1 Partial Differential Equation

- Stochastic Differential Equation
- Feynman-Kac
Two practical approaches to pricing a contingent claim:

- **Risk-neutral approach:** Simulate price paths of the underlying under the risk-neutral measure and use these paths to estimate the risk-neutral expected discounted payoff.
- **Partial differential equation approach:** Solve numerically the partial differential equation.

Consider a stochastic differential equation

\[ dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u), \]

with initial condition

\[ X(t) = x, \]

where \( \beta(u, x) \) and \( \gamma(u, x) \) are given functions, \( W(u) \) is a Brownian motion, \( 0 \leq t \leq u \leq T \) and \( x \in \mathbb{R} \).
Suppose there exists some $C$ such that

$$|\beta(u,x) - \beta(u,y)| + |\gamma(u,x) - \gamma(u,y)| \leq C|x - y|,$$

$$|\beta(u,x)| + |\gamma(u,x)| \leq C(1 + |x|)$$

for $0 \leq u \leq T$ and $x, y \in \mathbb{R}$. Then there exists a stochastic process solution $X$ such that

$$X(T) = x + \int_t^T \beta(u, X(u)) \, du + \int_t^T \gamma(u, X(u)) \, dW(u).$$

Let $\beta(u,x) = \beta_1(u) + \beta_2(u)x$ and $\gamma(u,x) = \gamma_1(u) + \gamma_2(u)x$. Then the linear stochastic differential equation

$$dX(u) = \left(\beta_1(u) + \beta_2(u)X(u)\right) \, du + \left(\gamma_1(u) + \gamma_2(u)X(t)\right) \, dW(u),$$

$X(t) = x,$

has the solution

$$X(T) = e^{\int_t^T \left(\beta_1(u) - \frac{1}{2}\gamma_2^2(u)\right) \, du + \int_t^T \gamma_2(u) \, dW(u)} \times \left[ x + \int_t^T e^{\int_r^T \left(\beta_2(v) - \frac{1}{2}\gamma_2^2(v)\right) \, dv - \int_r^T \gamma_2(v) \, dW(v)} \left(\beta_1(u) - \gamma_1(u)\gamma_2(u)\right) \, du + \int_t^T e^{\int_r^T \left(\beta_2(v) - \frac{1}{2}\gamma_2^2(v)\right) \, dv - \int_r^T \gamma_2(v) \, dW(v)} \gamma_1(u) \, dW(u) \right].$$
Example

The geometric Brownian motion
\[ dS(u) = \mu S(u) \, du + \sigma S(u) \, dW(u) \]
with initial condition
\[ S(t) = S, \]
has the solution
\[ S(T) = S e^{\left( \mu - \frac{1}{2} \sigma^2 \right) (T-t) + \sigma (W(T) - W(t))}. \]

The Hull-White interest rate process
\[ dr(u) = b(u) \left( a(u) - r(u) \right) du + \sigma(u) \, dW(u) \]
with initial condition
\[ r(t) = r, \]
has the solution
\[ r(t) = r e^{- \int_t^T b(u) \, du} + \int_t^T e^{- \int_t^v b(\nu) \, d\nu} a(u) b(u) \, du \\
+ \int_t^T e^{- \int_t^v b(\nu) \, d\nu} \sigma(u) \, dW(u). \]
Fix $T > 0$. Let $0 \leq t \leq T$ and $h(x)$ be a Borel-measurable function. Define

$$g(t, x) = \mathbb{E}^{t, x}\left(h(X(T))\right),$$

where $X(T)$ solves

$$dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u),$$
$$X(t) = x.$$

Let $X(u), u \geq 0$, be a solution of the stochastic differential equation

$$dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u),$$
$$X(0) = x.$$

Then

$$\mathbb{E}\left(h(X(T)) \mid \mathcal{F}(t)\right) = g(t, X(t)).$$

Hence solutions to stochastic differential equations are Markov processes.
Let $X(u), u \geq 0$, be a solution for

$$dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u).$$

with initial condition given at time 0. Fix $T > 0$. Let $0 \leq t \leq T$ and $h(x)$ be a Borel-measurable function. Define

$$g(t, x) = E^{t,x} (h(X(T))).$$

Then $g(t, X(t)), 0 \leq t \leq T$, is a martingale.

