( V_e \); if we could always do this then we would know what to do. We would alternate the value of \( V_e(x_i) \), and we will eventually succeed because \( \emptyset' \) is c.e. and the use \( \delta(x_i) \) is fixed. Unfortunately we only have partial control over \( V' \) and extraction can only be achieved by forcing a change in \( V \mid u \).

We can start another line of attack by trying to make \( \Gamma^{V'} \neq S \) true. We pick a follower \( z \) for \( S \), and for simplicity let us first consider the case where \( \Gamma \) is an \( m \)-reduction: i.e., \( z \in S \) iff \( g \in V' \) for some \( q \). We begin by making \( S(z) = 1 \), and wait for \( V'(q) = 1 \), i.e., \( \Phi'_q(q) \downarrow \) with some use \( u \). Note that while the uses on \( \Delta \) and \( \Gamma \) are bounded, this use \( u \) may be unbounded. We then begin the attack above by first waiting for \( \Delta^{S \oplus \emptyset'}_{e}(x_i) \downarrow = 0 \). We then enumerate an axiom \( \Phi'_{e_i}(x_i) \) with the same \( V' \)-use \( u \), and wait for a \( \emptyset' \)-change below \( \delta(x_i) \).
If no $\emptyset'$-change occurs then it is clear that we would succeed at $\Delta^0 \oplus \emptyset'(x_1) \neq V'(x_1)$. If on the other hand a $V'\upharpoonright u$ change occurs before a $\emptyset'$-change, then we would wait for $\Phi^{V'}_q(q) \downarrow$ again with a new use $u'$, and then make $\Phi^{V'}_{x_1}(x_1)$ converge with the same use $u'$. The point is that if $V$ changes infinitely often this way with no $\emptyset'$-change, then $V'(q) = 0$ and we would succeed via $\Gamma^{V'}(z_e) \neq S(z)$. Lastly if $\emptyset'$ changes then we would remove $z_e$ from $S$, and wait for $\Phi^{V'}_q(q)$ to become undefined again. This has to happen (unless already $\Gamma^{V'} \neq S$), and so at some point we will also get a divergence $\Phi^{V'}_{x_1}(x_1) \uparrow$. We can then repeat by making $S(z_e) = 1$ again.

Note that we only toggle $z_e$ in $S$ whenever $\emptyset'$ changes below $\emptyset$-use. Each time we toggle $z_e$ we will force the configuration $V'\upharpoonright \emptyset(z_e)$ to change. This can only move lexicographically right finitely many times (consecutively), hence after finitely much toggling of $z_e$, the configuration for $V'\upharpoonright \emptyset(z_e)$ will return to an earlier one. This makes all the $x_e$ (for all the $\tau$ on the right of the current $V'_\emptyset$-configuration) undefined, so that if $\tau \subset V'_\emptyset[s]$ holds again later we can use $x_e$ to cause further $\emptyset'$-changes.

The above works when diagonalizing against all c.e. sets. However if $V$ is $\Delta^0_2$ then whenever the configuration of $V'\upharpoonright \emptyset(z_e)$ returns to an earlier one, there is no guarantee that all $x_e$ (for $\tau$ on the right of the current $V'$-configuration) become undefined. However we can show that some amount of progress has been made because in this case, $V$ has to return to a previous $x_e$ axiom, and thus we will threaten $V$ to be not $\Delta^0_2$.

From the above discussion, the reader will notice that the different requirements act almost independently of one another. In fact all that a single requirement needs to know is the correct initial segment of $S$. When diagonalizing against all $\Delta^0_2$ sets, it may be possible for a requirement to flip $S$ infinitely often. This, however, does not necessitate re-picking the followers of lower priority requirements. The only reason

![Diagram](image-url)
why $R_2$ needs to pick a new $z_2$ is because $R_1$ has seen $\delta_1$, $\gamma_1$ converge, and wants to protect now a certain segment of $S$. This initialization happens only finitely often (despite $R_1$ flipping $S(z_1)$ infinitely often). Therefore it will be straightforward to combine the requirements, and will not require a tree argument as one usually expects in full approximation arguments.

