Discrete Version of Atiyah–Singer Index Theory from Lattice Gauge Theory

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Outline

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Background on Atiyah–Singer Index Theory

What it says; why it is important

General Context: Analysis and Topology

What is analysis? What is topology?

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Formulation of the Index Theorem (in the simplest cases)

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Atiyah–Singer Index Theory (1963-68)

A deep connection between analysis and topology:

analytic index = topological index

Analytic index: an integer associated with the solution space of certain partial differential equations (PDEs).

Topological index: an integer defined from topological data of the PDE.

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  - Gauss-Bonnet Theorem
  - Riemann–Roch Theorem
  - Hirzebruch Signature Theorem
- Important consequences in theoretical particle physics (quantum field theories).

Celebrated Status of A–S Index Theory

Recognized with Abel Prize for Atiyah and Singer in 2004

Sir Michael Atiyah
Oxford/Edinburgh

Isadore Singer
Caltech/M.I.T.

"The Atiyah-Singer index theorem is one of the great landmarks of twentieth-century mathematics, influencing profoundly many of the most important later developments in topology, differential geometry and quantum field theory."

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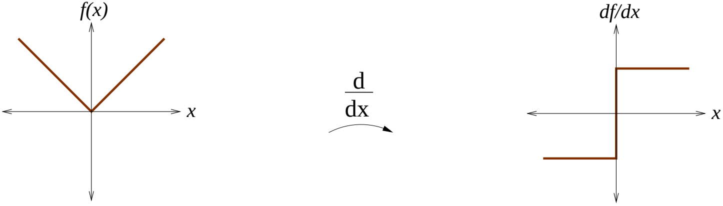

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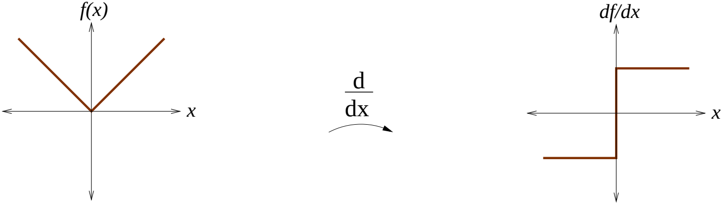

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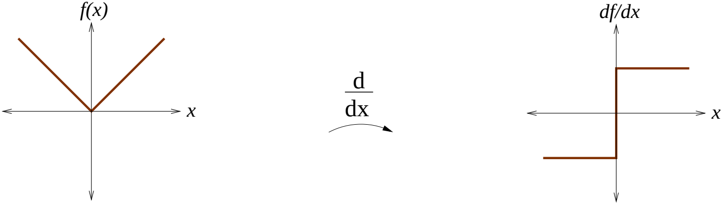

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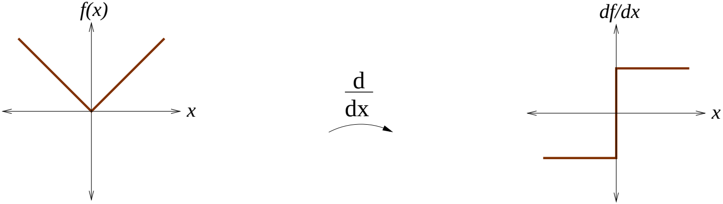

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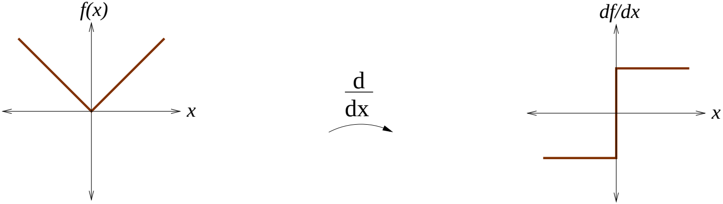

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$$\frac{d}{dx} : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \qquad \text{"Hilbert spaces"}$$

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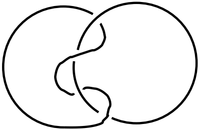

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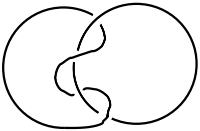

