An Insight into Differentiation and Integration

Differentiation

Differentiation is basically a task to find out how one variable is changing in relation to another variable, the latter is usually taken as a cause of the change.
For instance, $y = x^2$
We might consider x taking values: 1, 2, 3, ……. n, $n+1$
successively by increment 1, then y is increasing by an increment
$(n+1)^2 - n^2 = 2n+1$ at $x = n$
The average increment over the interval $(n,\ n+1)$ of x is

$$\frac{(n+1)^2 - n^2}{1} = 2n+1$$

To be more generally,
the average increment over the interval $(x,\ x+1)$ of x is $\frac{(x+1)^2 - x^2}{1} = 2x+1$

By decreasing the amount of increment of x to be $\frac{1}{2}$, the average increment over the interval $(x,\ x+\frac{1}{2})$ of x is

$$\frac{(x+\frac{1}{2})^2 - x^2}{\frac{1}{2}} = 2x+(\frac{1}{2})^2$$

Generalizing, the average increment over the interval $(x,\ x+\delta x)$ of x is

$$\frac{(x+\delta x)^2 - x^2}{\delta x} = 2x+(\delta x)^2$$

Going on this way, we then come to the idea that the instantaneous average increment at x is $2x$, when the interval becomes widthless. This is defined as the rate of change of x^2 with respective to x at any value of x.

To express the process of work, which we define as differentiation, and the result, we write

$$\frac{d}{dx}x^2 = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} = 2x$$

We have three fundamental results of differentiation:

$\frac{d}{dx}x^n = nx^{n-1}$ for positive integer n

$\frac{d}{dx}\sin x = \cos x$ (Based on the fundamental limit result $\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$), when x is in radians

$\frac{d}{dx}e^x = e^x$

From these, with the helps of a number of rules such as The Product Rules, The Quotient Rule, The Chain Rule, Implicit Function Rule… we deduce many standard results including

$\frac{d}{dx}x^n = nx^{n-1}$ for any rational number n (Finally, for any real number n)

Of course, we should be able to establish all other results from first principles, which means a work relying only on the definition

$$\frac{d}{dx}f(x) = \lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

Integration

Differentiation is important not only that it is a tool to track the change of one variable with respect to another, also, knowing of the derivative of a function leads us to the identification of the function itself.

For example, if we find that $\frac{d}{dx}f(x) = \frac{d}{dx}g(x) = u(x)$, then $f(x)$ and $g(x)$ can only differ by a constant

So, given $\frac{d}{dx}f(x) = u(x)$, to find $f(x)$, we first look for $g(x)$ which has the known derivative $u(x)$

The job is $\int u(x)dx$, which gives a result pending a constant

We write $f(x) = g(x) + c$

If we know that $f(a) = 0$, then $c = -g(a)$, $f(x) = g(x) - g(a)$. This is a definite result, which gives value of $f(x)$ at any value, for example, at $x = b$,

$f(b) = g(b) - g(a)$, for which we write $\int_a^b u(x)dx$

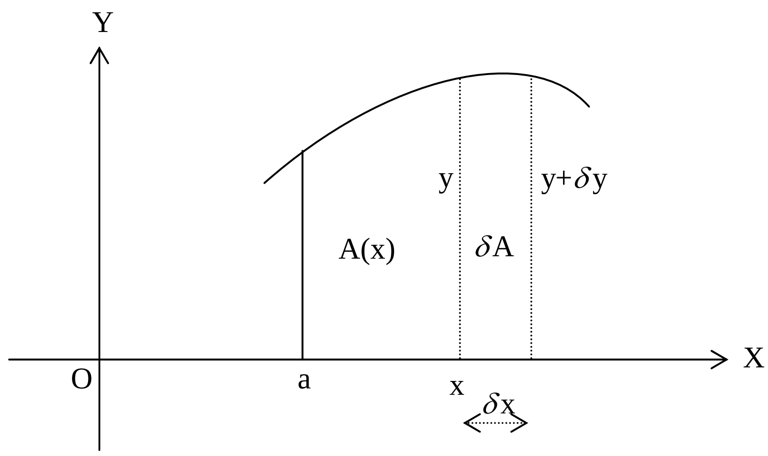
The value is definite and the work is called a definite integration

Cases of using derivative to find the original functions

A well-known case is finding distance traveled with knowledge of velocity v. If s is the distance traveled in time t, we have

$\frac{ds}{dt} = v$, based on the genuine definition of v

For some other cases, the derivative of a variable must be reasoned out, though they are usually taken intuitively. Take the case when we come to such work like finding area under a curve.



