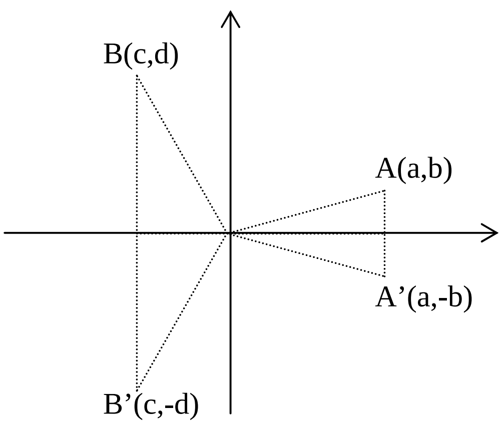
An Insight into Trigonometric Identities and Trigonometric Equations

Elementary Trigonometric Results

Recall definition of sine, cosine and tangent of an angle.
Let $\mathrm{P}(x, y)$ be a point on a circle with radius $r = \sqrt{x^2 + y^2}$ in the Cartesian Plane
As usual, $\angle\mathrm{XOP}$ is measured in the direction from $\overrightarrow{\mathrm{OX}}$ to $\overrightarrow{\mathrm{OP}}$, taking anticlockwise as positive.
Let $\theta = \angle\mathrm{XOP}$
Then $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

Relations between trigonometric functions of angles of opposite signs

Observe the following diagram where points A and A', B and B' are reflection of each other in the x-axis (think of $\mathrm{A'}$ to A as well as A to $\mathrm{A'}$)

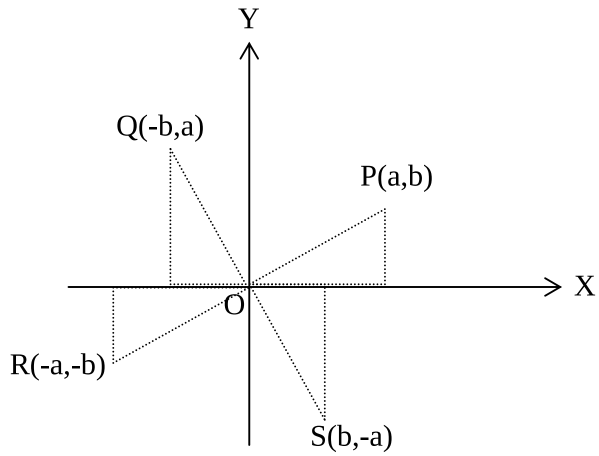


We can summarize the change of coordinates corresponding to change of sign of angle (namely a reflection) as this :
$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ -y \end{pmatrix}$ as $\theta \to -\theta$, for all values of θ, positive or negative
We thus have the result
$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$ ……….. (1)

Relations between trigonometric functions of angles by a difference of one right angle

The following diagram shows how coordinates are changed as a result of adding one right angle to a given one. The various cases are:

$\angle$XOP is changed to $\angle$XOQ, $\angle$XOQ is changed to $\angle$XOR ,………



We can summarize the change of coordinates as a result of the addition of one right angle as this:

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$ as $\theta \to \theta + \frac{\pi}{2}$, where θ is in any quadrant

We thus have

If $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$,

then $\sin(\theta + \frac{\pi}{2}) = \frac{x}{r} = \cos\theta$, $\cos(\theta + \frac{\pi}{2}) = \frac{-y}{r} = -\sin\theta$

So,

$\sin(\theta + \frac{\pi}{2}) = \cos\theta$, $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\tan(\theta + \frac{\pi}{2}) = -\tan\theta$

for all values of θ ………………………………………………………… (2)

(The reader should not just take $\angle$XOP as an example of θ, instead, think about $\angle$XOQ, $\angle$XOR and $\angle$XOS as well)

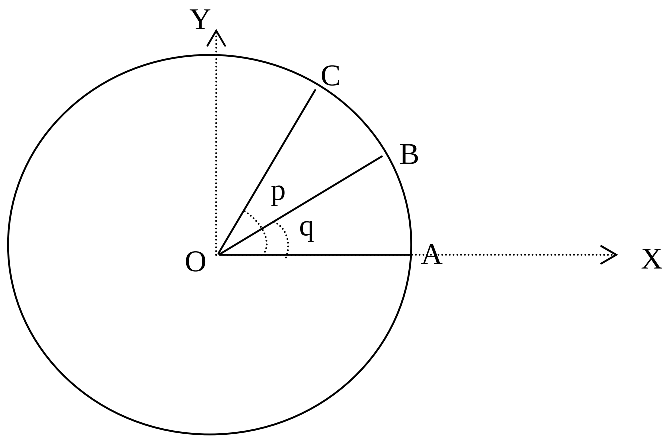
With results (1) and (2), which are valid for all values of θ, we deduce all results related to complimentary and supplementary angles.

