

Exact N -envelope-soliton solutions of the Hirota equation

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We discuss some properties of the soliton equations of the type $\partial u/\partial t = S[u, \bar{u}]$, where S is a nonlinear operator differential in x , and present the additivity theorems of the class of the soliton equations. On using the theorems, we can construct a new soliton equation through two soliton equations with similar properties. Meanwhile, exact N -envelope-soliton solutions of the Hirota equation are derived through the trace method.

Keywords: exact solutions, Hirota equation, solitons.

The trace method, which has been applied to the Korteweg-de Vries equation [1], modified Korteweg-de Vries equation [2], Kadomtsev-Petviashvili equation [3], sine-Gordon equation [4], [5] and Gz Tu equation [6], is useful for understanding these equations. The N -soliton solutions and some other results of these equations [7] have been derived through the trace method.

The present paper deals with an application of the trace method to the nonlinear partial differential equation as follows:

$$\frac{\partial}{\partial t} u + L_x u = N_x(u, \bar{u}) \quad (1)$$

where:

$$L_x u = \sum_{k=0}^{N_1} \alpha_k \frac{\partial^k}{\partial x^k} u,$$

$$N_x(u, \bar{u}) = \sum_{k=1}^{N_2} \beta_k \prod_{m=0}^{N_k} \left(\frac{\partial^m}{\partial x^m} u \right)^{r_{m,k}} \left(\frac{\partial^m}{\partial x^m} \bar{u} \right)^{s_{m,k}}$$

where α_k, β_k are complex constants, $r_{m,k}, s_{m,k}$ are nonnegative integers, $r_k = \sum_{m=0}^{N_k} r_{m,k}; s_k = \sum_{m=0}^{N_k} s_{m,k}; r_1 = r_2 = \dots = r_{N_2} = r; s_1 = s_2 = \dots = s_{N_2} = s$ and $d = r + s \geq 2; r, s$ satisfy one of the relations:

$$s \geq 1 \quad \text{for} \quad r = s + 1, \tag{i}$$

$$s = 0 \quad \text{for} \quad r \geq 2. \tag{ii}$$

Substituting the formal series

$$u = u^{(1)} + u^{(d)} + \dots + u^{((d-1)n+1)} + \dots \tag{2}$$

into Eq. (1), we obtain a set of equations for $u^{((d-1)n+1)}$ ($n = 0, 1, 2, \dots$):

$$\begin{aligned} \left(\frac{\partial}{\partial t} + L_x\right)u^{((d-1)n+1)} &= \sum_{l_1=1}^N \dots \sum_{l_{(d-1)n+1}=1}^N C^{(n)}(P_{l_1}, \bar{P}_{l_2}, \dots, \bar{P}_{l_{(d-1)n}}, P_{l_{(d-1)n+1}}) \\ &\times \phi_{l_1}^2 \bar{\phi}_{l_2}^2 \dots \bar{\phi}_{l_{(d-1)n}}^2 \phi_{l_{(d-1)n+1}}^2, \end{aligned} \tag{3}$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + L_x\right)u^{((r-1)n+1)} &= \sum_{l_1=1}^N \dots \sum_{l_{(r-1)n+1}=1}^N C^{(n)}(P_{l_1}, P_{l_2}, \dots, P_{l_{(r-1)n}}, P_{l_{(r-1)n+1}}) \\ &\times \phi_{l_1}^2 \phi_{l_2}^2 \dots \phi_{l_{(r-1)n}}^2 \phi_{l_{(r-1)n+1}}^2 \end{aligned} \tag{4}$$

where Eqs. (3) and (4) correspond to relations (i) and (ii), respectively,

$$\phi_k(x, t) = A_k(0) \exp(P_k x - \Omega_k t), \quad \Omega_k = \frac{1}{2} L_p(2P_k),$$

$$L_p(x) = \sum_{k=0}^{N_1} \alpha_k x^k,$$

$A_k(0)$ and P_k are complex constants ($k = 1, 2, \dots, N$), and $C^{(0)} = 0$.

