PRACTICAL FIRST-ORDER RELIABILITY COMPUTATIONS USING SPREADSHEET

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ABSTRACT: A practical procedure for reliability analysis involving correlated nonnormals was described in Low & Tang (1997), using object-oriented constrained optimization in the ubiquitous spreadsheet platform. Numerical derivatives and iterative search for the design point were automatic. The objective of this paper is to further illustrate the 1997 procedure for two geotechnical problems with relatively complicated performance functions, namely the bearing capacity of a rectangular foundation with eccentric and inclined loadings, and the stability of an anchored retaining wall. The results are compared with Monte-Carlo simulations.

INTRODUCTION

The Hasofer-Lind (1974) index and the first order reliability method (FORM) have been treated in Rackwitz and Fiessler (1978), Ditlevsen (1981), Shinozuka (1983), Ang and Tang (1984), Madsen et al. (1986), and Tichy (1993). The continuing interest in FORM is also evident in Melchers (1999), Haldar and Mahadevan (1999), Nowak and Collins (2000), for example. The potential inadequacies of FORM in some special cases have been recognized, and more refined alternatives proposed (e.g. Der Kiureghian et al. 1987). On the other hand, Rackwitz (2001) noted the usefulness and accuracy of FORM in most applications; some counterexamples were summarized, their rather extreme nature and artificiality noted, but their healthy role also admitted.

With respect to Hasofer-Lind index and FORM, a practical procedure was presented in Low and Tang (1997a). The Rackwitz-Fiessler equivalent normal transformation was used. The concepts of coordinate transformation and frame-of-reference rotation were not required. Correlation was accounted for by setting up the quadratic form directly. Iterative searching and partial derivatives were automatic using constrained optimization in the ubiquitous spreadsheet platform. The present paper further illustrates the procedure by applying it to two geotechnical problems with more involved performance functions. Results of reliability-index-based probability of failure are compared with Monte Carlo simulations.
CONSTRAINED OPTIMIZATION OF EQUIVALENT ELLIPSOID IN ORIGINAL SPACE

The matrix form of the Hasofer-Lind (1974) reliability index is:

$$\beta = \min_{x \in \mathcal{Y}} \sqrt{(x - m)^T C^{-1} (x - m)}$$  \hfill (1a)$$
or, equivalently,

$$\beta = \min_{x \in \mathcal{Y}} \left[ \frac{x_i - m_i}{\sigma_i} \right]^T \left[ R^{-1} \right] \left[ \frac{x_i - m_i}{\sigma_i} \right]$$  \hfill (1b)$$

where \(x\) is the vector of random variables, \(m\) the mean values, \(C\) the covariance matrix, \(R\) the correlation matrix, and \(\mathcal{Y}\) the failure domain. The classical method of computing \(\beta\) is intricate when correlation, autocorrelation, and nonnormal distributions are involved and when the performance function is complicated or implicit. These hitherto tedious problems can be solved, with relative ease and transparency, using spreadsheet-automated constrained optimization and the expanding ellipsoid perspective (Low & Tang 1997a). By this perspective, the quadratic form in Eq. 1 is visualized as a tilted multi-dimensional ellipsoid (centered at the mean \(m\) or equivalent mean \(m^e\)) in the original space of the random variables; there is no need to diagonalize the covariance or correlation matrix.

The quadratic form in Eq. 1 appears also in the negative exponent of the multivariate normal distribution. Hence, to minimize \(\beta\) (or \(\beta^2\) in the multivariate normal distribution) is to maximize the value of the multivariate normal probability density function, and to find the smallest ellipsoid tangent to the failure surface is equivalent to finding the most probable failure point. This perspective is consistent with Shinozuka (1983) that “the design point \(x^*\) is the point of maximum likelihood if \(x\) is Gaussian, whether or not its components are uncorrelated.”

The versatility of the spreadsheet constrained optimization approach is enhanced when used in combination with user-defined functions coded in the programming environment of a spreadsheet, for example the Visual Basic (VBA) programming environment of the Microsoft Excel spreadsheet software. This means that the performance function can be implicit, iterative, and based on numerical methods (e.g. Low 2001, Low et al. 2001a).

