Appendix

A Proof for Lemma 1

Lemma 1. *A coverage vector* $\mathbf{c}(t)$ *is locally optimal at* t_0 *if and only if it satisfies* $\sum_{i\in I} c_i(t_0) = m$.

Proof. The 'if' direction is shown by Kiekintveld et al[Kiekintveld *et al.*, 2009]. For the 'only if' direction, assume that there exists a coverage vector $\mathbf{c}(t)$ which is locally optimal at t_0 but does not satisfy the equation, i.e., $\sum_{i\in I} c_i(t_0) < m$. Since $m \le n$, and $\forall i \in \mathcal{T}, v_i(t) > 0$, we have $\forall i \in I, c_i(t_0) < 1$. Thus the defender can always decrease coverage on targets $j \in \mathcal{T} \setminus I$ and increase coverage on targets $i \in I$ to decrease the optimal attacker utility until $\sum_{i\in I} c_i(t_0) = m$. (During this process, the size of I may increase). □

B Proof for Theorem 2

Theorem 2. *COCO computes the optimal coverage vector.*

Proof. We show that in the last iteration of a time period including time point t_0, the set I is an attack set at t_0. Thus Theorem 2 is proved given Lemma 1. Let $\mathbf{c}(t_0) = \langle c_i(t_0)\rangle$ represent the coverages computed in the last iteration and let I be the corresponding set of targets. The previous iterations (Lines 8, 10) ensure that $\forall j \in \mathcal{T} \setminus I, c_j(t_0) = 0$.

We now show that $v_i(t_0)(1 - c_i(t_0)) > v_j(t_0)(\forall i \in I, \forall j \in \mathcal{T} \setminus I)$. Target j is not in set I only if there exists an iteration in which Line 2 leads to a coverage vector $\mathbf{c}'(t_0) = \langle c'_i(t_0), \forall i \in I'\rangle$ with $c'_j(t_0) < 0$. Here I' is the attack set in this iteration. Since each iteration does not add targets into the attack set in the previous iteration, we have that I is a subset of I'. Line 2 ensures that

1. $\sum_{i\in I} c_i(t_0) = \sum_{i\in I'} c'_i(t_0) = m$,
2. $v_{i_1}(t_0)(1 - c_{i_1}(t_0)) = v_{i_2}(t_0)(1-c_{i_2}(t_0)), \forall i_1, i_2 \in I$,
3. $v_{i_1}(t_0)(1 - c'_{i_1}(t_0)) = v_{i_2}(t_0)(1-c'_{i_2}(t_0)), \forall i_1, i_2 \in I'$.

Given that $\forall i \in I, c_i(t_0) \ge 0$ and $c'_j(t_0) < 0$, we have that $\forall i \in I$, $c_i(t_0) < c'_i(t_0)$, thus $v_i(t_0)(1 - c_i(t_0)) > v_i(t_0)(1 - c'_i(t_0)) = v_j(t_0)(1 - c'_j(t_0)) > v_j(t_0)$. The last inequality is due to the fact that $c'_j(t_0) < 0$. □

C Proof for Proposition 3

Proposition 3. *For* $t \in [\theta_k, \theta_{k+1}]$, $\mathbf{c}(t)$ *can be implemented by sampling from pure strategies in which resources are only transferred from targets* $u \in \Lambda$ *to targets* $v \in V$*, and each resource is transferred for at most once..*

Proof. Assume that $\mathbf{x}$ is a mixed defender strategy corresponding to coverage vector $\mathbf{c}(t)$. Given a time point t_0, let ϕ represent the set of pure strategies in which a resource is transferred from a target $v \in V$ to other targets $i \in \mathcal{T}$ at t_0; let ψ represent the set of pure strategies in which a resource is transferred from a target $i \in \mathcal{T}$ to a target $u \in \Lambda$ at t_0. We show that if in $\mathbf{x}$, $\exists S \in \phi$ or $S \in \psi$ with $x_S > 0$, then we can construct another mixed strategy $\mathbf{x}'$ corresponding to $\mathbf{c}(t)$ based on $\mathbf{x}$, in which $\forall S \in \phi$ or $S \in \psi$, $x'_S = 0$. In other words, the support pure strategies of $\mathbf{x}'$ only transfer resources from targets $u \in \Lambda$ to targets $v \in V$ at t_0.