2.2. The modular approach. We proceed in a general setting, and then obtain the main theorems as corollaries. We start by fixing a computable sequence $\{V_e\}_{e \in \mathbb{N}}$ of possible $\Delta^0_2$-approximations. That is, $V_{e,s}(x)$ is a computable function of $e, s, x$. We say that $V_e$ is $\Delta^0_2$ if $\lim_{s \to \infty} V_{e,s}(x)$ exists for all $x$ and $V_e(x)$ is this limit.

Let the natural approximation of the jump of $V_e$ be $V_{e,s}(n) = 1$ if $\Phi_{\delta_{e,s}}(n) \downarrow$ (as is customary we assume the hat trick, that there must be a divergence between consecutive convergences with different uses, see Soare [Soa87]). If $V_e$ is $\Delta^0_2$ then this serves as a natural $\Sigma^0_2$-approximation to the characteristic function of $V'_e$ in the sense that $V'_e(n) = \liminf_{s \to \infty} V_{e,s}(n)$ for every $n$. However when approximating $V'_e|x$ as a finite string, $V'_{e,s}|x$ is obviously not ideal because $V'_e|x$ might not be the lexicographically leftmost string specified by $V'_{e,s}|x$ at infinitely many $s$. Here we think of 0 as being to the left of 1. It is easy to fix this by delaying any entry of $n$ into the (approximation for the) jump in the following way.

We define another approximation $Q_e, x[s]$ for $V'_e|x$ this time by induction as follows: $0 \in Q_e[s]$ if $\Phi_{\delta_{e,s}}(0) \downarrow$. For $n > 0$, let $t < s$ be maximal such that $Q_{e,n}[n[t] = Q_{e,n}[n|s]$. If $\Phi_{\delta_{e,s}}(n) \downarrow$ for all $t < r < s$, and $V_e$ has been stable below the use during this period, declare $n \in Q_{e,s}[s]$, and declare $n \notin Q_{e,s}[s]$ otherwise. Hence if $V_e$ is $\Delta^0_2$ then the lexicographically leftmost segment $Q_{e,s}|x[s]$ specified infinitely often is the segment of the true jump $V'_e|x$. The “delayed” approximation $\{Q_e[x]|s\}_{s}$ will be used when deciding whether or not to act for a module, since it is correct infinitely often. Furthermore the delayed approximation $Q_e$ for $V'_e$ is obtained effectively in $e$.

We have infinitely many modules $M_{\theta,e}$ indexed by a finite binary string $\theta$ and $e \in \mathbb{N}$. Here $M_{\theta,e}$ works in a similar way to requirement $R_e$ above, and guesses that $\theta \subseteq S$. It outputs (effectively) an infinite binary sequence $m_{\theta,e}$ listing the stage by stage guesses as to whether our toggle point $z_{\theta,e}$ is in $S$, as well as a number $d_{\theta,e}$ such that if $V_e$ is $\Delta^0_2$, then

(P1) $m = \lim_{s \to \infty} m_{\theta,e}(s)$ exists,

(P2) if $\gamma_e$ and $\delta_e$ are total, then additionally $d_{\theta,e} \downarrow$ and we have either $\Gamma_e^V(z_{\theta,e}) \neq m$ or $V'_e \neq \delta_e^{(0^*\oplus 0^*(\ominus 0^*\oplus \theta^*))|d_{\theta,e}}$.

Note that undefined counts as being not equal. For the rest of this section the reader should think of the construction of each $M_{\theta,e}$ as being run at every stage in isolation. That is, at stage $s$ of the construction we evaluate the given parameters $Q_{e,s}, \gamma_e, \delta_e$ at stage $s$ and output $m_{\theta,e}(s)$. There are no interactions between different modules. In section 3 we will then show how to combine the independent modules into a construction of a set $S$.

2.3. The construction for $M_{\theta,e}$. Now we give the actions of the module $M_{\theta,e}$.

Step 1: Let $z_{\theta,e} = \emptyset$.

Step 2: Wait for $\gamma_e(z_{\theta,e}) \downarrow$. Using the recursion theorem, for each $\sigma \in 2^{V_{e,s}(z_{\theta,e})}$, let $x_{\sigma} > \gamma_e(z_{\theta,e})$ be a number that we control for $V'_e$. That is, we may enumerate
axioms for $\Phi^V_{\sigma}(x_\sigma)$ and also specify the use of the axioms. The recursion theorem allows us to obtain values for $x_\sigma$ effectively. In short, we call these axioms for $x_\sigma$. Note that $x_\sigma$ for different modules are different.