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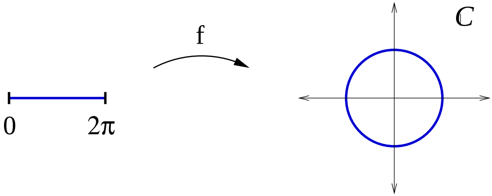

$$f(\theta) = e^{-i3\theta}$$

winding number = -3

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- More generally,

$$\pi_n(S^n) \simeq \mathbf{Z} \text{ for any } n$$

$\Rightarrow$ maps $S^n \to S^n$ classified by a "wrapping number"

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Consider the functions

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Simplest explicit choices: $\theta(x_1) = 2\pi Q x_1$ and

$$A_1(x_1, x_2) = -2\pi Q \quad \text{(constant)}\ , \qquad A_2(x_1, x_2) = 0$$

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An explicit realization is $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

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  - It can be non-zero

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Why is it surprising that index(D) can be non-zero?

Notations:

$$\Omega = \text{the vectorspace of functions } f(x_1, x_2)$$
$$V \subset \Omega \times \Omega = \text{the solution space to } D\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$

Set $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $\Gamma\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ -f_2 \end{pmatrix}$ maps $V \to V$

$$\to \quad \text{index}(D) = \text{trace}(\Gamma : V \to V)$$

Moreover, $\Gamma D = -D\Gamma$, hence trace of Γ vanishes on $V^\perp$:

$$\text{tr}(\Gamma) = \text{tr}(\Gamma D D^{-1}) = -\text{tr}(D\Gamma D^{-1}) = -\text{tr}(\Gamma) = 0$$

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A–S Index Theorem: $\boxed{\text{index}(D) = -Q}$

$Q =$ ‘winding number’ in boundary conditions of the PDE

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- Can generalize the Dirac operator D and its index to this setting. Index Theorem states:

$$\boxed{\mathrm{index}(D) = (-1)^n Q}$$

Outline

Challenge: Develop a discrete version of index theory

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– Do the index and topological data continue to be related?
I.e., is there a discrete index theory?

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- Important for gauge theories of particle physics.

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$\Rightarrow$ Must have index theorem in discrete setting to get correct masses from lattice QCD calculations.

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Nevertheless...a viable approach to discrete index theory found by physicists in framework of **lattice gauge theory**...

Discretization via Lattice

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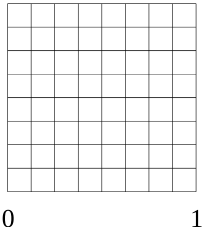

$N =$ number of lattice sites along each axis

$a =$ lattice spacing $\left(= \dfrac{1}{N}\right)$

Functions $f(x_1, x_2)$ defined on **lattice sites**; same boundary conditions as before.

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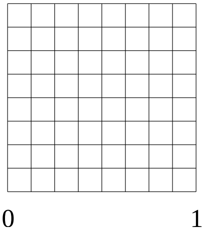

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Discretization of derivatives:

$$\frac{\partial f}{\partial x_1}(x_1, x_2) \;\rightarrow\; \frac{1}{2a}\Big[\, f(x_1 + a\,, x_2) \;-\; f(x_1 - a\,, x_2)\,\Big]$$

$\dfrac{\partial f}{\partial x_2}$ discretized analogously.

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- Already noted by previous general argument.
- More concrete explanation: The lattice PDE has **additional "rough" solutions** besides the continuum-like solutions.

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$\Rightarrow$ Need to extract the index in an indirect way: via "**spectral flow**".

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$\Rightarrow$ index(D) $=$ **Spectral flow** of $H(m)$ at $m = 0$:

Count ± 1 for each eigenvalue crossing with slope $\mp$.