### Proof

For $0 \leq s \leq t$,

$$E(g(t, X(t)) \mid \mathcal{F}(s)) = E(E(h(X(T)) \mid \mathcal{F}(t)) \mid \mathcal{F}(s))$$

$$= E(h(X(T)) \mid \mathcal{F}(s))$$

$$= g(s, X(s)).$$
Feynman-Kac Theorem

Consider a stochastic process $X$ that satisfies the stochastic differential equation

$$dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u).$$

Fix $T > 0$. Let $0 \leq t \leq T$ and $h(x)$ be a Borel-measurable function. Define

$$g(t, x) = \mathbb{E}^{t,x} \left( h(X(T)) \right).$$

Then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x) g_x(t, x) + \frac{1}{2} \gamma^2(t, x) g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x), \; \forall x.$$

Proof

Using the Itô-Doeblin formula,

$$d\left( g(t, X(t)) \right) = g_t(t, X(t)) \, dt + g_x(t, X(t)) \, dX(t) + \frac{1}{2} g_{xx}(t, X(t)) \, d[X, X](t)$$

$$= \beta(t, X(t)) \, dt + \gamma(t, X(t)) \, dW(t)$$

$$= \left( g_t(t, X(t)) + \beta(t, X(t)) g_x(t, X(t)) + \frac{1}{2} \gamma^2(t, X(t)) g_{xx}(t, X(t)) \right) \, dt$$

$$+ \gamma(t, X(t)) \, g_x(t, X(t)) \, dW(t).$$

Since $g(t, X(t))$ is a martingale,

$$= 0$$

and

$$+ \gamma(t, X(t)) \, g_x(t, X(t)) \, dW(t).$$
Consider a stochastic process $X$ that satisfies the stochastic differential equation
\[ dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u). \]

Fix $T > 0$. Let $0 \leq t \leq T$, $r$ be a constant and $h(x)$ be a Borel-measurable function. Define
\[ f(t, x) = \mathbb{E}^{t,x} \left( e^{-r(T-t)} h(X(T)) \right). \]

Then $f(t, x)$ satisfies the partial differential equation
\[ f_t(t, x) + \beta(t, x) f_x(t, x) + \frac{1}{2} \gamma^2(t, x) f_{xx}(t, x) = rf(t, x) \]
and the terminal condition
\[ f(T, x) = h(x), \quad \forall x. \]

**Proof**

Since $e^{-rt} f(t, x) = \mathbb{E}^{t,x} \left( e^{-rT} h(X(T)) \right)$ is a martingale,
\[
d(e^{-rt} f(t, X(t))) = -r e^{-rt} f(t, X(t)) \, dt + e^{-rt} \left( f_t(t, X(t)) + \frac{1}{2} f_{xx}(t, X(t)) \right) \, dt
\]
\[ = e^{-rt} \left( -r f(t, X(t)) + f_t(t, X(t)) + \beta(t, X(t)) f_x(t, X(t)) + \frac{1}{2} \gamma^2(t, X(t)) f_{xx}(t, X(t)) \right) \, dt
\]
\[ = 0 \text{ since } e^{-rt} f(t, x) \text{ is a martingale}
\]
\[ + e^{-rt} \gamma(t, X(t)) f_x(t, X(t)) \, dW(t). \]
Example

Consider an underlying risky asset $S$ whose price satisfies the geometric Brownian motion

$$dS(u) = r S(u) du + \sigma S(u) d\tilde{W}(u)$$

under the risk-neutral $Q$-measure, where $r$ is the constant risk-free rate and $\sigma$ is the constant volatility.

Let $C$ be an option on the underlying where the option payoff at maturity time $T$ is $h(S(T))$. Under the risk-neutral $Q$-measure, the value of the option at time $t$, $0 \leq t \leq T$, is

$$C(t, S(t)) = \mathbb{E}_Q^F \left( e^{-r(T-t)} h(S(T)) \right| F(t) \right).$$

Using the discounted Feynman-Kac theorem, $C(t, s)$ satisfies the partial differential equation

$$C_t(t, s) + r s C_s(t, s) + \frac{1}{2} \sigma^2 s^2 C_{ss}(t, s) = r C(t, s)$$

and the terminal condition

$$C(T, s) = h(s), \quad \forall s.$$

This is the Black-Scholes-Merton partial differential equation.
Example

Consider an interest rate process $r$ that satisfies
\[ dr(u) = \beta(u, r(u)) \, du + \gamma(u, r(u)) \, d\tilde{W}(u), \]
under the risk-neutral $\mathbb{Q}$-measure.