The reliability analyses of nonnormals in this paper have been done using Microsoft Excel XP (2002) and its built-in optimization program Solver. Excel 2000 (and earlier versions) may not achieve the same efficiency or accuracy because of the much narrower validity range of its NormSInv function (which is used to obtain the Rackwitz-Fiessler equivalent normal mean and standard deviations of nonnormals.)

RELIABILITY-BASED DESIGN OF WIDTH OF A WALL FOUNDATION

Tomlinson (1995, Example 2.2) analyzed deterministically the factor of safety against bearing capacity failure of a retaining wall foundation. Low and Tang (1997b) analyzed the problem, probabilistically, assuming correlated normal random variables.

The problem is re-examined below, this time assuming correlated nonnormals. The base (5 m \(\times\) 25 m) of the foundation is at a depth of 1.8 m in a silty sand with friction angle \(\phi = 25^\circ\), cohesion \(c = 15\ \text{kN/m}^2\), and unit weight \(\gamma = 21\ \text{kN/m}^3\). It carries a horizontal load (\(Q_h\)) of 300 kN/m run at 2.5 m above the base, and a centrally applied vertical load (\(Q_v\)) of 1100 kN/m run. The notations are shown in Fig. 1.
With respect to bearing capacity failure, the performance function \( \text{PerFunc} \) is:

\[
\text{PerFunc} = q_u - q
\]

(2a)

where

\[
q_u = c N_c s_c d_c i_c + p_0 N_q s_q d_q i_q + \frac{B'}{2} \gamma N_i s_i d_i i_i;
\]

\[
q = Q_u / B'
\]

(2b,c)

in which \( q_u \) is the ultimate bearing capacity, \( q \) the applied bearing pressure, \( c \) the cohesion of soil, \( p_0 \) the effective overburden pressure at foundation level, \( B' \) the effective width of foundation, \( \gamma \) the unit weight of soil below the base of foundation, and \( N_c, N_q, \) and \( N_i \) are bearing capacity factors, which are functions of the friction angle (\( \phi \)) of soil:

\[
N_c = \left( N_q - 1 \right) \cot(\phi);
\]

\[
N_q = e^{\pi \tan\phi} \tan^2 \left( \frac{45 + \phi}{2} \right);
\]

\[
N_i = 2(N_q + 1) \tan \phi
\]

(3a,b,c)

Several expressions for \( N_i \) exist. The above \( N_i \) is attributed to Vesic in Bowles (1996). The nine factors \( s_c, d_c, \) and \( i_c \) in Eq. 2b account for the shape and depth effects of foundation and the inclination effect of the applied load. The formulas for these factors (Cells J15:O15, and J17:L17) are based on Tables 4.5a and 4.5b of Bowles (1996).

The c, \( \phi \), \( Q_n \), and \( Q_u \) values in all equations refer to the x* column (cells H6:H9), which were initially given the mean values (15, 25, 300, 1100). The parameters of the nonnormal distributions in cells C6:F9 correspond to coefficients of variation equal to 0.20, 0.1, 0.15, 0.10 respectively. The random variables are correlated, as shown.

The illustrative reliability analyses in Figs. 1 and 2 involve only the normal, the lognormal, the type 1 extreme (Gumbel) and the beta distributions. Six other nonnormal distributions as listed in cells A23:A32 can be selected when needed. The cells labeled \( m^N \) and \( \sigma^N \) in Fig. 1 contain a specially-created function that performs the Rackwitz-Fiessler equivalent normal transformation. The key feature is that all distribution-specific equations for equivalent normal transformation are relegated to a short Excel macro code, which is called by cells beneath the headings \( m^N \) and \( \sigma^N \) in Fig. 1. The concise program code is easily transferable to other workbooks. It can also be converted to an Excel add-in. The function reads the values of its arguments from the columns labeled Distributions, Para1:Para4, and x* value. It is entered only in the first cells (I6 and J6) of the columns labeled \( m^N \) and \( \sigma^N \), and autofilled down the columns. There is no need to remember distribution-specific equations, since these are now decided at the program level. (The program code and other details are given in Low and Tang 2002.)