Wait for $\delta_\sigma(\max_\sigma x_\sigma) \downarrow$. Let $d_{\theta, e} = \delta_\sigma(\max_\sigma x_\sigma)$, and proceed to Step 3.

Step 3: We say that $s$ is a recovery stage if $\Gamma^\sigma_{r}(z_{\theta, e})[s] \downarrow = m_{\theta, e}[s]$ and $\Delta^V_{\theta, e}(z_{\theta, e}[s] \downarrow \downarrow = Q_e[s](1 + \max_\sigma x_\sigma)$.

For clarity of presentation, we assume that the enumeration of $Q_e$ is fixed and independent of our actions. In particular we do not follow the customary practice of using a slowdown lemma in the enumeration of the jump. That is, when we define some $\Phi^V_{\sigma}(x_\sigma)$ to converge, we do not assume that $Q_e(x_\sigma)$ responds instantly. If this computation we defined is indeed correct then this will be reflected eventually in $Q_e(x_\sigma)$ and we can just wait for it; on the other hand if $V_e$ changes before $Q_e(x_\sigma)$ responds, then we would have made some progress since the use on the axiom for $x_\sigma$ was based on some other “real” computation reflected earlier by $Q_e$. Consequently we say that $Q_e(x)$ is good at stage $s$, if $x$ is an index which we control by the recursion theorem, and $Q_e(x)[s] = 1$ iff there is a current axiom at $s$ which applies for $x$.

Let $x_\sigma$ have a mode $\sigma$-stage. We first make the following observation.

At all successive stages, $m_{\theta, e}(s)$ outputs the previous value unless $z_{\theta, e}$ is toggled in which case we flip $m_{\theta, e}(s)$.

Stage $s$: Let $\sigma = Q_e[z_{\theta, e}][s]$. If $x_\sigma$ has mode IN, and there is no axiom that currently applies for $x_\sigma$, we enumerate an axiom for $x_\sigma$ with use $V_{e,s}$ with $\max\{u(q, V_{e,s}) | \sigma(q) = 1\}$.

If $s$ is a recovery stage and $Q_e(x_\sigma)$ is good, we call $s$ a good recovery stage and proceed as follows.

Case 1: no axiom for $x_\sigma$ applies. Declare $x_\sigma$ to have mode IN for stage $s + 1$.

Case 2: an axiom for $x_\sigma$ applies. Declare $x_\sigma$ to have mode OUT for stage $s + 1$, and toggle $z_{\theta, e}$.

2.4. Verification. This completes the construction. If $\sigma \subseteq Q_e[s]$ then we will refer to $s$ as a $\sigma$-stage. We first make the following observation.

**Lemma 2.1.** At all stages $s$ after Step 2 is completed, if an axiom applies for $x_\tau$, then it has use $\max\{u(q, V_{e,s}) | \tau(q) = 1\}$ with all the uses defined. Moreover, if $x_\sigma$ has mode IN at stage $s$ and $\sigma = Q_e[z_{\theta, e}][s]$ then $u(q, V_{e,s}) \downarrow$ for every $q$ such that $\sigma(q) = 1$.

**Proof.** The first statement follows directly from the second, while the second statement follows from the fact that if $q \in Q_e[s]$ then $\Phi^V_{\sigma}(q)[s] \downarrow$.

**Lemma 2.2.** If $\sigma = Q_e[z_{\theta, e}][s]$ is to the left of $\tau$ and an axiom for $x_\tau$ currently applies (with use $u$), then $V_e[u]$ cannot have been stable since the last $\tau$-stage.

**Proof.** Since $\tau$ is to the right of $\sigma$ there is a least $q < z_{\theta, e}$ such that $\tau(q) = 1$ and $\sigma(q) = 0$. Since $\sigma(q) = 0$, we have $q \notin Q(s)$. We know $V_{e,s}$ extends $\eta$ for
some \(x_t\)-axiom \(\eta\) enumerated earlier (say at \(t\)). Hence \(\Phi_{q}^{V_e}(q)[t]\) ↓. This means that \(\Phi_{q}^{V_e}(q)[s] \downarrow\) which means that \(V_e\) cannot extend \(\eta\) at every stage between the last \(\sigma_{q[t]}\)-stage and \(s\) (otherwise \(q \in Q_e[s]\)).

We recall that we only enter Case 1 or 2 at good recovery stages.