We can obtain the solutions for Eqs. (3) and (4) in the following form:

$$u^{((d-1)n+1)} = \sum_{l_1=1}^N \dots \sum_{l_{(d-1)n+1}=1}^N \pi^{(n)} \phi_{l_1}^2 \bar{\phi}_{l_2}^2 \dots \bar{\phi}_{l_{(d-1)n}}^2 \phi_{l_{(d-1)n+1}}^2, \tag{5}$$

$$u^{((r-1)n+1)} = \sum_{l_1=1}^N \dots \sum_{l_{(r-1)n+1}=1}^N \pi^{(n)} \phi_{l_1}^2 \phi_{l_2}^2 \dots \phi_{l_{(r-1)n}}^2 \phi_{l_{(r-1)n+1}}^2 \tag{6}$$

where $\pi^{(0)} = 1$ and

$$\pi^{(n)} = C^{(n)} / [L_p(2P_{l_1} + 2\bar{P}_{l_2} + \dots + 2\bar{P}_{l_{(d-1)n}} + 2P_{l_{(d-1)n+1}}) - L_p(2P_{l_1}) - \bar{L}_p(2\bar{P}_{l_2}) - \dots - \bar{L}_p(2\bar{P}_{l_{(d-1)n}}) - L_p(2P_{l_{(d-1)n+1}})] \tag{7}$$

or

$$\pi^{(n)} = C^{(n)} / [L_p(2P_{l_1} + 2P_{l_2} + \dots + 2P_{l_{(r-1)n}} + 2P_{l_{(r-1)n+1}}) - L_p(2P_{l_1}) - L_p(2P_{l_2}) - \dots - L_p(2P_{l_{(r-1)n}}) - L_p(2P_{l_{(r-1)n+1}})] \tag{8}$$

Theorem 1. Let

$$\frac{\partial u}{\partial t} + L'_x u = N'_x(u, \bar{u}), \quad \frac{\partial u}{\partial t} + L''_x u = N''_x(u, \bar{u})$$

be two arbitrary equations that are defined by Eq. (1). If $r' = r''$, $s' = s''$, $\pi^{(n)} = \pi''^{(n)}$ ($n = 0, 1, 2, \dots$), then, for equation

$$\frac{\partial u}{\partial t} + L^*_x u = N^*_x(u, \bar{u})$$

(where $L^*_x = aL'_x + bL''_x$, $N^*_x(u, \bar{u}) = aN'_x(u, \bar{u}) + bN''_x(u, \bar{u})$, and a, b are two arbitrary real numbers), we have $\pi^{*(n)} = \pi'^{(n)} = \pi''^{(n)}$ ($n = 0, 1, 2, \dots$).

Proof. We consider the case (ii) by mathematical induction. Obviously $\pi^{*(0)} = \pi'^{(0)} = \pi''^{(0)} = 1$. Assume $\pi^{*(n)} = \pi'^{(n)} = \pi''^{(n)}$ ($n = 0, 1, 2, \dots, k$). When $n = k + 1$, from Eq. (4), $C^{*(k+1)} = aC'^{(k+1)} + bC''^{(k+1)}$, and from Eq. (8)

$$\begin{aligned} \pi^{*(k+1)} &= C^{*(k+1)} / \left[L_p \left(2 \sum_{m=1}^{(r-1)n+1} P_{l_m} \right) - \sum_{m=1}^{(r-1)n+1} L_p^*(2P_{l_m}) \right] \\ &= [aC'^{(k+1)} + bC''^{(k+1)}] / \left[aL'_p \left(2 \sum_{m=1}^{(r-1)n+1} P_{l_m} \right) + bL''_p \left(2 \sum_{m=1}^{(r-1)n+1} P_{l_m} \right) \right. \\ &\quad \left. - a \sum_{m=1}^{(r-1)n+1} L'_p(2P_{l_m}) - b \sum_{m=1}^{(r-1)n+1} L''_p(2P_{l_m}) \right] = \pi'^{(k+1)} = \pi''^{(k+1)}. \end{aligned}$$

For the case (i), we can prove it in the same manner.

We introduce two $N \times N$ matrices B and D whose elements are given respectively by $B_{mn} = [1/(P_m + P_n)]\phi_m(x, t)\phi_n(x, t)$, $D_{mn} = [1/(P_m + \bar{P}_n)]\phi_m(x, t)\bar{\phi}_n(x, t)$.

Theorem 2. Let

$$\frac{\partial u}{\partial t} + L'_x u = N'_x(u, \bar{u}), \quad \frac{\partial u}{\partial t} + L''_x u = N''_x(u, \bar{u})$$

be two arbitrary equations that are defined by Eq. (1). If they have respective solutions

$$u' = \text{Tr}[B'_x f(D' \bar{D}')] \quad (\text{or } \text{Tr}[B'_x g(B')]),$$

$$u'' = \text{Tr}[B''_x f(D'' \bar{D}'')] \quad (\text{or } \text{Tr}[B''_x g(B'')])$$

where f, g are arbitrarily derivable functions in the neighbourhood of zero, then, for equation

$$\frac{\partial u}{\partial t} + L^*_x u = N^*_x(u, \bar{u})$$

(where $L^*_x = aL'_x + bL''_x$, $N^*_x(u, \bar{u}) = aN'_x(u, \bar{u}) + bN''_x(u, \bar{u})$, and a, b are two arbitrary real numbers), we have solution

$$u^* = \text{Tr}[B^*_x f(D^* \bar{D}^*)] \quad (\text{or } \text{Tr}[B^*_x g(B^*)]).$$