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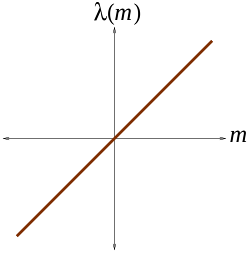

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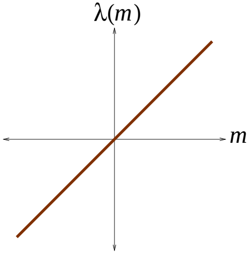

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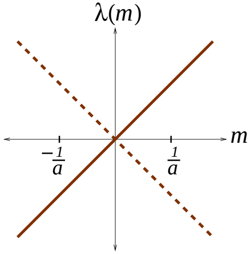

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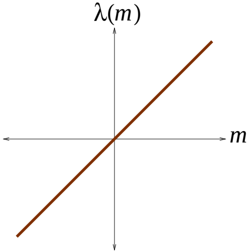

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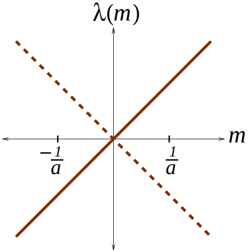

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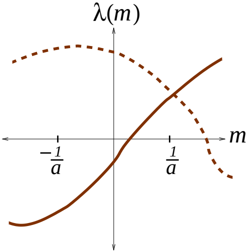

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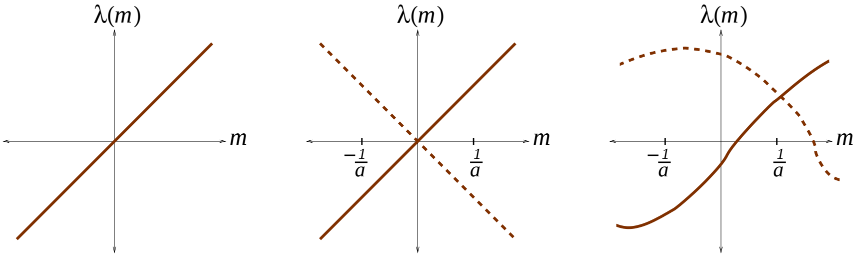

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Lattice with $D \to D + a\Delta$

$\Rightarrow$ **In lattice setting, index can be defined from the spectral flow of $H(m)$ associated with eigenvalue crossings near $m=0$.**

Index in the Lattice Setting

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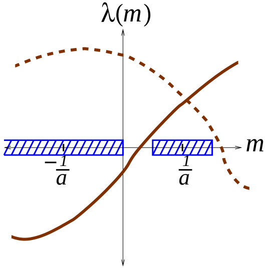
Theorem [M. Lüscher, 1998]: When the lattice gauge fields satisfy an **approximate smoothness condition**, eigenvalue crossings are **excluded** in a region around $m = \frac{1}{a}$, and also for $m < 0$:



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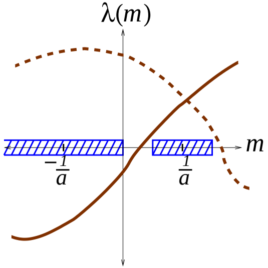

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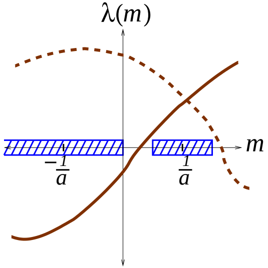

$\Rightarrow$ Notion of eigenvalue crossing “near $m = 0$” is well-defined.

$\Rightarrow$ **Lattice index is well-defined via spectral flow in the “near $m = 0$” region.**

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Progress so far: Continuum limit result [D. Adams, 1999]:

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when lattice gauge fields are transcripts of smooth continuum gauge fields.

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Families index theory relates the family of P.D.E. solution spaces to **topological data** encoded in the family of gauge fields.

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Atiyah–Singer Families Index Theorem states:

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$\mathbf{ch}(\cdot)$ = "**Chern character**" of a vector bundle.

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Families Index Theorem is important for evaluating anomalies in chiral gauge theories of particle physics.

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- Define Q_Y for families of lattice gauge fields satisfying an appropriate approx smoothness condition.
- Prove general lattice version of Families Index Theorem.
  –This will show that "anomaly cancellation" is working correctly in lattice formulation of chiral gauge theories.

Outline

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  - Need to deal with *spurious solutions*
    $\Rightarrow$ Of interest for numerical solution of PDEs

- **What has been done**:
  - Most of the ingredients for discrete index theory in lattice setting have been developed.
  - Continuum index theory reproduced.
- **The challenges ahead**:
  - Prove the lattice index theorems. (*Current research*)
  - Extend from lattice gauge theory setting to general discretizations of general manifolds. (*Long-term goal*)