If we define $\mathrm{A}(x)$ to be the area under the curve from $x = a$ to x

Intuitively, we take $\frac{d\mathrm{A}}{dx} = y$, so A=$\int y dx$, Area $\mathrm{A}(x)$ can then be found

To be a bit more rigorous

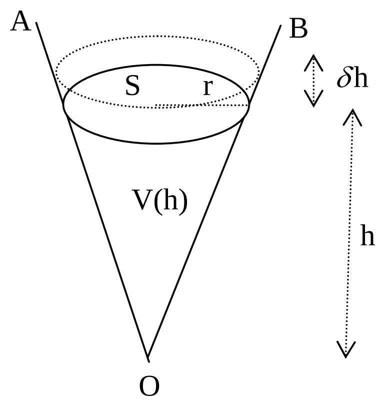
We first claim that $\delta\mathrm{A}$ lies between $y\delta x$ and $(y+\delta y)\delta x$,

So, $\frac{\delta\mathrm{A}}{\delta x}$ lies between y and $(y+\delta y)$

$$\frac{d\mathrm{A}}{dx} = \lim_{\delta x \to 0} \frac{\delta\mathrm{A}}{\delta x} = y$$

Take another case, for finding the volume V(h) of a right circular cone of height h

Denoting $\mathrm{V}(h)$ to be the volume of the circular cone, of the same shape, with a height h, considered as a variable, S to be the surface area of the circular cross-section at height h,



We claim that δV lies between Sδh and $(\mathrm{S}+\delta\mathrm{S})\delta h$

$\dfrac{\delta \mathrm{V}}{\delta h}$ lies between S and $(\mathrm{S}+\delta\mathrm{S})$

so $\dfrac{d\mathrm{V}}{dh} = \lim\limits_{\delta h \to 0} \dfrac{\delta \mathrm{V}}{\delta h}$ = S .

By similarity, $r = kh,$ for a constant k

$$\mathrm{V}=\int \mathrm{S}dh = \int \pi (kh)^2\, dh = \pi k^2 \frac{h^3}{3} + c$$

$$= \frac{1}{3}\pi (kh)^2 h + c = \frac{1}{3}\pi r^2 h + c = \frac{1}{3}\pi r^2 h \quad \text{if we take V = 0 for } h = 0$$

Exercise (without solution attached)

1. Differentiate $\tan x$ with respect to x from first principles.
   (You may use the result $\lim\limits_{\delta x \to 0} \dfrac{\sin \delta x}{\delta x} = 1$)

2. Base on the result $\dfrac{d}{dx} x^n = nx^{n-1}$ for positive integer n, and the rules of differentiation, prove that $\dfrac{d}{dx} x^{\frac{1}{3}} = \dfrac{1}{3} x^{-\frac{2}{3}}$

   Prove also that $\dfrac{d}{dx} x^{\frac{2}{3}} = \dfrac{2}{3} x^{-\frac{1}{3}}$

3 A particle is moving vertically upwards with velocity $v = 50t - 5t^2$
   Show that it is moving downwards after $t = 10$
   Find the distance of the particle from its position at $t = 0$

(i) When $t = 8$

(ii) When $t = 12$

$$s = \int_0^{12} (50t - 5t^2)dt \; = [25t^2 - \frac{5}{3}t^3]_0^{12} = 720$$

4. Given that $y = \sec x^0$, find $\frac{dy}{dx}$

Given a reason why in Calculus, radian measure is preferred rather than degree measure.

5. A kite is at a horizontal distance l from the flyer and a vertical distance h above the same. When it is rising up at a velocity v, the flyer let off more string to keep the horizontal distance constant.
Assuming the string to be straight, and the angle of elevation of the kite to be θ radians , find the rate of change of θ with respective to time t

6. Let $(e^*)^x = (1 + \frac{x}{n})^n$, where n is a large number

Show that $\frac{d}{dx}(e^*)^x \approx (e^*)^x$

Exercise (with solution attached)

1. Differentiate $\tan x$ with respect to x from first principles.

   (You may use the result $\lim_{\delta x \to 0} \dfrac{\sin \delta x}{\delta x} = 1$)

Solution:

$$\frac{\tan(x+\delta x)-\tan x}{\delta x} = \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\delta x[\cos(x+\delta x)\cos x]} =$$

$$\frac{\sin(x+\delta x - x)}{\delta x[\cos(x+\delta x)\cos x]} = \frac{\sin \delta x}{\delta x}\frac{1}{\cos(x+\delta x)\cos x}$$