We first consider pure strategies $S \in \phi$ with $x_S > 0$. $\forall S \in \phi$, we have

$$\lim_{t\to t_0^-} q_v(S, t_0) = 1; \lim_{t\to t_0^+} q_v(S, t_0) = 0. \tag{20}$$

Since the coverage function of target v monotonically increases during (θ_k, θ_{k+1}), we have

$$\begin{aligned}
c'_v(t_0) &= \lim_{\Delta t\to 0} \frac{c_v(t_0 + \Delta t) - c_v(t_0 - \Delta t)}{2 \cdot \Delta t} \\
&= \lim_{\Delta t\to 0} \frac{\int_{S\in\mathcal{S}} x_S q_v(S, t_0 + \Delta t) - \int_{S\in\mathcal{S}} x_S q_v(S, t_0 - \Delta t)}{2 \cdot \Delta t} \\
&= \lim_{\Delta t\to 0} \frac{\int_{S\in\mathcal{S}/\phi} x_S f_1 + \int_{S\in\phi} x_S \cdot 0 - \int_{S\in\mathcal{S}/\phi} x_S f_2 - \int_{S\in\phi} x_S \cdot 1}{2 \cdot \Delta t} \\
&\quad (f_1 = q_v(S, t_0 + \Delta t), f_2 = q_v(S, t_0 - \Delta t)) \\
&= \lim_{\Delta t\to 0} \frac{\int_{S\in\mathcal{S}/\phi} x_S f_1 - \int_{S\in\mathcal{S}/\phi} x_S f_2 - \int_{S\in\phi} x_S}{2 \cdot \Delta t} \\
&\ge 0.
\end{aligned} \tag{21}$$

Thus we have $\int_{S\in\mathcal{S}/\phi} x_S q_v(S, t_0 + \Delta t) - \int_{S\in\mathcal{S}/\phi} x_S q_v(S, t_0 - \Delta t) > 0$. Given that $q_i(S, t)$ are piecewise constant functions with values in $\{0, 1\}$, there must exist pure strategies S with $x_S > 0$ in which

$$\lim_{t\to t_0^-} q_v(S, t_0) = 0; \lim_{t\to t_0^+} q_v(S, t_0) = 1. \tag{22}$$

Let φ be the set of all such strategies. In these strategies, a resource is transferred from some target $j \in \mathcal{T}$ to target v at time t_0. For other strategies $S \notin \phi$ and $S \notin \varphi$, $q_v(S, t)$ is constant over $(t_0 - \Delta t, t_0 + \Delta t)$. Thus we have

$$c'_v(t_0) = \lim_{\Delta t\to 0} \frac{\int_{S\in\varphi} x_S - \int_{S\in\phi} x_S}{2 \cdot \Delta t} \ge 0. \tag{23}$$

It indicates that

$$\int_{S\in\varphi} x_S - \int_{S\in\phi} x_S \ge 0. \tag{24}$$

Since our goal is to construct a mixed strategy $\mathbf{x}'$ corresponding to $\mathbf{c}(t)$ in which no pure strategies in ϕ or ψ is used, we begin with constructing a mixed strategy $\mathbf{x}^1$ in which no pure strategies in ϕ is used. First, choose a subset $\varphi' \subseteq \varphi$ such that

$$\int_{S\in\varphi'} y_S = \int_{S\in\phi} x_S, \tag{25}$$

$$0 \le y_S \le x_S, \forall S \in \varphi'. \tag{26}$$

Given Eq. 24, there must exist a set φ' which leads to a feasible solution to the above program. Let ϕ^i be a subset of ϕ in which a resource is transferred from target v to target i at time t_0. Thus

$$\int_{S\in\phi} x_S = \sum_{i\in\mathcal{T}} \int_{S\in\phi^i} x_S. \tag{27}$$

Here are the steps of constructing $\mathbf{x}^1$.

1. $\forall S \in \phi$ with $x_S > 0$, convert S to S' by deleting the transfer made from target v at time t_0. Set $x^1_{S'} = x_S$.
2. $\forall i \in \mathcal{T}$, choose a subset $\varphi_i \subseteq \varphi'$ such that

$$\int_{S\in\varphi_i} z_S^{\varphi_i} = \int_{S\in\phi^i} x_S, \tag{28}$$

$$0 \leq z_S^{\varphi_i} \leq y_S, \forall S \in \varphi_i, \tag{29}$$

$$\sum_{i\in\mathcal{T}} z_S^{\varphi_i} = y_S, \forall S \in \varphi'. \tag{30}$$

Given Eqs. 25 and 27, there must exist a set φ_i which leads to a feasible solution to the above program. $\forall S \in \varphi_i$, convert S to S' by changing the destination of the transfer made at time t_0 from target v to target i. Set $x^1_{S'} = z_S^{\varphi_i}$.

(The intuition behind the first two steps is that, if a resource is transferred from target $v \in V$ to target $i \in \mathcal{T}$ at time t_0 with a probability of p_1 while a resource is transferred from target $j \in \mathcal{T}$ to target $v \in V$ at time t_0 with a probability of p_2, then we can get rid of the transfer made from target v by transferring a resource directly from target j to target i instead of target v with a probability of p_1. This is implementable because $p_2 \geq p_1$ given Eq. 24).
3. For other pure strategies $S \in \mathcal{S} \setminus (\phi \cup \varphi')$, set $x^1_S = x_S$.
4. Repeat steps 1-3 until no resource is transferred from targets $v \in V$ to other targets at t_0.

After the steps, the probability that a target is protected at time t_0 given $\mathbf{x}^1$ is the same as it is given $\mathbf{x}$. Based on $\mathbf{x}^1$, we can now construct the mixed strategy $\mathbf{x}'$ corresponding to $\mathbf{c}(t)$ in which no pure strategies in ϕ or ψ is used. Note that no pure strategies in ϕ is used to implement $\mathbf{x}^1$. We now consider strategies $S \in \psi$ with $x^1_S > 0$. Given that $\forall u \in \Lambda, c_u(t)$ monotonically decreases during (θ_k, θ_{k+1}), there must exist pure strategies S with $x^1_S > 0$, in which a resource is transferred from target u to other targets. $\mathbf{x}'$ can be constructed in similar steps as is shown above. Thus no pure strategies in ϕ or ψ is used to implement $\mathbf{x}'$, while $\mathbf{x}'$ still corresponds to $\mathbf{c}(t)$.

Based on the above proof, a resource will not be transferred from target $v \in V$ to other targets during (θ_k, θ_{k+1}). Thus a resource is transferred at most once in this time period. ☐

D Proof for Lemma 5

Lemma 5. *If $|\Lambda| > 2$ and $|V| > 2$, there exists feasible $Z((u,v),t)$ satisfying $\lim_{\Delta t\to 0} \frac{Z((u,v),t+\Delta t)-Z((u,v),t)}{\Delta t} > 0, \forall u, \forall v, \forall t \in [\theta_k, \theta_{k+1}]$.*

Proof. To prove Lemma 5 is to prove that if we change the '$\geq$' in Eq. 11 to '$>$', an LP consisting of Eqs. 8 - 11 has feasible solutions. We prove it by constructing an equivalent LP and showing that the constructed LP has feasible solutions. Given Eq. 12, for any time point $t \in [\theta_k, \theta_{k+1}]$, we can change the variables in Eqs. 8 - 11 from $Z((u,v),t)$ to $z^i_{uv}(t)$. Thus we get the following LP'.