The set-up of Fig. 1 is convenient for trying different combinations of distributions: one only needs to change the distribution names in cells A6 to A9, modify the parameters in cells C6 to F9, initialize x* values to mean values (according to the relationships shown in cells H23 to H32), input changes (if any) to the correlation matrix, and invoke Solver. More random variables can be treated by adding additional rows beneath row number 9, and expand the correlation matrix accordingly.

The 4-parameter \((\alpha, \lambda, \text{min}, \text{max})\) beta distribution, used in Fig. 1, can model a variety of nonsymmetric PDF curves of a random variable within a user-defined minimum-maximum range. Three established equations (functions of \( \alpha, \lambda, \text{min} \) and \( \text{max} \)) exist for the mean, mode, and standard deviation of a beta variate. For known \( \text{max}, \text{min}, \mu \) (or mode) and \( \sigma \), one can use Solver to solve two of the three equations for the appropriate values of \( \alpha \) and \( \lambda \). This has been done in Fig. 1 to obtain the \( \alpha \) and \( \lambda \) values in cells C7 and D7. If the standard deviation is taken to be 1/6 of the range of \((\text{max} - \text{min})\), with mean at \( 0.5*(\text{max} + \text{min}) \), then \( \alpha = \lambda = 4 \), as in case 3 of Fig. 2 and also in Fig. 3. In these cases the beta distribution is symmetric and resembles the normal distribution.
\[ q_u = c N_c d c_i c + p_o N_q d q_i q + \frac{B'}{2} \gamma N_s d s_i s \]

**Units:** m, kN/m, kN/m², kN/m³, degrees, radians, as appropriate.

<table>
<thead>
<tr>
<th>Distributions</th>
<th>Para1</th>
<th>Para2</th>
<th>Para3</th>
<th>Para4</th>
<th>(x^*)</th>
<th>(m)</th>
<th>(g)</th>
<th>(n)</th>
<th>B</th>
<th>L</th>
<th>D</th>
<th>(\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lognormal</td>
<td>c</td>
<td>15</td>
<td>3</td>
<td></td>
<td>15.58</td>
<td>14.68</td>
<td>3.086</td>
<td>0.29</td>
<td>4.50</td>
<td>25</td>
<td>1.8</td>
<td>21</td>
</tr>
<tr>
<td>BetaDist</td>
<td>(\phi)</td>
<td>13.8</td>
<td>9.2</td>
<td>10</td>
<td>35</td>
<td>21.00</td>
<td>2.605</td>
<td>-1.55</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>ExtValue1</td>
<td>(Q_h/m)</td>
<td>300</td>
<td>45</td>
<td></td>
<td>419.7</td>
<td>238.3</td>
<td>86.84</td>
<td>2.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ExtValue1</td>
<td>(Q_v/m)</td>
<td>1100</td>
<td>110</td>
<td></td>
<td>1171</td>
<td>1068</td>
<td>131.5</td>
<td>0.782</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Correlation matrix R**

| \(c\) | \(-0.5\) | 0 | 0 |
|\(\phi\) | 0 | 0 | 0 |
|\(Q_h/m\) | 0 | 0 | 0 |
|\(Q_v/m\) | 0 | 0 | 1 | 0.5 |

**Meaning of OPTIONS**

| Normal     | mean = StDev Para1 |
| Lognormal  | mean = StDev Para1 |
| ExtValue1  | mean = StDev Para1 |
| Uniform    | Use BetaDist(1,1,min,max) mean = 0.5*(min+max) |
| Exponential| mean = Para1 |
| Gamma      | \(\alpha\), \(\lambda\) mean = \(\alpha^*\lambda\) |
| Weibull    | \(\alpha\), \(\lambda\) mean = \(\lambda/\alpha+1\) |
| Triangular | min, mode, max mean = \((\text{min+mode+max})/3\) |
| BetaDist   | \(\alpha\), \(\lambda\), min, max mean = \((\text{min+max-min})^*\alpha / (\alpha+\lambda)\) |
| PERTDist   | min, mode, max mean = \((\text{min+4*mode+max})/6\) |