So, $\lim_{\delta x \to 0} \dfrac{\tan(x+\delta x)-\tan x}{\delta x} = \lim_{\delta x \to 0}[\dfrac{\sin \delta x}{\delta x}\dfrac{1}{\cos(x+\delta x)\cos x}] = \sec^2 x$

2. Base on the result $\dfrac{d}{dx}x^n = nx^{n-1}$ for positive integer n, and the rules of differentiation, prove that $\dfrac{d}{dx}x^{\frac{1}{3}} = \dfrac{1}{3}x^{-\frac{2}{3}}$

   Prove also that $\dfrac{d}{dx}x^{\frac{2}{3}} = \dfrac{2}{3}x^{-\frac{1}{3}}$

Solution:

Let $y = x^{\frac{1}{3}}$, then $y^3 = x$, $3y^2\dfrac{dy}{dx} = 1$, $\dfrac{dy}{dx} = \dfrac{1}{3y^2} = \dfrac{1}{3x^{\frac{2}{3}}} = \dfrac{1}{3}x^{-\frac{2}{3}}$

For proving of $\dfrac{d}{dx}x^{\frac{2}{3}} = \dfrac{2}{3}x^{-\frac{1}{3}}$, let $z^3 = x^2$

3. A particle is moving vertically upwards with velocity $v = 50t - 5t^2$

   Show that it is moving downwards after $t = 10$

   Find the distance of the particle from its position at $t = 0$

   (iii) When $t = 8$

   (iv) When $t = 12$

Solution:

$v = 50t - 5t^2 = 5t(10-t) < 0$ when $t > 10$

So, the particle is moving downwards after $t = 10$

(i) If s is the distance moved upwards after t

$\dfrac{ds}{dt} = v = 50t - 5t^2$, noting that this is true for all t,

that is whether $t < 10$ or $t \geq 10$

$$s = \int_0^8 (50t - 5t^2)dt = [25t^2 - \frac{5}{3}t^3]_0^8 = 746\frac{2}{3}$$

(ii) $\dfrac{ds}{dt} = v = 50t - 5t^2$, even though $t \geq 10$

$$s = \int_0^{12} (50t - 5t^2)dt = [25t^2 - \frac{5}{3}t^3]_0^{12} = 720$$

4. Given that $y = \sec x^0$, find $\dfrac{dy}{dx}$

Given a reason why in Calculus, radian measure is preferred rather than degree measure.

Solution

Let $y = \sec x^0 = \sec u$, where u radians = x degrees. Then $180u = x\pi$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)\frac{\pi}{180} = (\sec x^0 \tan x^0)\frac{\pi}{180}$$

As seen from the above, $\dfrac{d}{dx}\sec x^0 = (\sec x^0 \tan x^0)\dfrac{\pi}{180}$

Whereas $\dfrac{d}{du}\sec u^{\text{rd}} = \sec u \tan u$

5. A kite is at a horizontal distance l from the flyer and a vertical distance h above the same. When it is rising up at a velocity v, the flyer let off more string to keep the horizontal distance constant.

Assuming the string to be straight, and the angle of elevation of the kite to be θ radians, find the rate of change of θ with respective to time t

Solution

We have $\tan\theta = \dfrac{h}{l}$, where h is a variable and l is a constant

Though an expression for $\dfrac{d\theta}{dt}$ is what we are looking for, we need not write $\theta = \tan^{-1}(\dfrac{h}{l})$, and do $\dfrac{d\theta}{dt}$. We can choose to do indirectly

Differentiating both sides of this equation, we get $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{1}{l}\dfrac{dh}{dt} = \dfrac{1}{l}v$

So $\dfrac{d\theta}{dt} = (\cos^2\theta)\dfrac{1}{l}v$, though the result is not totally in terms of the variable h.

6. Let $(e^*)^x = (1+\dfrac{x}{n})^n$, where n is a large number

Show that $\dfrac{d}{dx}(e^*)^x \approx (e^*)^x$

Solution

$$\frac{d}{dx}(e^*)^x = \frac{d}{dx}(1+\frac{x}{n})^n = n(1+\frac{x}{n})^{n-1}(\frac{1}{n}) = (1+\frac{x}{n})^{n-1} = (1+\frac{x}{n})^n\frac{1}{(1+\frac{x}{n})}$$

$$\approx (1+\frac{x}{n})^n = (e^*)^x, \text{ as } (1+\frac{x}{n}) \approx 1$$