**Lemma 2.3.** If Step 3 is started and \(V_e\) is \(\Delta^0_e\), then there are only finitely many good recovery stages. Consequently \(z_{\theta,e}\) is toggled only finitely often.

**Proof.** Assume for a contradiction that there are infinitely many good recovery stages. Let \(s\) be the stage by which \(\theta'\) has settled on \(d_{\theta,e}\). There are at most two possible configurations of \((\theta^{-}m_{\theta,e}^{-\omega}0^\omega\oplus\theta')\mid d_{\theta,e}\) after \(s\), which differ on the value of \(m_{\theta,e}\). Every good recovery stage after \(s\) is either a \(\sigma_0\)-stage or a \(\sigma_1\)-stage, where \(\sigma_1\) corresponds to the configuration with \(m_{\theta,e} = i\).

We first claim that there is a stage \(s_1 > s\) such that Case 1 applies. Suppose not. Then at every good recovery stage after \(s\), we must have Case 2 applies. whence \(z_{\theta,e}\) is toggled. Thus as we visit the good recovery stages after stage \(s\), we must be alternating between the two configurations \(\sigma_0\) and \(\sigma_1\), in order to recover the toggles. Also, after we have our first good recovery stage with configuration \(\sigma_1\) after stage \(s\), we give \(x_{\eta}\) mode OUT. Since we are assuming we never enter Case 1 after stage \(s\), this means that \(x_{\eta}\) will remain in mode OUT for the duration of the construction. In particular, it follows that only finitely many axioms are enumerated for \(x_{\eta}\). Let \(t > s\) be a stage where \(V_e\mid\max\{\text{of the } x_{\eta}\text{ axioms}\}\) is stable. Let \(t_1 > t\) be a good recovery stage with configuration \(\sigma_1\). Since we were in case 2, an axiom for \(x_{\eta}\) applied at stage \(t_1\). Let \(t_2 > t_1\) be a good recovery stage with configuration \(\sigma_0\). Since \(t_1 > t\), the axiom for \(x_{\eta}\) still applied at stage \(t_2\). Now since \(\sigma_0 = Q_e[z_{\theta,e}][t_2]\) is to the left of \(\sigma_1\), Lemma 2.2 shows that \(V_e\) could not have been stable on the \(x_{\eta}\) axiom since the previous \(\sigma_1\)-stage, giving the desired contradiction.

The above contradiction shows that \(x_{\eta}\) exists. Suppose \(x_{\eta}\) is a \(\tau\)-stage. Let \(s_2 > s_1\) be the next good recovery stage (we want to get a contradiction). Since \(z_{\theta,e}\) is not toggled by the actions at \(s_1\), it follows that the configuration of \((\theta^{-}m_{\theta,e}^{-\omega}0^\omega\oplus\theta')\mid d_{\theta,e}\) at the beginning of \(s_2\) is the same as at the beginning of \(s_1\). Hence \(s_2\) is also a \(\tau\)-stage. Since \(x_{\eta}\) receives mode \(\text{IN}\) at \(s_1\), it follows that \(x_{\eta}\) has mode \(\text{IN}\) at the beginning of \(s_2\), where an axiom for \(x_{\eta}\) will be enumerated (if there is not already one). At \(s_1\), \(Q_e(x_{\eta})\) must be \(0\) because of its goodness, which means that at \(s_2\), \(Q_e(x_{\eta})\) must be again \(0\) since \(s_2\) is a recovery stage and a \(\tau\)-stage. But an axiom for \(x_{\eta}\) applies at stage \(s_2\), and \(s_2\) is a good stage, so \(Q_e(x_{\eta}) = 1,\) a contradiction.

**Lemma 2.4.** \(M_{\theta,e}\) satisfies \((P1)\) and \((P2)\).

**Proof.** If \(V_e\) is \(\Delta^0_e\), then \((P1)\) holds by Lemma 2.3. To show \((P2)\) holds as well we assume that \(\delta_{\eta}, \gamma_{\eta}\) are total (hence Step 3 is started). Let \(\sigma = V_e[V_e(z_{\theta,e})]\), the true segment of \(V_e\). Also let \(r = V_e(x_{\eta})\). Hence there are infinitely many \(\sigma\)-stages \(s\) where \(Q_e(x_{\eta})[s] = r\). We claim that \(Q_e(x_{\eta})\) is good at almost every such stage.

For every \(p\) such that \(\sigma(p) = 1\), \(p\) must be in the real \(V_e\) and so it is easy to see that once \(V_e\) is stable below these uses, any \(x_{\eta}\)-axiom we enumerate applies forever. Hence we only enumerate finitely many axioms for \(x_{\eta}\). If \(V_e\) extends one of these axioms then at almost every stage \(V_e[s]\) extends the axiom and also \(Q_e(x_{\eta}) = 1\). If \(V_e\) extends none of these axioms then at almost every \(\sigma\)-stage where \(Q_e(x_{\eta})[s] = 0\), we have \(V_e[s]\) extending none of these axioms. Hence \(Q_e(x_{\eta})\) is good at almost every \(\sigma\)-stage \(s\) where \(Q_e(x_{\eta})[s] = r\).
Assume for a contradiction that the last condition in (P2) fails. By Lemma 2.3 let \( s_0 \) be a stage by which \((θ_m^{\emptyset \cup \emptyset'})]\( d_{θ,e} \) has settled. There are infinitely many stages after \( s_0 \) where \( Q_{t}[1 + \max \{ x_e \}] \) is correct, and each of these is a recovery stage. By the above paragraph we will have infinitely many good recovery stages, contradicting Lemma 2.3.

We make a further comment. If we further assumed that \( \{ V_e \} \) is a c.e. approximation for every \( e \), then the function \((θ, e) \mapsto \lim_s m_{θ,e}[s] \) is \( ω + 1 \)-c.e. To see this, suppose \( s \) is a stage where we toggled \( z_{θ,e} \). Follow the proof of Lemma 2.3 and see that after \( s \), as long as there is no change to the \( θ' \) portion of \( (θ_m^{\emptyset \cup \emptyset'})\) \( d_{θ,e} \), we only toggle \( z_{θ,e} \) at most 4 times under Case 2 before Case 1 must apply at a good recovery stage. The second paragraph in the proof of Lemma 2.3 shows that \( z_{θ,e} \) is never toggled again, unless there is a change to the \( θ' \) portion of \( (θ_m^{\emptyset \cup \emptyset'})\) \( d_{θ,e} \). Hence, if \( V_e \) is c.e., then \( z_{θ,e} \) will be toggled no more than 2\( d_{θ,e} \) many times.

§3. The failure of the analogs of jump inversion. Towards proving our main theorems, the module \( M_{θ,e} \) will meet requirement \( R_e \), provided that \( θ \) is indeed the initial segment of the characteristic function of \( S \). We now show how to combine the modules in such a way that for each \( e \), there is a successful \( M_{θ,e} \) module.

Given a sequence \( \{ V_e \} \). We apply the previous section to get \( m_{θ,e}, d_{θ,e} \). Hence \( m_{θ,e}(s) \) is the result after running module \( M_{θ,e} \) for \( s \) many steps. We now use the result from the different modules to specify an approximation \( S[s] \) in the following way. First we order the finite binary strings: \( λ < 1 < 0 < 11 < 10 < 01 < 00 < 111 < \cdots \). Hence \( < \) refers to the ordering obtained by first considering increasing length, and then reverse lexicographic ordering. For each \( η \) we will have an associated binary string \( θ_η \), and the corresponding \( z_{θ_η}[s] \) as defined in module \( M_{θ_η}[s] \). That is, \( z_{θ_η}[s] = |θ_η| \). For convenience, we will let \( z_η \) denote \( z_{θ_η}[η] \). We will arrange it so that for \( η' < η \), we have \( z_{η'} < z_η \). Basically \( z_η \) serves as a pointer, and points to a location of \( S \) where \( S[z_η][s] \) will be approximated by the digits of \( m_{θ_η}[s] \). Each \( η \) codes a guess as to the membership of \( z_{η'} \) in \( S \) for \( η' < η \). We will have \( θ_η \) represent the \( η \)-guess as to the correct initial segment \( S[|θ_η]| \). As we give the stage by stage construction of \( S \), we will move the pointers \( z_η \), but each \( z_η \) will only be moved finitely often. Although \( z_η \) is defined to be the length of \( θ_η \), in practice we will define \( z_η \) first, and later define \( θ_η \). At every stage \( s \), if \( y \) is not being pointed at \( \{i.e., y \neq z_η \text{ for any } η \} \), then we will have \( S(y)[s] = 0 \).

We give a few notations to be used. Define \( T_s \) to be a string of finite length, which can be thought of as the current approximation to the “true strategies”. Loosely speaking, only those \( z_η \) where strategy \( η < T \) will be the important ones; the other \( z_η \) with \( η \) not on \( T \) are just red herrings; they are the artifacts produced by our wrong guesses. \( T_s \) is defined inductively by: \( T_s(n) = S[z_{T_{s−1}}(n)][s] \). Proceed this way until we hit the first undefined \( z_η \).

At stage \( s \) to read the next digit of \( m_{θ,e} \) means to do the obvious thing: if this is the first time we encounter this instruction then we output \( m_{θ,e}(0) \). Otherwise output \( m_{θ,e}(k + 1) \) where \( m_{θ,e}(k) \) was the previous digit read by the construction.

Construction of \( S \): at \( s = 0 \) make every \( z_η, θ_η \) undefined. At stage \( s > 0 \), only finitely many \( z_η, θ_η \) have been defined at the end of stage \( s − 1 \). Go through all such \( η \) in increasing order, and for each we (inductively) update \( θ_η \) and specify \( S[z_η][s] \). For \( z_η \) we let \( θ_η = η \) and set \( S(z_η)[s] = \) the next digit of \( m_{θ_η,0} \).
Now assume that $S(z) \in [s]$ has been defined for all $\eta' \prec \eta$. We define $\theta_\eta$ as follows. For $y < z$ such that $y \neq z_\eta$ for any $\eta'$, set $\theta_\eta(y) = 0$. If $y < z_\eta$ is such that $y = z_\eta'$, then necessarily $\eta' \prec \eta$. If $\eta'$ is lexicographically to the right of $\eta'|\eta'|$ then set $\theta_\eta(z_\eta') = 0$. If $\eta'$ is left of $\eta'|\eta'|$ then set $\theta_\eta(z_\eta') = S(z_\eta')[s]$. Otherwise $\eta' = \eta'|\eta'|$ and we let $\theta_\eta(z_\eta') = \eta'|\eta'|'$. Next we define $S(z_\eta')[s]$ in the following way. Note first of all that $T_s|\eta|$ can be evaluated at this point. If $T_s|\eta| = \eta$ then we let $S(z_\eta') [s]$ be the next digit of $m_{\theta_\eta}[\eta]$. If $T_s|\eta|$ is left of $\eta$ then let $S(z_\eta)[s] = 0$. Otherwise if $T_s|\eta|$ is right of $\eta$ we let $S(z_\eta')[s] = S(z_\eta)[s - 1]$.

If some $d_{\theta_\eta}[\eta]$ has converged at stage $s$, we make all $z_\eta', \theta_\eta'$ undefined for all $\eta' \succ \eta$ and go to the next stage. Otherwise the above stops naturally when we find some least $\eta$ with $z_\eta$ not defined at stage $s - 1$. We then pick a fresh value for $z_\eta$ and set $S(z_\eta)[s] = 0$.

Finally let $S(x) = \liminf, S(x)[s]$. It is clear that $z_\eta$ eventually settles on a final value for each $\eta$, and also that $|T_s| \rightarrow \infty$. Let $T$ be the lefmost path specified infinitely often by $T_s$. We first show that $T$ actually reflects the correct $\eta'$:

**Lemma 3.1.** For every $\eta \subset T$, we have $\theta_\eta$ eventually settles, $\theta_\eta \subset S$ and $S(z_\eta) = T(|\eta|) = \liminf m_{\theta_\eta}[\eta]$.

**Proof.** We proceed inductively on $|\eta|$. The statement clearly holds if $|\eta| = 0$ so take $|\eta| > 0$. After $z_\eta$ settles, the value of $\theta_\eta$ and also $S(z_\eta)$ will be decided on the places $\{z_\eta', \eta' \prec \eta\}$. There are three cases. If $\eta'$ is right of $\eta'|\eta'|$ then $\theta_\eta(z_\eta')$ is always 0, while at infinitely many stages $s$, $T_s \supset \eta$ which makes $S(z_\eta')[s] = 0$ infinitely often. If $\eta'$ is left of $\eta'|\eta'|$ then $T_s$ is right of $\eta'$ at every stage after some $s_0$. Hence $S(z_\eta')[s] = S(z_\eta)[s_0]$ for all $s > s_0$ and also $\theta_\eta(z_\eta')$ will agree with $S(z_\eta)[s_0]$. Finally if $\eta' \subset \eta$ then inductively let $\theta_\eta'$ be the limit value. It is easy to see that the value of $\theta_\eta(z_\eta') = \eta'|\eta'|$ is the desired set. Hence $\theta_\eta$ eventually settles and $\theta_\eta \subset S$.

Since $\eta$ is on $T$. Hence for almost all $s$ we have $T_s$ is right of $\eta$ (where $S(z_\eta)[s]$ is unchanged from the previous stage) or $T_s \supset \eta$ (in which the next digit of $m_{\theta_\eta}[\eta]$ is read). Hence $S(z_\eta) = \liminf m_{\theta_\eta}[\eta]$. To see that this value is the same as $T(|\eta|)$, observe that $T(|\eta|) = \liminf \{T_s(|\eta|) \mid T_s \supset \eta\} = \liminf \{S(z_\eta)[s] \mid T_s \supset \eta\} = \liminf m_{\theta_\eta}[\eta]$.

**Lemma 3.2.** For every $e$. If $V_e$ is $\Delta^0_3$, then $V_e \not\equiv_{\text{wtt}} S \oplus \emptyset'$.

**Proof.** We assume that $V_e = \Delta^0_3 \oplus \emptyset'$ and $S = \Gamma^I_{e,e}$ with use bounded by $\delta_{e,e}$ (which are total). Let $\eta = T(e)$. By Lemmas 2.4 and 3.1 $S_T[\eta] = \liminf m_{\theta_\eta}[\eta]$ where $\theta_\eta \subset S$. Since $d_{\theta_\eta,e} \downarrow$ then the initialization in the construction of $S$ ensures that in fact $(S \oplus \emptyset') \cap d_{\theta_\eta,e} = (\eta^{-1} \lim m_{\theta_\eta,e} \ominus 0^e \oplus \emptyset') \cap d_{\theta_\eta,e}$. A contradiction to the last condition of $(P2)$ follows.

We now obtain as corollaries, the following three statements.

**Theorem 3.3.** For any computable sequence of $\Delta^0_3$ sets $\{V_e\}_{e \in N}$ (given by their $\Delta^0_3$ indices), there exists a $\Delta^0_3$ set $S \geq_T \emptyset'$ such that for every $e$, $V_e \not\equiv_{\text{wtt}} S$.

**Proof.** Apply the results of the past two sections, and $\ominus \emptyset'$ is the desired set. Note that $S$ is $\Delta^0_3$ because of $(P2)$ and the fact that it is easy to prove that $\{T_s\}$ itself is a $\Delta^0_3$ approximation.

**Theorem 3.4.** There exists an $\omega + 1$-c.e. set $S >_T \emptyset'$ such that there is no c.e. set $A$ with $A' \equiv_{\text{wtt}} S$.
Proof. Theorem 3.3 gives a us a $\Delta^0_2$ set $S$. To see that $S$ can be made $\omega + 1$-c.e., use the fact that every module will reach a limit and modify the construction of $S$ slightly to ensure that each time $S|z_\eta[s]$ changes we also reset $z_\eta$. It is not hard to see that the ensuing approximation for $S$ will be $\omega + 1$-c.e.. We sketch the reason why, and leave the details to the reader. The value $S(z_\eta)[s]$ depends directly on the value of $T_s[|\eta|]$, which in turn depend on $S(z_\eta)[s]$ where $v \prec \eta$. As long as $S|z_\eta[s]$ remains unchanged, we will either output 0 for $S(z_\eta)[s]$, or the digits of $m_{\eta[|\eta|]}$. $\theta_\eta$ will also not change as long as $S|z_\eta[s]$ remains fixed. Hence the number of changes in $S(z_\eta)$ is at most the number of flips in $m_{\eta[|\eta|]}$ (until $z_\eta$ is cancelled). This number can be computed by the comments after Lemma 2.4.

Theorem 3.5. There exists a $\Sigma^0_2$ set $S \ni \emptyset'$ such that there is no $\Delta^0_2$ set $A$ with $A' \equiv_{m} S$.

Proof. Use a list of all possible $\Delta^0_2$ indices.

References