**Figure 1.** Reliability analysis of spread foundation; correlated nonnormals.
Figure 2. Reliability indices of spread foundation of width $B = 4.5$ m, for different assumed correlated nonnormal distributions.
The nonnormals in the lower four cases of Fig. 2 have the same means and standard deviations as the normal variates of the first case. The different computed $\beta$ indices are due to the different assumed distributions. For the cases in hand, the results are in good agreement (as shown) with Monte Carlo simulations using @RISK version 4.0. (The type 1 extreme distribution function in @RISK is ExtValue($a$, $b$), where $a$ and $b$ are related to the mean and standard deviation $\sigma$ of the type 1 extreme variates $Q_v$ and $Q_h$ of cases 4 and 5 by: $a = \text{mean} - 0.45005\sigma$, and $b = 0.7797\sigma$.)

Theoretically, the correlation matrix $R$ in Eq. 1b refers to the Pearson correlation between equivalent normal variates. By comparing with @RISK, one assumes that the Spearman rank correlation used in @RISK is approximately the same as the equivalent normal Pearson correlation in $R$. This can be verified by transforming the non-normal realizations from @RISK to standard normal realizations using $u = \Phi^{-1}[F(x)]$, and computing their Pearson correlation. Based on 5000 realisations from the non-normal variables in Fig. 1, the equivalent normal Pearson correlation between $c$ and $\phi$ is found to be equal to $-0.495$, as compared to Spearman correlation of $-0.475$. The corresponding correlations for $Q_h$ and $Q_v$ are 0.500 and 0.479, respectively. Accounting for statistical uncertainty, the results are in agreement because normal scores are used to induce rank correlation in @RISK as suggested by Iman and Conover (1982).

All the five cases in Fig. 2 have the same foundation width $B = 4.5$ m. If a $\beta$ index of 3.0 is desired, the required foundation widths $B$ are (rounded to two decimal places) 4.51 m, 4.51 m, 4.42 m, 4.77 m and 4.78 m, respectively.

In reliability-based design, the trial design values (width $B$, in this case) should be such that $\text{PerFunc} > 0$ initially when $x^* = \text{mean}$. This ensures that the spreadsheet’s optimization search will start from a point that is not within the failure domain in order to obtain a positive reliability index. The need to distinguish positive $\beta$ index from negative $\beta$ index was discussed in Tichy (1993) and Low et al. (2001b). In addition, the correlation matrix has to be consistent, as discussed in Ditlevsen (1981) and Low (2001), for example.

RELIABILITY ANALYSIS OF ANCHORED SHEET-PILE WALL

Figure 3 shows one of the retaining wall problems analyzed in Low et al. (2001b), except that correlated nonnormal distributions are now assumed instead of the correlated normals used earlier. The rationales for the random variables and their correlations are discussed in the 2001 paper. The nonnormals in Fig. 3 have the same mean values and standard deviations as the normal variates of the 2001 paper, and the same correlation matrix. The performance function (PerFn1), based on the free-earth support method, is a nonlinear and lengthy function of the nine $x^*$ values.

The $x^*$ column in Fig. 3 were initially given their mean values (1.5, 6.4, 2.4, 17, 20, 10, 36, 18, 4.01). Excel XP’s Solver was then invoked, with its “Use Automatic Scaling” option activated, to obtain the $\beta$ index of 2.734. For the all-normal-variate case of 2001, the $\beta$ index was 2.50. The difference is due to different assumed distributions.