For example,

$\cos(180^0 - \theta)$ = $\cos(90^0 + 90^0 - \theta)$ = - $\sin(90^0 - \theta)$ = - $\cos(-\theta)$ = - $\cos\theta$

which is true for all values of θ

Formulae for addition and subtraction of angles.



The above figure shows two angles p and q of any size, given by $\angle$AOC and $\angle$AOB respectively, where A, B and C are points on a circle, centre O, radius r. (It need not be $p > q$ as depicted)

Let OA be lying on the OX axis and so the angles are measured using OX as initial direction.

We find $C(r\cos p, r\sin p)$, $B(r\cos q, r\sin q)$

By using the cosine rule, we have, for all possible positions of B and C,

$2(OB)(OC)\cos(p-q) = (OB)^2 + (OC)^2 - (BC)^2$

So,

$2r^2\cos(p-q) = r^2 + r^2 - [(r\cos p - r\cos q)^2 + (r\sin p - r\sin q)^2]$

$= 2r^2 - r^2[\cos^2 p - 2\cos p\cos q + \cos^2 q] - r^2[\sin^2 p - 2\sin p\sin q + \sin^2 q]$

$= 2r^2 - r^2[1 + 1 - 2\cos p\cos q - 2\sin p\sin q]$

$= 2r^2[\cos p\cos q + \sin p\sin q]$

So, $\cos(p-q) = \cos p\cos q + \sin p\sin q$, for all values of p and q ……………… (3)

From results (1) , (2) and (3) we can establish all formulae related to additions and subtractions of angles,

For example

$\cos(p+q) = \cos(p - -q) = \cos p\cos -q + \sin p\sin -q = \cos p\cos q - \sin p\sin q$

Trigonometric Equations

Let us investigate the three simple equations

$\sin\theta = a,$ where $-1 \le a \le 1$

$\cos\theta = b,$ where $-1 \le b \le 1$

$\tan\theta = c,$ for all values of c

First note that all three equations have two solutions in any interval of length 2π, for example, we can find two solutions in each of $(0.5\pi,\ 2.5\pi)$, $(-0.9\pi,\ 1.1\pi)$ etc
So, to begin with, it is usually convenient to obtain two in the range of
$[-\pi, \pi]$
Next, we note that if α is a solution, say to $\sin\theta = a$, where $-1 \le a \le 1$, then
$\sin\alpha = a$, $\sin(\alpha + 2\pi) = a$, $\sin(\alpha - 2\pi) = a$, $\sin(\alpha + 2n\pi) = a$, where n is any integer
So, if α, β are two solutions to $\sin\theta = a$, where $-1 \le a \le 1$, in the interval $[-\pi, \pi]$, then the complete solution set is $\theta = \alpha + 2n\pi$, $\beta + 2n\pi$, though of course, it works out to be $\theta = n\pi + (-1)^n(\alpha)$, which is equivalent to $\theta = n\pi + (-1)^n(\beta)$
(More discussions are left as an exercise)

Exercise (without answers attached)

Recall these results:

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$,
for all values of θ ………………………….. (1)

$\sin(\theta + \frac{\pi}{2}) = \cos\theta$, $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\tan(\theta + \frac{\pi}{2}) = -\tan\theta$
for all values of θ ……………………… (2)
$\cos(p - q) = \cos p \cos q + \sin p \sin q$, for all values of p and q ……………… (3)

1. Testify the rule :
$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$ as $\theta \to \theta + 90^0$, where $P(x, y)$ is the point on a circle of radius r, $\theta = \angle XOP$, for the following cases

2. By using Results (1) and (2) above, express the following in terms of $\sin\theta$, $\cos\theta$ and $\tan\theta$ only
(i) $\sin(180^0 - \theta)$ (ii) $\cos(180^0 - \theta)$ (iii) $\tan(180^0 - \theta)$
(iv) $\sin(180^0 + \theta)$ (v) $\cos(180^0 + \theta)$ (vi) $\tan(180^0 + \theta)$

3. By using Results (1) and (2) above, express the following in terms of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\cot\theta$ only
(i) $\sin(270^0 - \theta)$ (ii) $\cos(270^0 - \theta)$ (iii) $\tan(270^0 - \theta)$
(iv) $\sin(270^0 + \theta)$ (v) $\cos(270^0 + \theta)$ (vi) $\tan(270^0 + \theta)$

4. By using Results (1), (2) and (3) above, obtain the formulae
$\sin(p + q) = \sin p \cos q + \cos p \sin q$ and
$\sin(p - q) = \sin p \cos q - \cos p \sin q$

5. $\alpha,\ \beta$ are two different solutions of the equation
$\sin\theta = a,$ where $|a| \le 1$
State a relation between α and β if
(i) α and β are in the same quadrant
(ii) α and β are in different quadrant
(iii) there is no information about the quadrants of α and β.

6. $\alpha,\ \beta$ are two different solutions of the equation
$\cos\theta = a,$ where $|a| \le 1$
State a relation between α and β

7. $\alpha,\ \beta$ are two different solutions of the equation
$\tan\theta = a$
What is the values of $\alpha - \beta$?

8 In the solution for the equation $\cos(2x - 70^0) = 0.5$, values of x required for are in the interval $(-150^0,\ 570^0)$
How many possible values of x are there?

Exercise (with answers attached)

Recall these results:

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$,
for all values of θ ………………………… (1)

$\sin(\theta + \frac{\pi}{2}) = \cos\theta$, $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$ and $\tan(\theta + \frac{\pi}{2}) = -\tan\theta$
for all values of θ ……………………… (2)
$\cos(p - q) = \cos p\cos q + \sin p\sin q$, for all values of p and q ……………… (3)

1. Testify the rule :
   $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$ as $\theta \to \theta + 90^0$, where $P(x, y)$ is the point on a circle of radius r, $\theta = \angle XOP$, for the following cases
   (i) $\theta = 240^0$ (ii) $\theta = -240^0$ (iii) $\theta = 500^0$

2. By using Results (1) and (2) above, express the following in terms of $\sin\theta$, $\cos\theta$ and $\tan\theta$ only
   (i) $\sin(180^0 - \theta)$ (ii) $\cos(180^0 - \theta)$ (iii) $\tan(180^0 - \theta)$
   (iv) $\sin(180^0 + \theta)$ (v) $\cos(180^0 + \theta)$ (vi) $\tan(180^0 + \theta)$

Answers
(i) $\sin\theta$ (ii) $-\cos(\theta)$ (iii) $-\tan\theta$
(iv) $-\sin\theta$ (v) $-\cos\theta$ (vi) $\tan\theta$

3. By using Results (1) and (2) above, express the following in terms of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\cot\theta$ only

   (i) $\sin(270^0 - \theta)$ (ii) $\cos(270^0 - \theta)$ (iii) $\tan(270^0 - \theta)$
   (iv) $\sin(270^0 + \theta)$ (v) $\cos(270^0 + \theta)$ (vi) $\tan(270^0 + \theta)$

Answers:
(i) $-\cos\theta$ (ii) $-\sin\theta$ (iii) $\cot\theta$
(iv) $-\cos\theta$ (v) $\sin\theta$ (vi) $-\cot\theta$

4. By using Results (1), (2) and (3) above, obtain the formulae
   $\sin(p + q) = \sin p\cos q + \cos p\sin q$ and
   $\sin(p - q) = \sin p\cos q - \cos p\sin q$

Solution

$$\sin(p + q) = -\cos(\frac{\pi}{2} + p + q) = -[\cos(\frac{\pi}{2} + p)\cos q - \sin(\frac{\pi}{2} + p)\sin q\,]$$
$$= \sin p\cos q + \cos p\sin q$$
$$\sin(p - q) = -\cos(\frac{\pi}{2} + p - q) = -[\cos(\frac{\pi}{2} + p)\cos -q - \sin(\frac{\pi}{2} + p)\sin -q$$

$= \sin p \cos q - \cos p \sin q$

5. $\alpha,\ \beta$ are two different solutions of the equation
$\sin\theta = a,$ where $|a| \le 1$
State a relation between α and β if
(i) α and β are in the same quadrant
(ii) α and β are in different quadrant
(iv) there is no information about the quadrants of α and β.

Answers:
(i) $\beta = 2n\pi + \alpha,$ where $n \in \mathbb{Z}$
(ii) $\beta = 2n\pi + \pi - \alpha,$ where $n \in \mathbb{Z}$ $\pi - \alpha$
(iii) $\beta = \pi - \alpha,\ 2\pi + \alpha,\ 3\pi - \alpha, ...$ $-\pi - \alpha,\ -2\pi + \alpha,\ -3\pi - \alpha, ...$
So, $\beta = n\pi + (-1)^n \alpha,$ where $n \in \mathbb{Z}$

6. $\alpha,\ \beta$ are two different solutions of the equation
$\cos\theta = a,$ where $|a| \le 1$
State a relation between α and β

Answer
$\beta = 2n\pi \pm \alpha,$ where $n \in \mathbb{Z}$

7. $\alpha,\ \beta$ are two different solutions of the equation
$\tan\theta = a$
What is the values of $\alpha - \beta$?

Answer
$\alpha - \beta = n\pi,$ for $n \in \mathbb{Z}$

8 In the solution for the equation $\cos(2x - 70^0) = 0.5$, values of x required for are in the interval $(-150^0,\ 570^0)$
How many possible values of x are there?

Answer
The interval length for x is $570^0 - -150^0 = 720^0 = 2 \times 360^0$
So, the interval length for $(2x - 70^0)$ is 4×360^0
So, there are 8 possible values for $(2x - 70^0)$, and therefore also 8 possible values for x.