$$LP': \sum_{v\in V} z^u_{uv}(t) = 1, \forall u \in \Lambda \tag{31}$$

$$\sum_{u\in\Lambda} z^v_{uv}(t) = 1, \forall v \in V \tag{32}$$

$$\sum_{v\in V} z^i_{uv}(t) = 0, \forall i \in \mathcal{T} \setminus u, \forall u \in \Lambda \tag{33}$$

$$\sum_{u\in\Lambda} z^i_{uv}(t) = 0, \forall i \in \mathcal{T} \setminus v, \forall v \in V \tag{34}$$

$$\text{Eqs. 10 and 11, change '}\geq\text{' in Eq. 11 to '}>\text{'.} \tag{35}$$

Based on Eq. 12, Eqs. 31 - 34 lead to $Z((u,v),t)$ satisfying Eqs. 8 - 9. Eqs. 10 and 11 naturally have variables $z^i_{uv}(t)$ when we replace $Z((u,v),t)$ with the right side of Eq. 12. Thus LP' is equivalent to Eqs. 8 - 11.

The number of variables in LP' is $nv = |\Lambda| \cdot |V| \cdot |\Lambda + V|$, while the number of functions in LP' is $nf = 2|\Lambda| + 2|V| + 2|\Lambda| * |V|$. If $|\Lambda| > 2$ and $|V| > 2$, then $nv > nf$, thus LP' has a feasible solution. ☐

E Proof for Proposition 4

Proposition 4. *There exist parameter functions which lead to a feasible expression of $Z((u,v),t)$ satisfying that each $z^i_{uv}(t)$ is piecewise constant during $[\theta_k, \theta_{k+1}]$.*

Proof. If $|\Lambda| \leq 2$ or $|V| \leq 2$, based on Eqs. 8 and 9, $Z((u,v),t)$ has a unique value at any $t \in [\theta_k, \theta_{k+1}]$ and each $z^i_{uv}(t)$ is constant over $[\theta_k, \theta_{k+1}]$. If $|\Lambda| > 2$ and $|V| > 2$, let $Z((u,v),t)$ be an expression based on parameter functions $z^i_{uv}(t_0)$, which are solutions to LP' when $t = t_0$. We now show that such $Z((u,v),t)$ satisfies Eqs. 8 - 11 not only at t_0, but during a time period around t_0. Note that Eqs. 31 - 34 ensures $Z((u,v),t)$ to satisfy Eqs. 8 - 9 for all $t \in [\theta_k, \theta_{k+1}]$. In addition, given the form of coverage functions (Line 2 in Algorithm 1), $Z((u,v),t)$ is a quotient of polynomial functions. The sign of $\frac{\mathrm{d}Z((u,v),t)}{\mathrm{d}t}$ does not change continuously. Given that $\frac{\mathrm{d}Z((u,v),t)}{\mathrm{d}t} > 0$ at t_0 and $Z((u,v),t_0) \geq 0$, we have that $Z((u,v),t) > 0$ in a time period after t_0. Therefore, $Z((u,v),t)$ based on constant parameter functions $z^i_{uv}(t_0)$ satisfies Eqs. 8 - 11 in a time period around t_0, which indicates that piecewise constant parameter functions can lead to feasible expression of $Z((u,v),t)$ for $t \in [\theta_k, \theta_{k+1}]$. ☐

F Proof for Theorem 6

Theorem 6. *Algorithm 2 computes a feasible expression of $Z((u,v),t), \forall t \in [\theta_k, \theta_{k+1}]$ after finite loops.*

Proof. First, based on Line 3 and Line 9 in Algorithm 2, the resulted $Z((u,v),t)$ is feasible, i.e., satisfying Eqs. 8 - 10, for all $t \in [\theta_k, \theta_{k+1}]$. Given that the constraint of Eq. 11 is set as strict '$>$', we have $t_m > t_0$ in each round. Thus the iteration terminates after finite loops. ☐

G Proof for Lemma 9

Lemma 9. *Assume that transfer time between any target pair d_{ij} is multiplier of δ. If $\mathbf{x}$ is an equilibrium strategy of G_d, it is an equilibrium strategy of G_b.*

Proof. Note that this does not apply directly from Lemma 8, since the strategy space of G_d is a subset of the strategy space of G_b.We prove it by showing that there must exist an equilibrium strategy of G_b that can be mapped to an equilibrium of G_d.