For $t \geq 0$, define the discount process $D$ by
\[ D(t) = e^{-\int_0^t r(u) \, du}. \]

Consider a zero-coupon bond that pays $1$ at time $T$. The value of the bond at time $t$, $0 \leq t \leq T$, is given by
\[ B(t, T) = \mathbb{E}^\mathbb{Q} \left( e^{-\int_t^T r(u) \, du} \bigg| \mathcal{F}(t) \right) = \frac{1}{D(t)} \mathbb{E}^\mathbb{Q} \left( D(T) \bigg| \mathcal{F}(t) \right). \]

Since $r$ is given by a stochastic differential equation, it is a Markov process, i.e.
\[ B(t, T) = f(t, r(t)). \]
Consider the discounted bond price process where
\[ d(D(t)f(t, r(t))) = D(t)
\left(-r(t)f(t, r(t)) + f_t(t, r(t))
+ \beta(t, r(t))f_r(t, r(t)) + \frac{1}{2}\gamma^2(t, r(t))f_{rr}(t, r(t))\right) dt
+ \gamma(t, r(t))f_r(t, r(t)) d\tilde{W}(u). \]

Since the discounted bond price is a $Q$-martingale, $f(t, r)$ satisfies the partial differential equation
\[ f_t(t, r) + \beta(t, r)f_r(t, r) + \frac{1}{2}\gamma^2(t, r)f_{rr}(t, r) = rf(t, r) \]
with terminal condition
\[ f(T, r) = 1, \quad \forall r. \]

Consider the Hull-White interest rate model where
\[ dr(t) = b(t)(a(t) - r(t)) dt + \sigma(t) d\tilde{W}(t), \quad t \geq 0. \]

Let $f(t, r)$ be the time $t$ value of a zero-coupon bond that pays $1$ at time $T$ where $0 \leq t \leq T$. Then
\[ f_t(t, r) + b(t)(a(t) - r) f_r(t, r) + \frac{1}{2}\sigma^2(t)f_{rr}(t, r) = rf(t, r) \]
with terminal condition
\[ f(T, r) = 1, \quad \forall r. \]
The solution is
\[ f(t, r) = e^{-rt} C(t, T) - A(t, T) \]

where
\[
A(t, T) = \int_t^T \left( a(u)b(u)C(u, T) - \frac{1}{2} \sigma^2(u)C^2(u, T) \right) du,
\]
\[
C(t, T) = \int_t^T e^{-\int_t^u b(v) dv} du.
\]

The zero-coupon bond that pays $1 at time \( T \) has a value at time \( t \), \( 0 \leq t \leq T \), given by
\[ B(t, T) = f(t, r(t)). \]

Consider a call option on a zero-coupon bond that pays $1 at time \( T_2 \) where the call has strike price \( K \) and maturity at time \( T_1 \), \( T_1 < T_2 \). The value of the call at time \( t \), \( 0 \leq t \leq T_1 \), is
\[
C(t, r(t)) = \frac{1}{D(t)} \mathbb{E}^Q \left( \left( D(T_1)(B(T_1, T_2) - K) \right)^+ \bigg| \mathcal{F}(t) \right).
\]

The value of the call satisfies
\[
C_t(t, r) + b(t)(a(t) - r)C_r(t, r) + \frac{1}{2} \sigma^2(t)C_{rr}(t, r) = r C(t, r)
\]
with terminal condition
\[
C(T_1, r) = (B(T_1, T_2) - K)^+, \quad \forall r.
\]
Multi-Dimensional Itô-Doebelin Formula

Let $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^m \mapsto \mathbb{R}$ be a $C^{1,2}\ldots,2$-function, and $X$ be a $m$-dimensional Itô process with

$$dX(t) = \beta(t, X(t)) \, dt + \gamma(t, X(t)) \, dW(t)$$

where $\beta = (\beta_i)_{m \times 1}$, $\gamma = (\gamma_{ij})_{m \times d}$ and $W = (W_t)_{d \times 1}$ is a $d$-dimensional Brownian motion. Then

$$df = \left( f_t + \sum_{i=1}^m \beta_i f_{x_i} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \gamma_{ik} \gamma_{jk} f_{x_i x_j} \right) \, dt + \sum_{i=1}^m \sum_{k=1}^d \gamma_{ik} f_{x_i} dW_k \quad \overset{\text{def}}{=} \mathcal{L}f \, dt + \nabla_x f \cdot dW,$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^m \beta_i(t, X(t)) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij}(t, X(t)) \frac{\partial^2}{\partial x_i \partial x_j},$$

$$\nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} \right)^\top$$

and

$$\sigma(t, X(t)) = \left( \gamma(t, X(t)) \gamma(t, X(t))^\top \right).$$
Consider a $m$-dimensional stochastic process $X$ that satisfies the stochastic differential equation
\[ dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u) \]
with initial condition $X(t) = x$, where $\beta$ is a $m$-dimensional and $\gamma$ is a $m \times d$-dimensional adapted process, and $W$ is a $d$-dimensional Brownian motion.

