1. (10 pts) Give proofs in $\text{Ni}$ (see Troelstra & Schwichtenberg’s “Basic Proof Theory”) of the following formulas:

$A \rightarrow (B \rightarrow A)$

$A \rightarrow A \lor B$

$(A \rightarrow C) \rightarrow [(B \rightarrow C) \rightarrow (A \lor B \rightarrow C)]$

$A \rightarrow (B \rightarrow A \land B)$

$\bot \rightarrow A$

2. (10 pts) Same as Question 1, but do it in $\text{G1i}$. 

3. (10 pts) Prove in $\text{Ni}$ ($\neg
\neg A \rightarrow \neg
\neg B$) $\rightarrow \neg
\neg (A \rightarrow B)$. Hint: First construct deductions of $\neg
\neg A$ and of $\neg B$ from the assumption $\neg(A \rightarrow B)$.

4. (5 pts) Consider the following version of “disjunction property”:

For any $\Gamma$, $A$ and $B$, $\text{G1i} \vdash \Gamma \Rightarrow A \lor B$ implies either $\text{G1i} \vdash \Gamma \Rightarrow A$ or $\text{G1i} \vdash \Gamma \Rightarrow B$.

This statement is not true in general. Show a counterexample where this property fails.

5. (15 pts) Consider the class formulas generated by the following grammar:

$\mathcal{F} := p | \mathcal{F} \land \mathcal{F} | \mathcal{G} \rightarrow \mathcal{F}$

$\mathcal{G} := p | \mathcal{F} \rightarrow \mathcal{G} | \mathcal{G} \land \mathcal{G} | \mathcal{G} \lor \mathcal{G}$

where $p$ is a propositional variable. Show that the disjunction property stated in Question 4 is true when $\Gamma$ is a multiset of $\mathcal{F}$-formulas and $A$ and $B$ are $\mathcal{G}$-formulas.

6. (10 pts) Prove the following statement: $\text{G1i} + \text{Cut} \vdash \Gamma \Rightarrow A$ if and only if $\text{Ni} \vdash \Gamma \Rightarrow A$. Consider only the propositional case (i.e., no quantifiers in the formulas).

7. (10 pts) An inference rule is said to be invertible if provability of its conclusion implies provability of its premise(s), and vice versa. Show that the $R \rightarrow$ rule of $\text{G1i}$ is invertible. Hint: there is an easy proof using cut and the cut-elimination theorem for $\text{G1i}$.

8. (5 pts) Prove that in $\text{G1i}$, the axiom $A \Rightarrow A$, for arbitrary formula $A$, can be derived from a more restricted axiom $P \Rightarrow P$, where $P$ is an atomic proposition (i.e., it has no logical connectives in it).

9. (5 pts) Do there exist simply typed lambda-terms $t$ and $u$ such that $(t \ u)$ reduces to $t$? If yes, give example terms, otherwise, prove it.

10. (15 pts) Let us extend $\text{G1i}$ with a new ternary operator, $O(\ldots)$, whose inference rules are as follows:

$$\frac{A,B,\Gamma \Rightarrow D \quad C,\Gamma \Rightarrow D}{O(A,B,C),\Gamma \Rightarrow D} \quad OL$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow O(A,B,C)} \quad OR_1$$

$$\frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow O(A,B,C)} \quad OR_2$$

Let us call this extended proof system $\text{G1i}^+$. Is the proof system consistent, i.e., does there exist a formula $A$ such that $\text{G1i}^+ \vdash A$ and $\text{G1i}^+ \vdash \neg A$?