We first define the following map Φ that maps each pure strategy S in G_b to another pure strategy S': for each resource transfer in S, we adjust its starting time from t to $r\delta$, where $t \in (r\delta, (r+1)\delta)$ and $r \in \mathbb{N}$. Furthermore, we map a mixed strategy $\mathbf{x}$ to another one $\mathbf{x}'$, such that if $\mathbf{x}$ uses pure strategy S with probability x_S, then $\mathbf{x}'$ uses S' with a probability of $x'_{S'} = \int_S x_S I(\Phi(S) = S')$, where $I(\Phi(S) = S')$ indicates whether S can be mapped to S' by Φ. Thus we obtain $\mathbf{x}'$ where all resource transfers start at $r\delta$, $r \in \mathbb{N}$. Note that $\mathbf{x}'$ is also a mixed strategy of G_d.

Next, we show that if $\mathbf{x}$ is an equilibrium strategy of G_b, $\mathbf{x}'$ is an equilibrium strategy of G_b and G_d. Note that the defender will not be better off in G_d than G_b since her strategy space in G_d is a subset of that in G_b. Thus we only need to show that $\mathbf{x}'$ is an equilibrium strategy of G_b, then it follows readily that $\mathbf{x}'$ is also an equilibrium strategy of G_d.

Since $\mathbf{x}$ is the equilibrium strategy of G_b, we have $U_d^{G_b}(\mathbf{x}) \geq U_d^{G_b}(\mathbf{x}')$. We now show that $U_d^{G_b}(\mathbf{x}) \leq U_d^{G_b}(\mathbf{x}')$, thus $U_d^{G_b}(\mathbf{x}) = U_d^{G_b}(\mathbf{x}')$, and $\mathbf{x}'$ is the equilibrium strategy of G_b. Let $\mathbf{c} = \langle c_i(t) \rangle$ and $\mathbf{c}' = \langle c'_i(t) \rangle$ represent the coverage vectors corresponding to $\mathbf{x}$ and $\mathbf{x}'$ respectively. We show that, within time interval $[r\delta, (r+1)\delta)$, $r \in \mathbb{N}$, the minimum coverage for a target in $\mathbf{c}$ is no larger than that for this target in $\mathbf{c}'$. Since the players' payoffs are determined by minimum coverages in the time intervals (as the value functions are constant over each time interval), it follows readily that $U_d^{G_b}(\mathbf{x}) \leq U_d^{G_b}(\mathbf{x}')$. Given $\mathbf{c}$ and $\mathbf{c}'$, we have

$$
\begin{aligned}
&\min_{t \in [r\delta, (r+1)\delta)} c_i(t) \\
=& \min_{t \in [r\delta, (r+1)\delta)} \int_S x_S \cdot q_i(S,t) \\
=& \min_{t \in [r\delta, (r+1)\delta)} \Big(c_i(r\delta) + \int_S x_S \cdot \big(q_i(S,t) - q_i(S, r\delta) \big) \Big) \\
=& \min_{t \in [r\delta, (r+1)\delta)} \Big(c_i(r\delta) + \int_{S\text{: a transfer to } i \text{ ends in } (r\delta, t] \text{ by } S} x_S \; - \\
& \quad \int_{S\text{: a transfer away from } i \text{ starts in } (r\delta, t] \text{ by } S} x_S \Big) \\
\leq& \min_{t \in [r\delta, (r+1)\delta)} \Big(c_i(r\delta) + \int_{S\text{: a transfer to } i \text{ ends in } (r\delta, (r+1)\delta) \text{ by } S} x_S \\
& \quad - \int_{S\text{: a transfer away from } i \text{ starts in } (r\delta, t] \text{ by } S} x_S \Big) \\
=& \min_{t \in [r\delta, (r+1)\delta)} \Big(c'_i(r\delta) - \int_{S\text{: a transfer away from } i \text{ starts in } (r\delta, t] \text{ by } S} x_S \Big) \\
=& \min_{t \in [r\delta, (r+1)\delta)} c'_i(t)
\end{aligned}
$$