Proof. Since f, g are arbitrarily derivable functions in the neighbourhood of zero, f, g can be expanded into power series in convergence region. Correspondingly, u', u'' can be expanded into power series. Comparing the coefficients, we have $r' = r''$, $s' = s''$, $\pi^{(n)} = \pi''^{(n)}$ ($n = 0, 1, 2, \dots$). From Theorem 1, we obtain

$$u^* = \text{Tr}[B^*_x f(D^* \bar{D}^*)] \quad (\text{or } \text{Tr}[B^*_x g(B^*)]).$$

On using Theorems 1 and 2, we can construct a new soliton equation through two soliton equations with similar properties. As an example, we use the trace method to solve the Hirota equation [8] as follows:

$$i\psi_t + i3\alpha|\psi|^2\psi_x + \rho\psi_{xx} + i\sigma\psi_{xxx} + \delta|\psi|^2\psi = 0 \tag{9}$$

where α, ρ, σ and δ are positive real constants with the relation $\alpha/\sigma = \delta/\rho = \lambda$. In one limit of $\alpha = \sigma = 0$, the equation becomes the nonlinear Schrödinger equation [9] that describes a plane self-focusing and one-dimensional self-modulation of waves in nonlinear dispersive media

$$i\psi_t + \rho\psi_{xx} + \delta|\psi|^2\psi = 0. \tag{10}$$

In another limit of $\rho = \delta = 0$ the equation for real ψ , becomes the modified Korteweg–de Vries equation [10], [11]

$$\psi_t + 3\alpha\psi^2\psi_x + \sigma\psi_{xxx} = 0. \tag{11}$$

Hence, the present solutions reveal the close relation between classical solitons and envelope solitons. Substituting the formal series

$$\psi = \psi^{(1)} + \psi^{(3)} + \dots + \psi^{(2n+1)} + \dots \tag{12}$$

into Eq. (9), we obtain a set of equations for $\psi^{(2k+1)}$ ($k = 0, 1, 2, \dots$):

$$i\psi_t^{(1)} + \rho\psi_{xx}^{(1)} + i\sigma\psi_{xxx}^{(1)} = 0, \tag{13}$$

$$i\psi_t^{(3)} + \rho\psi_{xx}^{(3)} + i\sigma\psi_{xxx}^{(3)} = -i3\alpha\psi^{(1)}\bar{\psi}^{(1)}\psi_x^{(1)} - \delta\psi^{(1)}\bar{\psi}^{(1)}\psi^{(1)}, \tag{14}$$

⋮

$$\begin{aligned} & i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)} \\ &= -i3\alpha \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \psi^{(2l+1)}\bar{\psi}^{(2m+1)}\psi_x^{(2n-2l-2m-1)} \\ & - \delta \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \psi^{(2l+1)}\bar{\psi}^{(2m+1)}\psi^{(2n-2l-2m-1)} \end{aligned} \tag{15}$$

⋮

We can solve the set of equations iteratively:

$$\psi^{(1)} = \sum_{l_1=1}^N \phi_{l_1}^2(x, t), \tag{16}$$

$$\psi^{(3)} = -\frac{\lambda}{8} \sum_{l_1=1}^N \sum_{l_2=1}^N \sum_{l_3=1}^N \frac{1}{(P_{l_1} + \bar{P}_{l_2})(\bar{P}_{l_2} + P_{l_3})} \phi_{l_1}^2(x, t)\bar{\phi}_{l_2}^2(x, t)\phi_{l_3}^2(x, t) \tag{17}$$

where $\phi_k(x, t) = A_k(0)\exp(P_kx - \Omega_k t)$, $\Omega_k = -2i\rho P_k^2 + 4\sigma P_k^3$, $A_k(0)$ and P_k are complex constants relating respectively to the amplitude and phase of the k -th soliton

($k = 1, 2, \dots, N$). We introduce two $N \times N$ matrices B and D whose elements are given respectively by:

$$B_{mn} = \left[\frac{1}{P_m + P_n} \right] \phi_m(x, t) \phi_n(x, t), \quad D_{mn} = \left[\frac{1}{P_m + \bar{P}_n} \right] \phi_m(x, t) \bar{\phi}_n(x, t).$$

With matrices B and D , $\psi^{(1)}$ and $\psi^{(3)}$ are expressed as:

$$\psi^{(1)} = \text{Tr}[B_x], \tag{18}$$

$$\psi^{(3)} = -\frac{\lambda}{8} \text{Tr}[B_x(D\bar{D})]. \tag{19}$$

In general, we can prove that

$$\psi^{(2n+1)} = (-1)^n \frac{\lambda^n}{8^n} \text{Tr}[B_x(D\bar{D})^n], \quad n = 0, 1, 2, \dots \tag{20}$$

satisfies Eq. (15).