The solution $X$ to the stochastic differential equation is a Markov process.

Fix $T > 0$. Let $0 \leq t \leq T$ and $h : \mathbb{R}^m \to \mathbb{R}$ be a Borel-measurable function. Define
\[ g(t, x) = \mathbb{E}^{t,x}(h(X(T))) , \]
\[ f(t, x) = \mathbb{E}^{t,x}(h(e^{-r(T-t)}X(T))) . \]

Then
\[ dg(t, X(t)) = \mathcal{L}g(t, X(t)) \, dt + \nabla_x g(t, X(t)) \cdot \gamma(t, X(t)) \, dW(t) \]
\[ d(e^{-rt}f(t, X(t))) = \mathcal{L}(e^{-rt}f(t, X(t))) \, dt \]
\[ + \nabla_x (e^{-rt}f(t, X(t))) \cdot \gamma(t, X(t)) \, dW(t) . \]
Consider an underlying risky asset $S$ whose price satisfies the geometric Brownian motion

$$dS(u) = rS(u)\, du + \sigma S(u)\, d\tilde{W}(u)$$

under the risk-neutral $Q$-measure, where $r$ is the constant risk-free rate and $\sigma$ is the constant volatility.

Let $C$ be an option on the underlying where the option payoff at maturity time $T$ is

$$C(T) = \left( \frac{1}{T} \int_0^T S(u)\, du - K \right)^+. $$

For $0 \leq t \leq T$, define

$$Y(t) = \int_0^t S(u)\, du,$$

$$X(t) = \begin{pmatrix} S(t) \\ Y(t) \end{pmatrix}.$$

Then

$$dX(t) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} X(t)\, du + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} X(t)\, d\tilde{W}(u).$$

Thus $X(t)$ is a 2-dimensional Markov process.
Verification

Since
\[ dS(t) = rS(t) \, dt + \sigma S(t) \, d\tilde{W}(t), \]
\[ dY(t) = S(t) \, dt, \]
we have
\[ d\begin{pmatrix} S(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} r & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S(t) \\ Y(t) \end{pmatrix} \, dt + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(t) \\ Y(t) \end{pmatrix} \, d\tilde{W}(t). \]

The value of the option at time \( t, 0 \leq t \leq T_1 \), is
\[ C(t, X(t)) = \frac{1}{D(t)} \mathbb{E}^Q \left( D(T) C(X(T)) \mid \mathcal{F}(t) \right). \]

The value of the option satisfies the partial differential equation
\[ C_t(t, s, y) + r s C_s(t, s, y) + s C_y(t, s, y) + \frac{1}{2} \sigma^2 s^2 C_{ss}(t, s, y) = r C(t, s, y) \]
and the terminal condition
\[ C(T, s, y) = \left( \frac{y}{T} - K \right)_+, \quad \forall y. \]
Since
\[ D(t) C(t, X(t)) = \mathbb{E}_Q^F \left( D(T) C(X(T)) \mid \mathcal{F}(t) \right), \]
using the Itô-Doeblin formula,
\[
d \left( D(t) C(t, X(t)) \right) = \mathcal{L}(D(t) C(t, X(t))) dt + \nabla_x (D(t) C(t, X(t))) \cdot \gamma(t, X(t)) dW(t), \]
where
\[
\mathcal{L}(D(t) C(t, X(t))) = -r D(t) C(t, X(t)) + \frac{1}{2} \sigma^2 S^2(t) C_{SS}(t, X(t)) + r D(t) S(t) C_S(t, X(t)) + D(t) S(t) C_Y(t, X(t)) + D(t) C_i(t, X(t)) + r D(t) S(t) C_S(t, X(t)) + \frac{1}{2} \sigma^2 D(t) S^2(t) C_{SS}(t, X(t)).
\]
Hence,
\[
-r C(t, X(t)) + C_i(t, X(t)) + r S(t) C_S(t, X(t)) + S(t) C_Y(t, X(t)) + \frac{1}{2} \sigma^2 S^2(t) C_{SS}(t, X(t)) = 0.
\]