The last two equalities hold since all transfers ending (respectively, starting) in $(r\delta, (r+1)\delta)$ in $\mathbf{x}$ ends (respectively, starts) at $r\delta$ in $\mathbf{x}'$. □

H Proof for Theorem 10

Theorem 10. *Let $\mathbf{x}'$ be an equilibrium strategy of G_d and $\mathbf{x}$ be an equilibrium strategy of G, we have*

$$U_d^G(\mathbf{x}') - U_d^G(\mathbf{x}) \geq -\epsilon,$$

where $U_d^G(\mathbf{x}')$ is the defender utility in G if the defender plays $\mathbf{x}'$ and the attacker responds the best, while $U_d^G(\mathbf{x})$ is the optimal defender utility in G.

Proof. This is because

$$
\begin{aligned}
& U_d^G(\mathbf{x}') - U_d^G(\mathbf{x}) \\
=& [U_d^G(\mathbf{x}') - U_d^{G_b}(\mathbf{x}')] + [U_d^{G_b}(\mathbf{x}') - U_d^{G_b}(\mathbf{x})] + \\
& [U_d^{G_b}(\mathbf{x}) - U_d^G(\mathbf{x})] \\
\geq& [U_d^G(\mathbf{x}') - U_d^{G_b}(\mathbf{x}')] + [U_d^{G_b}(\mathbf{x}) - U_d^G(\mathbf{x})] \\
\geq& - |U_d^G(\mathbf{x}') - U_d^{G_b}(\mathbf{x}')| - |U_d^G(\mathbf{x}) - U_d^{G_b}(\mathbf{x})|
\end{aligned}
$$

where the first inequality is due to $U_d^{G_b}(\mathbf{x}') - U_d^{G_b}(\mathbf{x}) \geq 0$ since $\mathbf{x}'$ is a better strategy than $\mathbf{x}$ for G_d, hence G_b (Lemma 8). Given that $\mathbf{x}'$ is the equilibrium strategy of G_d, the corresponding coverage functions are piecewise constant in any time interval $[t_{k-1}, t_k], \forall k = 1, \ldots, |\zeta|$. Thus the players' payoffs are determined by the maximum value of the value functions of targets. Given that the value of value functions in G_b in a time interval is equal to the maximum value of the corresponding value functions in G in this time interval (Eq. 14), we have $U_d^G(\mathbf{x}') = U_d^{G_b}(\mathbf{x}')$. Since $v_i(t)$ differs from $v'_i(t)$ by at most ϵ for any i, t, we have $U_d^G(\mathbf{x}) - U_d^{G_b}(\mathbf{x}) \leq \epsilon$. Therefore,

$$U_d^G(\mathbf{x}') - U_d^G(\mathbf{x}) \geq 0 - \epsilon = -\epsilon. \tag{36}$$

□

I Proof for Theorem 11

Theorem 11. *A mixed strategy of G_d is an m-unit fractional flow, and vise versa*

Proof. It is straightforward that a mixed strategy of G_d is an m-unit fractional flow. The other direction is due to the following conclusion in graph theory: in a directed graph with integer capacities, the vertices of the polytope of m-unit fractional flows are precisely all the m-unit integer flows. Therefore, any m-unit fractional flow can be decomposed as a convex combination (thus a distribution) over $0-1$ integer flows (i.e., pure strategies) in our case. □