With the definitions of matrices B and D

$$\begin{aligned} \psi^{(2n+1)} &= \\ &= (-1)^n \frac{\lambda^n}{8^n} \sum_1 \dots \sum_{2n+1} \frac{\phi_1^{2-2} \phi_2^{2-2} \dots \phi_{2n}^{2-2} \phi_{2n+1}^2}{(P_1 + \bar{P}_2)(\bar{P}_2 + P_3) \dots (P_{2n-1} + \bar{P}_{2n})(\bar{P}_{2n} + P_{2n+1})}. \end{aligned} \tag{21}$$

Here and in the following we simplify the expressions by writing $1, 2, \dots, 2n+1$ instead of $l_1, l_2, \dots, l_{2n+1}$. There should be no confusion about this. We have

$$\begin{aligned} &i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)} \\ &= (-1)^n \frac{\lambda^n}{2^{3n-1}} \sum_1 \dots \sum_{2n+1} \{4i\sigma[(P_1 + \bar{P}_2 + \dots + \bar{P}_{2n} + P_{2n+1})^3 \\ &\quad - (P_1^3 + \bar{P}_2^3 + \dots + \bar{P}_{2n}^3 + P_{2n+1}^3)] + 2\rho[(P_1 + \bar{P}_2 + \dots + \bar{P}_{2n} + P_{2n+1})^2 \\ &\quad - (P_1^2 - \bar{P}_2^2 + \dots - \bar{P}_{2n}^2 + P_{2n+1}^2)]\} \\ &\quad \times \frac{\phi_1^{2-2} \phi_2^{2-2} \dots \phi_{2n}^{2-2} \phi_{2n+1}^2}{(P_1 + \bar{P}_2)(\bar{P}_2 + P_3) \dots (P_{2n-1} + \bar{P}_{2n})(\bar{P}_{2n} + P_{2n+1})}. \end{aligned} \tag{22}$$

Substituting two identities

$$\begin{aligned}
 & (k_1 + k_2 + \dots + k_{2n} + k_{2n+1})^3 - (k_1^3 + k_2^3 + \dots + k_{2n}^3 + k_{2n+1}^3) \\
 &= 3 \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} [(k_1 + \dots + k_{2l+1})(k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3}) \\
 &+ (k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3})(k_{2l+2m+3} + \dots + k_{2n+1})], \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 & (k_1 + k_2 + \dots + k_{2n} + k_{2n+1})^2 - (k_1^2 + k_2^2 + \dots + k_{2n}^2 + k_{2n+1}^2) \\
 &= 2 \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} [(k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3})] \quad (24)
 \end{aligned}$$

into Eq. (22) and using Eq. (20) for $\psi^{(2k+1)}$ ($k < n$), we obtain

$$\begin{aligned}
 & i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)} \\
 &= -\delta \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \psi^{(2l+1)} \bar{\psi}^{(2m+1)} \psi^{(2n-2l-2m-1)} \\
 & - i\frac{3}{2}\alpha \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \bar{\psi}^{(2m+1)} (\psi^{(2l+1)} \psi^{(2n-2l-2m-1)})_x.
 \end{aligned}$$

Therefore we obtain the N -envelope-soliton solution for Eq. (9) in the following form:

$$\psi = \text{Tr} \left\{ \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^k}{8^k} [B_x (D\bar{D})^k] \right\} = \text{Tr} \left[B_x \left(1 + \frac{\lambda}{8} D\bar{D} \right)^{-1} \right] \quad (25)$$

where $\|D\bar{D}\| < 8/\lambda$ in a certain region. In particular, for $N = 1$, we obtain the one-envelope-soliton solution

$$\begin{aligned} \psi(x, t) = & \frac{A_1(0)}{2} \operatorname{sech}[(P_1 + \bar{P}_1)x - (\Omega_1 + \bar{\Omega}_1)t + \eta] \\ & \times \exp[(P_1 - \bar{P}_1)x - (\Omega_1 - \bar{\Omega}_1)t - \eta] \end{aligned} \quad (26)$$

where

$$\eta = \frac{1}{2} \ln \left(\frac{\lambda |A_1(0)|^4}{8(P_1 + \bar{P}_1)^2} \right).$$

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