Monte Carlo simulation with 500,000 trials was performed using @RISK version 4.0 for the correlated nonnormal distributions of Fig. 3. The percentage failure (PerFn1 $\leq 0$) was 0.31%, compared with 0.312% based on the reliability index value of 2.734. Monte Carlo simulation for the all-normal-variate case of Low et al. (2001b) showed a failure rate of 0.63%, compared with 0.62% based on the reliability index of 2.50 in that case. Such good agreements may not always be expected, for there are cases where the agreements were less remarkable than those between more refined methods (e.g., SORM) and Monte Carlo simulations.
Boxed cells contain equations

\[ a \quad h_1 \quad h_2 \quad \gamma \quad \gamma_{\text{sat}} \quad \sigma_s \quad \phi' \quad \delta \quad d \]

\[ \chi^2 \quad mN \quad \sigma^N \quad nx \quad \text{crmatrix (Correlation matrix)} \]

| BetaDist | a 4 4 1.275 1.725 1.497 1.5 0.082 | 1 0 0 0 0 0 0 0 0 | a |
| BetaDist | h_1 4 4 5.44 7.36 6.541 6.402 0.344 | 0 1 -0.5 0 0 0 0 0 | h_1 |
| BetaDist | h_2 4 4 2.04 2.76 2.427 2.4 0.131 | 0 -0.5 1 0 0 0 0 0 -0.5 | h_2 |
| Lognormal | \gamma 17 0.85 | 16.117 16.96 0.805 -1.043 | 0 0 0 1 0.5 0.5 0 0 | \gamma |
| Lognormal | \gamma_{\text{sat}} 20 1 | 18.489 19.92 0.924 -1.547 | 0 0 0 0.5 0.5 0.5 0 | \gamma_{\text{sat}} |
| Lognormal | \sigma_s 10 1 | 10.009 9.95 0.998 0.059 | 0 0 0 0 0 1 0 | \sigma_s |
| Lognormal | \phi' 36 3.6 | 27.738 34.83 2.767 -2.564 | 0 0 0 0.5 0.5 0 | \phi' |
| Lognormal | \delta 18 1.8 | 14.584 17.58 1.455 -2.06 | 0 0 0 0 0 | \delta |
| BetaDist | d 4 4 3.409 4.613 3.864 4.006 0.209 | 0 0 0 0.5 0.5 0 0 | d |

\[ =K_a \cos \delta \]

\[ \begin{align*}
K_a & = 0.3288 \\
K_{\text{sat}} & = 0.318 \\
K_p & = 4.358 \\
K_p' & = 4.218
\end{align*} \]

\[ \text{PerFn1} \]

\[ \beta = -2 \times 10^{-11} \]

\[ =\text{sqrt}((\text{mmult(\text{transpose(nx), mmult(minverse(crmatrix), nx))}))} \]

\[ \text{Surcharge } \sigma_s = 10 \text{ kN/m}^2 \]

No surcharge above anchorage

\[ \begin{align*}
A & \quad h_1 \\
\gamma & \quad \phi', \delta \\
\text{Water table} & \quad h_2 \\
\text{No surcharge above anchorage} & \quad \sigma_s \\
\text{Surcharge } \sigma_s & = 10 \text{ kN/m}^2
\end{align*} \]

**Figure 3. Reliability-based design of anchored wall; correlated nonnormals.**
SUMMARY AND CONCLUSIONS

The spreadsheet constrained optimization approach for reliability analysis of correlated nonnormals was demonstrated in Low & Tang (1997a). This paper further illustrates the procedure for two common geotechnical problems: the bearing capacity of a rectangular foundation with inclined and eccentric loadings, and the rotational stability of an anchored sheet-pile wall. The computed reliability indices differ somewhat, depending on the assumed distributions. For the problems in hand, the probabilities of failure inferred from reliability indices are in good agreements with Monte Carlo simulations using the commercial package @RISK which generates correlated random numbers based on the distribution-free Spearman rank order correlation coefficients.

In general, for correlated nonnormals, the success and accuracy of the present approach may depend on the nature of the performance functions, the number of random variables, the validity range of the Excel NormSInv function (for equivalent normal transformation), and the capability of the nonlinear constrained optimization program. For the cases of correlated nonnormals analyzed in this paper, Excel XP and its built-in standard Solver are found to be efficient and accurate (as far as FORM is concerned).

REFERENCES: